CATEGORIES $\mathcal{O}$

Contents

The big purpose of the entire course is to study the category of rational representations of connected reductive groups in positive characteristic. This category shares many common features with an easier and more classical representation theoretic category: the BGG category $\mathcal{O}$, which consists of certain, generally speaking, infinite dimensional representations of a complex semisimple Lie algebra. So we discuss the categories $\mathcal{O}$ in some detail before proceeding to the rational representations.

1. Definition, Verma and simple modules

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$ so that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. We have the triangular decompositions $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, and hence

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}). \tag{1.1}$$

For $\lambda \in \mathfrak{h}^*$ we define Verma modules as follows $\Delta(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module with action via $\mathfrak{b} \twoheadrightarrow \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$. Thanks to (1.1), $\Delta(\lambda)$ is a free rank one $U(\mathfrak{n}_-)$-module, and, moreover, $\Delta(\lambda) \cong U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ as a $U(\mathfrak{b}_-)$-module. Its weight decomposition

$$\Delta(\lambda) = \bigoplus_{\mu \leqslant \lambda} \Delta(\lambda)_\mu, \tag{1.2}$$

where $<$ refers to the standard order on Λ, and $\dim \Delta(\lambda)_\lambda = 1$. Any proper $\mathfrak{b}_-$- and hence $\mathfrak{g}$-submodule of $\Delta(\lambda)$ is contained in $\bigoplus_{\mu<\lambda} \Delta(\lambda)_\mu$. It follows that $\Delta(\lambda)$ has a unique simple quotient, denote it by $L(\lambda)$. All weights of the kernel of $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ are $< \lambda$. We note that $\lambda \neq \lambda' \Rightarrow L(\lambda) \ncong L(\lambda')$.

Definition 1.1. *Let $\mathcal{O}$ denote the category of finitely generated $U(\mathfrak{g})$-modules M such that*

(1) *M has weight decomposition, $M = \bigoplus_{\mu\in\Lambda} M_\mu$ for the $\mathfrak{h}$-action (where Λ stands for the weight lattice of $\mathfrak{g}$). If $M_\mu \neq \{0\}$, we say that μ is a* weight *of M.*
(2) *The set of weights of M is bounded from above, i.e., there is a finite collection $\lambda_1, \ldots, \lambda_k \in \Lambda$ such that $M_\mu \neq \{0\} \Rightarrow \mu \leqslant \lambda_i$ for some i.*

Note that $\dim M_\mu < \infty$ for all $M \in \mathcal{O}, \mu \in \Lambda$.

Exercise 1.2. *Every object in $\mathcal{O}$ admits a finite filtration by quotients of Verma modules.*

Exercise 1.3. *The assignment $\lambda \mapsto L(\lambda)$ identifies Λ with the set* $\operatorname{Irr}(\mathcal{O})$ *of isomorphism classes of simples in $\mathcal{O}$.*

1.1. **Harish-Chandra isomorphism and block decomposition.** Let $Z(\mathfrak{g})$ denote center of $U(\mathfrak{g})$.

Theorem 1.4 (Harish-Chandra)**.** *$Z(\mathfrak{g})$ is identified with $\mathbb{C}[\mathfrak{h}^*]^W$, where the action of W on $\mathfrak{h}^*$ is given by $w \cdot \lambda := w(\lambda + \rho) - \rho$.*

Let us explain how this identification work. For $z \in Z$, let $f_z \in \mathbb{C}[\mathfrak{h}^*]^W$ be the corresponding element. We need to explain how to compute $f_z(\lambda)$.

Exercise 1.5. *z preserves $\Delta(\lambda)_\lambda$ and hence acts on $\Delta(\lambda)$ by a scalar.*

The value $f_z(\lambda)$ is this scalar.

Example 1.6. *Let $\mathfrak{g} = \mathfrak{sl}_2$. We have $W = S_2 = \{e, s\}$, $\rho = 1$ and $Z(\mathfrak{g})$ is freely generated by the Casimir element*

$$C = ef + fe + \frac{h^2}{2} = 2fe + h + \frac{h^2}{2}.$$

We see that C acts on $\Delta(\lambda)$ via $(\lambda(\lambda+2))/2$. The dotted action is given by $s \cdot \lambda = s(\lambda+1) - 1 = -\lambda - 2$. So $\mathbb{F}[\lambda]^W = \mathbb{F}[\lambda(\lambda+2)]$ and everything matches.

Let us proceed to (infinitesimal) blocks.

Definition 1.7. *For $W \cdot \lambda \in W\backslash\Lambda$, let $\mathcal{O}_{W\cdot\lambda}$ denote the full subcategory of $\mathcal{O}$ consisting of all modules $M \in \mathcal{O}$ such that every $z \in Z(\mathfrak{g})$ with $f_z(\lambda) = 0$ acts on M nilpotently.*

Example 1.8. *We have $\Delta(\mu) \in \mathcal{O}_{W\cdot\lambda} \Leftrightarrow \mu \in W \cdot \lambda$. The same is true for the objects $L(\mu)$.*

Exercise 1.9. *For every $\lambda \in \Lambda$, the object $\Delta(\lambda)$ has finite length.*

Using Exercises 1.2,1.9, we deduce the following result.

Proposition 1.10. *Every object in $\mathcal{O}$ has finite length. Moreover, $\mathcal{O} = \bigoplus_{\chi \in W\backslash\Lambda} \mathcal{O}_\chi$.*

Exercise 1.11. *If $\lambda + \rho \in \Lambda$ is antidominant, then $\Delta(\lambda) = L(\lambda)$ (in fact, it is an "if and only if" statement).*

Note that $\mathrm{Irr}(\mathcal{O}_{W\cdot\lambda}) \xrightarrow{\sim} W \cdot \lambda$. In particular, it is identified with W when $\lambda+\rho$ is regular, i.e. $\langle \lambda+\rho, \alpha^\vee\rangle \neq 0$ for all $\alpha \in \Delta$. In this case, we call $\mathcal{O}_{W\cdot\lambda}$ a *regular block*. Otherwise, we say that $\mathcal{O}_{W\cdot\lambda}$ is a *singular block*.

2. Duality

Let $\tau\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ denote the Cartan involution: it acts as -1 on $\mathfrak{h}^*$ and swaps each e_i with f_i. In particular, $\tau^2 = \mathrm{id}$.

Let M be a $\mathfrak{g}$-module such that $M = \bigoplus_{\mu\in\Lambda} M_\mu$ with $\dim M_\mu < \infty$ for all μ. We define another such $\mathfrak{g}$-module $M^\vee$ as follows: $M^\vee := \bigoplus_\mu M_\mu^*$ the action is given by $\langle x \cdot n, m\rangle = \langle n, -\tau(x)m\rangle$ for $n \in M^\vee, m \in M$.

Exercise 2.1. *Prove the following properties of $\bullet^\vee$:*

1. $(M^\vee)^\vee \cong M$,
2. $M_\mu^\vee = M_\mu^*, \forall \mu$, *– this is a reason why we twist with τ.*
3. $L(\lambda)^\vee \cong L(\lambda)$.

Since every object in $\mathcal{O}$ has finite length, from (1) and (3) we deduce

Proposition 2.2. *$\bullet^\vee$ is an equivalence $\mathcal{O} \xrightarrow{\sim} \mathcal{O}^{opp}$. Moreover, for each λ, the equivalence $\bullet^\vee$ restricts to $\mathcal{O}_{W\cdot\lambda} \xrightarrow{\sim} \mathcal{O}^{opp}_{W\cdot\lambda}$.*

Definition 2.3. *For $\lambda \in \Lambda$, define the* dual Verma module $\nabla(\lambda)$ *as $\Delta(\lambda)^\vee$.*

Remark 2.4. *Let us give a category theoretical characterization of the objects $\Delta(\lambda)$ and $\nabla(\lambda)$. Set $\mathcal{O}_{\not>\lambda} := \{M \in \mathcal{O} | M_\mu \neq \{0\} \Rightarrow \mu \not> \lambda\}$. Note that, for $M \in \mathcal{O}$, we have $\mathrm{Hom}(\Delta(\lambda), M) = \{m \in M_\lambda | \mathfrak{n}m = 0\}$. So, for $M \in \mathcal{O}_{\not>\lambda}$, this becomes $\mathrm{Hom}(\Delta(\lambda), M) = M_\lambda$ because the action of $\mathfrak{n}$ increases weights. We deduce that $\Delta(\lambda)$ is a projective object in $\mathcal{O}_{\not>\lambda}$, and, moreover, it is the projective cover of $L(\lambda)$. Dually, $\nabla(\lambda)$ is the injective envelope of $L(\lambda)$ in $\mathcal{O}_{\not>\lambda}$.*

Exercise 2.5. *Prove the following:*

1. $L(\lambda) \hookrightarrow \nabla(\lambda)$,
2. $\dim \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda\mu}$,
3. *A nonzero homomorphism $\Delta(\lambda) \to \nabla(\lambda)$ is a composition $\Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda)$.*

3. Projective functors and projective objects

3.1. **Tensor products with finite dimensional representations.** Let V be a finite dimensional $\mathfrak{g}$-module.

Exercise 3.1. *If $M \in \mathcal{O}$, then $M \otimes V \in \mathcal{O}$.*

So we get a functor $T_V : \mathcal{O} \to \mathcal{O}, T_V(M) := M \otimes V$. Let us discuss its properties.

Exercise 3.2. *The following are true:*

(1) *T_V is biadjoint to T_{V^*}, in particular, it is exact.*
(2) *T_V commutes with the duality functor.*

Let us understand the structure of $\Delta(\lambda) \otimes V$.

Lemma 3.3. *Let $v_1, \ldots, v_m$ be a weight basis of V with weights $\mu_1, \ldots, \mu_m$. Suppose that the vectors are ordered in the decreasing order, i.e., $\mu_i \leqslant \mu_j \Rightarrow i \geqslant j$. Then there is a filtration $\{0\} = F_0 \subset F_1 \subset \ldots \subset F_m = V \otimes \Delta(\lambda)$ with $F_i/F_{i-1} = \Delta(\lambda + \mu_i)$.*

Proof. Note that

$$\Delta(\lambda) \otimes V = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda) \otimes V \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C}_\lambda),$$

where the last isomorphism holds because of the following chain of isomorphisms:

$$\operatorname{Hom}_{\mathfrak{g}}((U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda) \otimes V, \bullet) \simeq \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, V^* \otimes \bullet) \simeq$$
$$\simeq \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V^* \otimes \bullet) \simeq \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V, \bullet) \simeq \operatorname{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (\mathbb{C}_\lambda \otimes V), \bullet).$$

By our ordering of weights, $V_{\leqslant i} := \operatorname{Span}(v_j | j \leqslant i)$ is a $\mathfrak{b}$-submodule of V. We set $F_i := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V_{\leqslant i} \otimes \mathbb{C}_\lambda)$. Thanks to triangular decomposition (1.1), $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bullet \simeq \mathcal{U}(\mathfrak{n}_-) \otimes_{\mathbb{C}} \bullet$ is an exact functor, so F_i/F_{i-1} is indeed $\Delta(\lambda + \mu_i)$. $\square$

Example 3.4. *Let $\mathfrak{g} = \mathfrak{sl}_2$. Consider the object $P(-2) := \mathbb{C}^2 \otimes \Delta(-1)$. We have $\mu_1 = 1, \mu_2 = -1$ and hence an exact sequence*

$$0 \to \Delta(0) \to P(-2) \to \Delta(-2) \to 0.$$

3.2. **Projective functors.** Now we want to cook functors $\mathcal{O}_{W\cdot\lambda} \to \mathcal{O}_{W\cdot\lambda'}$ out of T_V. Note that we have the inclusion functor $\iota_{W\cdot\lambda} : \mathcal{O}_{W\cdot\lambda} \hookrightarrow \mathcal{O}$ and its biadjoint, the projection functor $\pi_{W\cdot\lambda} : \mathcal{O} \twoheadrightarrow \mathcal{O}_{W\cdot\lambda}$. Often we abuse the notation and write $\iota_\lambda, \pi_\lambda$ instead of $\iota_{W\cdot\lambda}, \pi_{W\cdot\lambda}$.

By a *projective functor* we mean a functor of the form

$$\pi_{\lambda'} \circ T_V \circ \iota_\lambda.$$

Note that it has a biadjoint, commutes with duality.

Exercise 3.5. *Use Lemma 3.3 to deduce that $\pi_{\lambda'} \circ T_V \circ \iota_\lambda(\Delta(\lambda))$ is filtered with successive quotients $\Delta(\lambda + \mu)$, where μ runs over the weights of V such that $\lambda + \mu \in W \cdot \lambda'$.*

3.3. **Translation functors.** This is a special case of projective functors, where we pick "the smallest interesting" module V. Let us pick λ, λ' such that $\lambda+\rho, \lambda'+\rho$ are dominant (there is a unique element with this property in each W-orbit). Pick a dominant element ν in $W(\lambda'-\lambda)$ (here we consider the usual action) and set $V = L(\nu)$, this is a finite dimensional $\mathfrak{g}$-module.

Definition 3.6. *The* translation functor $T_{\lambda'\leftarrow\lambda} : \mathcal{O}_{W\cdot\lambda} \to \mathcal{O}_{W\cdot\lambda'}$ *is* $\pi_{\lambda'} \circ T_V \circ \iota_\lambda$.

Let us introduce the following notation. For $\lambda + \rho \in \Lambda^+$ by C_λ we denote the (closed) face of the positive Weyl chamber $\mathbb{R}_{\geqslant 0}\Lambda^+$ that contains $\lambda + \rho$ i.e. $C_\lambda = \mathbb{R}_{\geqslant 0}\Lambda^+ \cap \bigcap_{\alpha_i^\vee, \langle \lambda+\rho, \alpha_i^\vee\rangle = 0} \ker \alpha_i^\vee$. For example, C_0 is the whole positive Weyl chamber, while $C_{-\rho} = \{0\}$. We want to examine the behavior of the functors $T_{\lambda'\leftarrow\lambda}$ in the case when one of $C_\lambda, C_{\lambda'}$ contains the other.

Proposition 3.7. *Suppose that $C_\lambda = C'_\lambda$. Then we have*

$$T_{\lambda'\leftarrow\lambda}(\Delta(w\cdot\lambda)) = \Delta(w\cdot\lambda'), T_{\lambda'\leftarrow\lambda}(\nabla(w\cdot\lambda)) = \nabla(w\cdot\lambda'), T_{\lambda'\leftarrow\lambda}(L(w\cdot\lambda)) = L(w\cdot\lambda').$$

Moreover, $T_{\lambda'\leftarrow\lambda}$ and $T_{\lambda\leftarrow\lambda'}$ are mutually quasi-inverse equivalences.

Proposition 3.8. *Suppose that $C_\lambda \supset C'_\lambda$. Then we have*

$$T_{\lambda'\leftarrow\lambda}(\Delta(w\cdot\lambda)) = \Delta(w\cdot\lambda'), T_{\lambda'\leftarrow\lambda}(\nabla(w\cdot\lambda)) = \nabla(w\cdot\lambda'),$$

while for w longest in $w\,\mathrm{Stab}_W(\lambda+\rho)$ we have

$$T_{\lambda'\leftarrow\lambda}(L(w\cdot\lambda)) = \begin{cases} L(w\cdot\lambda'), & \textit{if } w \textit{ is longest in } w\,\mathrm{Stab}_W(\lambda'+\rho), \\ 0, & \textit{else.} \end{cases}$$

In particular, various computations for $\mathcal{O}_{W\cdot\lambda'}$ (such as the multiplicities of simples in the Vermas) can be deduced from the analogous computations for the *principal block* $\mathcal{O}_{W\cdot 0}$.

Example 3.9. *Let $\mathfrak{g} = \mathfrak{sl}_3, \lambda = 0, \lambda' = -\omega_1$ (the first fundamental weight i.e. $\langle\omega_1, \alpha_1^\vee\rangle = 1, \langle\omega_1, \alpha_2^\vee\rangle = 0$). Then $\mathrm{Stab}_W(\lambda'+\rho) = \langle s_1\rangle$ and $T_{\lambda'\leftarrow\lambda}$ kills precisely the simples $L(0), L(s_2\cdot 0), L(s_2s_1\cdot 0)$.*

Proposition 3.10. *Suppose that $C_\lambda \subset C'_\lambda$. Then $T_{\lambda'\leftarrow\lambda}(\Delta(w\cdot\lambda))$ is filtered by all different Vermas $\Delta(wu\cdot\lambda')$ with $u \in \mathrm{Stab}_W(\lambda+\rho)$. The order of terms is increasing with respect to the weight from bottom (sub) to top (quotient), compare with Lemma 3.3.*

Hint: deduce this proposition from Proposition 3.8 using adjointness.

Example 3.11. *In the setting of the previous example,*

$$0 \to \Delta(0) \to T_{0\leftarrow-\omega_1}\Delta(-\omega_1) \to \Delta(s_1\cdot 0) \to 0.$$

Exercise 3.12. *If $C_{\lambda''} \subset C_{\lambda'} \subset C_\lambda$, then $T_{\lambda''\leftarrow\lambda} \cong T_{\lambda''\leftarrow\lambda'} \circ T_{\lambda'\leftarrow\lambda}$ and $T_{\lambda\leftarrow\lambda''} \cong T_{\lambda\leftarrow\lambda'} \circ T_{\lambda'\leftarrow\lambda''}$.*

3.4. **Reflection functors.** Recall that we write I for the set of simple roots. For $\alpha_i I$, let ω_i denote the corresponding fundamental weight.

Definition 3.13. *Set* $\Theta_i := T_{0\leftarrow -\omega_i} \circ T_{-\omega_i \leftarrow 0}$. *This is a* reflection functor $\mathcal{O}_{W\cdot 0} \to \mathcal{O}_{W\cdot 0}$.

Exercise 3.14. Θ_i *is self biadjoint and commutes with* $\bullet^\vee$.

The following result is a consequence of Propositions 3.8 and 3.10.

Proposition 3.15. *The object* $\Theta_i \Delta(w\cdot 0)$ *is filtered by* $\Delta(w \cdot 0)$ *and* $\Delta(ws_i \cdot 0)$. *The larger of the two weights labels the sub.*

3.5. **Projective objects.** One application of reflection functors is to construct projective objects in $\mathcal{O}_{W\cdot 0}$. Note that 0 is the maximal weight that can occur in an object of $\mathcal{O}_{W\cdot 0}$. So, by Remark 2.4, $\Delta(0)$ is a projective in $\mathcal{O}_{W\cdot 0}$. And since each Θ_i is left adjoint to an exact functor (again, Θ_i), it maps projectives to projectives.

Theorem 3.16. $\mathcal{O}_{W\cdot 0}$ *has enough projectives. Moreover, if* $P(w\cdot 0)$ *denotes the projective cover of* $L(w \cdot 0)$ *and* $w = s_{i_1} \dots s_{i_k}$ *is a reduced expression, then*

$$\Theta_{i_\ell}\Theta_{i_{\ell-1}} \dots \Theta_{i_1}\Delta(0) = P(w\cdot 0) \oplus \bigoplus_{w' \prec w} P(w' \cdot 0)^{\oplus ?}.$$

As a hint for the second part $\Theta_{i_\ell}\Theta_{i_{\ell-1}} \dots \Theta_{i_1}\Delta(0) \twoheadrightarrow \Delta(w\cdot 0)$, while the kernel is filtered with $\Delta(w'\cdot 0)$, where $w' \prec w$.

Example 3.17. *The object* $P(-2) := \mathbb{C}^2 \otimes \Delta(-1) = T_{0\leftarrow -1}\Delta(-1)$ *from Example 3.4 is indecomposable and hence it is the projective cover of* $L(-2)$.

Exercise 3.18. *Prove that, for any* $\mathfrak{g}$, $P(-2\rho) = T_{0\leftarrow -\rho}\Delta(-\rho)$.

Exercise 3.19. *Prove that any* $\mathcal{O}_{W\cdot\lambda}$ *has enough projectives for all* $\lambda \in \Lambda$.

Remark 3.20. *We see that* $\mathcal{O}_{W\cdot\lambda}$ *has finitely many simples and enough projectives, while all objects have finite length. This precisely means that* $\mathcal{O}_{W\cdot\lambda}$ *is equivalent to the category of finite dimensional modules over a finite dimensional associative (unital) algebra.*

4. Highest weight structure and tilting objects

Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{C}$ be an abelian category equivalent to A-mod for a finite dimensional associative $\mathbb{K}$-algebra A.

Suppose $\mathrm{Irr}(\mathcal{C})$, the set of isomorphism classes of irreducibles in $\mathcal{C}$ is equipped with a partial order $\leqslant$. Let $\mathcal{C}_{\not> L}$ denote the Serre span of all simples L' with $L' \not> L$ in $\mathcal{C}$. For $L \in \mathrm{Irr}(\mathcal{C})$, we write Δ_L, ∇_L for the projective cover and the injective hull of L. These are called the *standard* and *costandard* objects, respectively.

Exercise 4.1. *Prove the following:*

(1) $\mathrm{Ext}^1(\Delta_L, \Delta_{L'}) \neq 0 \Rightarrow L < L'$.
(2) $\mathrm{Ext}^1(\nabla_L, \nabla_{L'}) \neq 0 \Rightarrow L > L'$.
(3) $\mathrm{Ext}^1(\Delta_L, \nabla_{L'}) = 0$.

Definition 4.2. $(\mathcal{C}, \leqslant)$ *is called a highest weight category if for every* $L \in \mathrm{Irr}(\mathcal{C})$ *the following two conditions hold:*

(1) *The kernel of* $\Delta_L \twoheadrightarrow L$ *lies is* $\mathcal{C}_{<L}$.
(2) *There is a projective object* $\tilde{P}_L \in \mathcal{C}$ *such that* $\tilde{P}_L \twoheadrightarrow \Delta_L$ *and the kernel is filtered by* $\Delta_{L'}$ *with* $L' > L$.

Exercise 4.3. *Prove that if* (1) *holds for* $\mathcal{C}$*, then:*
$\mathrm{Hom}(\Delta_L, \Delta_{L'}) \neq 0 \Rightarrow L \leqslant L'$ *and* $\mathrm{End}(\Delta_L) = \mathbb{K}$.
$\mathrm{Hom}(\nabla_L, \nabla_{L'}) \neq 0 \Rightarrow L \geqslant L'$ *and* $\mathrm{End}(\nabla_L) = \mathbb{K}$.
$\dim \mathrm{Hom}(\Delta_L, \nabla_{L'}) = \delta_{L,L'}$.

Remark 4.4. *In fact, modulo (1), the subcategory* $\mathcal{C}^\Delta$ *of all standardly filtered (=filtered by standards) is closed under taking the direct summands. So in (2) we can require that* $P(L) = P_L$*, the projective cover of* L*. A more interesting equivalent condition is that, modulo (1), (2) is equivalent to* $\mathrm{Ext}^i(\Delta_L, \nabla_{L'}) = 0$ *for all* $i > 0$.

The next theorem follows from Theorem 3.16 and the hint for its proof.

Theorem 4.5. *Category* $\mathcal{O}_{W\cdot 0}$ *is highest weight with respect to the standard order on weights. The standards are* $\Delta(w \cdot 0)$*'s, while the costandards are* $\nabla(w \cdot 0)$.

In fact, we can take a weaker order as well. Let us identify $\mathrm{Irr}(\mathcal{O}_{W\cdot 0})$ with W via $w \mapsto L(w \cdot (-2\rho))$. Then we can take the Bruhat order on W for the highest weight order. This also follows from Theorem 3.16.

Exercise 4.6. *Prove that* $\mathcal{O}_{W\cdot\lambda}$ *is highest weight with respect to the standard order on weights for any* λ.

Exercise 4.7. *Let* $\mathcal{C}$ *be a highest weight category, in particular,* P_L *is standardly filtered. Then the multiplicity* $[P_L : \Delta_{L'}]$ *of* $\Delta_{L'}$ *in any filtration of* P_L *by standards coincides with* $\dim(\mathrm{Hom}(P_L, \nabla_{L'}))$ *which in turn coincides with the multiplicity* $[\nabla_{L'} : L]$ *of* L *in (the Jordan-Hölder filtration of)* $\nabla_{L'}$*. This is the so called BGG reciprocity.*

Thanks to $\bullet^\vee$, in the category $\mathcal{O}$, we have $[\Delta(\lambda) : L(\mu)] = [\nabla(\lambda) : L(\mu)]$. Together with the previous exercise, this gives the classical BGG reciprocity:

Theorem 4.8. $[P(\lambda) : \Delta(\mu)] = [\Delta(\mu) : L(\lambda)]$ *for all* $\lambda, \mu \in \Lambda$.

Let us now discuss tilting objects.

Definition 4.9. *An object in $\mathscr{C}$ is called* tilting *if it is both standardly and costandardly filtered.*

Let us explain why we care about tiltings. Since there are no higher exts between standard and costandard objects, for any two tiltings T, T', we have $\mathrm{Ext}^i(T, T') = 0$. Moreover, the indecomposable tiltings are in a natural bijection with $\mathrm{Irr}(\mathcal{C})$: for any L, there is a unique indecomposable tilting T_L subject to the following two equivalent properties:

- There is a monomorphism $\Delta_L \hookrightarrow T_L$ whose cokernel is filtered with $\Delta_{L'}$ for $L' < L$.
- There is an epimorphism $T_L \twoheadrightarrow \nabla_L$ whose kernel is filtered with $\nabla_{L'}$ for $L' < L$.

It follows that the indecomposable tiltings form a full exceptional collection in $\mathcal{C}$, just like projectives and injectives. But tiltings are more symmetric: for example, when $\mathcal{C}$ has a contravariant duality fixing the simples (like $\mathcal{O}_{W\cdot\lambda}$ does), every tilting is self-dual.

For the category $\mathcal{O}_{W\cdot 0}$, we have the following analog of Theorem 3.16:

Exercise 4.10. *If $w = s_{i_1} \dots s_{i_k}$ is a reduced expression, then*

$$\Theta_{i_k}\Theta_{i_{k-1}} \dots \Theta_{i_1}\Delta(-2\rho) = T(w \cdot (-2\rho)) \oplus \bigoplus_{w' \prec w} T(w' \cdot (-2\rho))^{\oplus ?}.$$

5. Parabolic categories $\mathcal{O}$

It will be useful for us to consider a generalization of $\mathcal{O}$, parabolic categories $\mathcal{O}$ (the reason is that the category of rational representations of a reductive group in characteristic p is a "modular analog" of a suitable *affine* parabolic category $\mathcal{O}$).

Let $J \subset I$. This gives a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_J$ and its Levi $\mathfrak{l} = \mathfrak{l}_J$.

Definition 5.1. *We define the parabolic category $\mathcal{O}_J$ as the full subcategory of $\mathcal{O}$ consisting of all modules, where $\mathfrak{l}$ acts locally finitely (=every vector lies in a finite dimensional $\mathfrak{l}$-submodule).*

Remark 5.2. *For $J = \varnothing$ we recover the category $\mathcal{O}$, while for $J = I$ we get the category of all finite dimensional $\mathfrak{g}$-modules.*

Exercise 5.3. *$\mathcal{O}_J$ is a Serre subcategory.*

Definition 5.4. *Let Λ_J^+ be the set of dominant weights for $\mathfrak{l}$, i.e., $\lambda \in \Lambda$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for any $\alpha \in J$.*

Definition 5.5. *For $\lambda \in \Lambda_J^+$ we define the parabolic Verma module $\Delta_J(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} L_J(\lambda)$, where $L_J(\lambda)$ is the (finite dimensional) irreducible representation of $\mathfrak{l}$ with highest weight λ.*

Exercise 5.6. $\Delta(\lambda) \twoheadrightarrow \Delta_J(\lambda)$.

Let $\mathfrak{m}$ denote the nilpotent radical of $\mathfrak{p}$ and let $\mathfrak{m}_-$ be the nilpotent radical of the opposite parabolic $\mathfrak{p}_-$ so that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}, \mathfrak{p}_- = \mathfrak{l} \oplus \mathfrak{m}_-$ and $\mathfrak{g} = \mathfrak{m}_- \oplus \mathfrak{p}$. Similarly to the case of usual Verma modules, we get an isomorphism $\Delta_J(\lambda) \cong U(\mathfrak{m}_-) \otimes L_J(\lambda)$ of $U(\mathfrak{p}_-)$-modules.

The properties $\mathcal{O}_J$ are very similar to those of the usual category $\mathcal{O}$. Let us list these properties as an exercise.

Exercise 5.7. *The following hold:*

1. $\Delta_J(\lambda) \in \mathcal{O}_J$ *for all* $\lambda \in \Lambda_J^+$. *Moreover,* $L(\lambda)$ *is the unique irreducible quotient of* $\Delta_J(\lambda)$.
2. $\mathcal{O}_J$ *is the Serre span in* $\mathcal{O}$ *of* $L(\lambda)$ *with* $\lambda \in \Lambda_J^+$.
3. *The category* $\mathcal{O}_J$ *is preserved by* $\bullet^\vee$ *and by all projective functors.*
4. *For any* $\lambda \in \Lambda_J^+$, *every projective functor maps* $\Delta_J(\lambda)$ *to an object that admits a filtration by parabolic Vermas.*
5. *If* $\lambda \in \Lambda^+$, *then* $\Delta_J(\lambda)$ *is projective in* $\mathcal{O}_J$.
6. $\Theta_i \Delta_J(w \cdot 0)$ *is filtered with* $\Delta_J(w \cdot 0), \Delta_J(ws_i \cdot 0)$ *if* $ws_i \cdot 0 \in \Lambda_J^+$ *and is zero else.*
7. *The category* $\mathcal{O}_{J,W\cdot\lambda} := \mathcal{O}_J \cap \mathcal{O}_{W\cdot\lambda}$ *is highest weight with respect to the standard order on weights.*