$\otimes$ product on $\mathcal{O}_\kappa$ and equivalence of braided monoidal categories

1) coinvariants
2) $\otimes$ - product construction
3) construction of equivalence

1) Coinvariants

Def Let $S \subseteq \mathbb{P}^1$ a finite set of pts

1) Let $\mathfrak{g}_{out,S} = \mathfrak{g} \otimes \mathbb{C}[\mathbb{P}^1 \setminus S]$

2) Denote by $\hat{\mathfrak{g}}_{\kappa,S}$ the central extension

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}}_{\kappa,S} \to \bigoplus_S \mathfrak{g}((t)) \to 0$$

given as the quotient of $\bigoplus_S \hat{\mathfrak{g}}_\kappa$ by the kernel of the addition map $\bigoplus_S \mathbb{C} \to \mathbb{C}$ the central subalgebra of $\bigoplus_S \hat{\mathfrak{g}}_\kappa$

3) A coordinate system at S is a family $\{\phi_s\}$, where $\phi_s : \hat{\mathbb{P}}^1_s \xrightarrow{\sim} \hat{D}$ an isomorphism of the completion of $\mathbb{P}^1$ at s with the formal disk $\hat{D}$

Given a coordinate system we get a map.

$$\mathfrak{g}_{out,S} \longrightarrow \bigoplus_S \mathfrak{g}((t))$$

$$\underset{\mathbb{C}[\mathbb{P}^1 \setminus S]}{g} \underset{\mathfrak{g}}{x} \longmapsto (\phi_s(g)x)$$

As the sum of residue of a differential form on $\mathbb{P}^1$ is 0.

we get the map lift $\mathfrak{g}_{out,S} \longrightarrow \hat{\mathfrak{g}}_{\kappa,S}$.

<u>Def</u> Given a family $\{V_s \in \mathcal{O}_\kappa\}$ of the <u>same level</u>,

we get $\hat{\mathfrak{g}}_{\kappa,S}$ representation $\bigotimes_S V_s$

Denote $C(\{V_s\},\{\phi_s\})$ as the coinvariants of the

induced $\mathfrak{g}_{out,S}$-representation $\bigotimes_S V_s \Big/ \mathfrak{g}_{out,S} \bigotimes_S V_s$.

<u>Prop</u> Let $\{M_s^\kappa\}$ generalized Weyl modules. Then

$$C(\{M_s^\kappa\},\{\phi_s\}) = \bigotimes M_s \Big/ \mathfrak{g}(\bigotimes M_s)$$

$\Rightarrow V_s \in \mathcal{O}_\kappa \quad C(\{V_s\},\{\phi_s\})$ f. dim

Sketch of pf: Note that for every $\{ f_s \in \mathbb{C}((t)) \}$

$\exists\, g \in \mathbb{C}[\mathbb{P}^1 \setminus S]$ st $f_s - \phi_s(g) \in \mathbb{C}[[t]]$

M^{χ} a generalized Weyl module is spanned by

$$(t^{-k_1} x_1) \qquad (t^{-k_r} x_r)\, m$$

By induction

$$\bigotimes_s M_s \twoheadrightarrow C(\{ M_s^{\chi} \}, \{ \phi_s \}).$$

If $g \in \mathbb{C}[\mathbb{P}^1 \setminus S]$ has no poles on $\mathbb{P}^1$ then it is

constant, so $g \in \mathfrak{g}_{out,S}$ preserves $\otimes M_s$.

2) $\dot{\otimes}$ - product on $\mathcal{O}_\kappa$.

We define a $\dot{\otimes}$ - product on $\mathcal{O}_\kappa$, st

$$\mathrm{Hom}_{\hat{g}_\kappa}(X, D(V_1 \dot{\otimes} V_2)) = C(\{X, V_1, V_2\}, \{\varphi_s\})^*$$

<u>Def</u> Given S, $\{\varphi_s\}$ a coordinate system and $s_0 \in S$

a) Γ is a central extension of $g_{out,S}$ using the residue at the pt $s_0 \in S$ ie.

$$[gx, fy] = gf[x,y] + \mathrm{Res}_{s_0}(f\,dg)\,\kappa(x,y)\,\mathbb{1}.$$

b) There is a Lie alg map $\Gamma \to \hat{g}_{\kappa, S \setminus s_0}$.

$$\mathbb{1} \mapsto -\mathbb{1}.$$

using that the sum of all residues is 0.

c) For $\{V_s\}_{s \in S \setminus s_0}$ a family in $\mathcal{O}_\kappa$ of <u>same level</u> we get a $\hat{g}_{\kappa, S \setminus s_0}$ representation $\bigotimes_{S \setminus s_0} V_s$

and thus Γ - representation

d) $V_{s_0} \in \mathcal{O}_x$ we get a Γ representation using ϕ_{s_0}.

<u>Construction of $\otimes$-prod.</u>

$Z = \mathrm{Hom}\left(\bigotimes_{s \neq s_0} V_s, \mathbb{C}\right)$ inherits a Γ-action

Define $G_N \subseteq \mathcal{U}\Gamma$ spanned by

$(g_1 x_1) \qquad (g_N x_N) \qquad \phi_{s_0}(g_i) \in t\,\mathbb{C}[[t]]$

$x_i \in \mathfrak{g}$.

Let $Z^K \subseteq Z$ the space of $\varphi \in Z$ st.

$G_K \varphi = 0$.

$W = Z^\infty = \bigcup Z^K$.

We construct a $\hat{\mathfrak{g}}_x$ - action

For $fx \in \hat{\mathfrak{g}}_x$, $z \in Z^N \subseteq W$ choose $g \in \mathbb{C}[\mathbb{P}^1 \setminus S]$

$f - \phi_{s_0}(g) \in t^N \mathbb{C}[[t]]$

Define $fx \cdot z = gx \cdot z$ this is independent of the choice of g

Also if $\varphi_{s_0}(y) \in t^{-n}\mathbb{C}[[t]]$

$$y x \cdot z \in Z^{N+n}.$$

Define $\bigotimes_{s \in S \setminus s_0} V_s := D(W)$

Rmk: Note that we need to check W is dualizable.
This follows from the following Proposition,
which shows $\mathrm{Hom}(V_\lambda^{\varkappa}, W)$ is finite dimensional
We also need it is non-zero for finitely
many λ's, which can be checked for $V_s = N_s^{\varkappa}$
by exactness and then it is clear

$$\mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes N_s, \mathbb{C}) \neq 0 \text{ for finitely many } \lambda.$$

Prop $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}\left(X, D\left(\dot{\bigotimes}_{s\in S\setminus s_1} V_s\right)\right) = C\left(\{X, V_s\}_{s\in S\setminus s_0}, \{\phi_s\}_{s\in S}\right)^*$

for $X, V_s \in \mathcal{O}_\kappa$.

Pf: $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}\left(X, D\left(\dot{\bigotimes}_{s\in S\setminus s_0} V_s\right)\right) = \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(X, W)$.

$$\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(X, W) = \mathrm{Hom}_\Gamma(X, W) \quad \text{by smoothness}.$$

$$\mathrm{Hom}_\Gamma(X, W) = \mathrm{Hom}_\Gamma(X, Z)$$

this follows $\forall y \in X \ \exists N$ st $G_N y = 0$.

$$\begin{aligned}\mathrm{Hom}_\Gamma(X, Z) &= \mathrm{Hom}_\Gamma\left(X, \mathrm{Hom}\left(\bigotimes V_s, \mathbb{C}\right)\right) \\ &= \mathrm{Hom}_\Gamma\left(X \otimes \bigotimes V_s, \mathbb{C}\right) \\ &= \mathrm{Hom}_{\mathfrak{g}_{out,S}}\left(X \otimes \bigotimes V_s, \mathbb{C}\right) = C\left(\{X, X_s\}, \{\phi_s\}\right)^*\end{aligned}$$

Lemma $\dot{\otimes} N_s^{\chi} = (\otimes N_s)^{\chi}$ for irrational level $N_s \in \mathrm{Rep}\, G$.

Pf: $\mathrm{Hom}((N_s^*)^{\chi}, D(N_s^{\chi})) = C(\{(N_s^*)^{\chi}, N_s^{\chi}\}, \{t\})^*$

For N_s simple, N_s^{χ} simple $\quad$ 1-dim.

$$(N_s^*)^{\chi} = D(N_s^{\chi}).$$

Rmk: for χ irrational $\mathcal{O}_{\chi}$ semisimple.

$$\mathrm{Hom}_{\hat{g}_{\chi}}(X^{\chi}, D(\dot{\otimes}(N_s)^{\chi})) = C(\{X^{\chi}, (N_s)^{\chi}\}, \{p_s\})^*$$

$$= \mathrm{Hom}_{g}(X, \otimes N_s^*) \cong \mathrm{Hom}_{\hat{g}}(X^{\chi}, (\otimes N_s^*)^{\chi}).$$

so $\quad D(\dot{\otimes} N_s^{\chi}) = (\otimes N_s^*)^{\chi}$

so $\quad \dot{\otimes} N_s^{\chi} = (\otimes N_s)^{\chi}$

Theorem For $\kappa \notin \mathbb{Q}_{\geq 0}$ the category $(\mathcal{O}_\kappa, \dot{\otimes})$ is a rigid braided monoidal category, with dual D.

Pf: omitted

Rmk: To understand the braiding consider:

$C(\{V_s\}, \{\phi_s\})$ as S varies

this gives a local system on $(\mathbb{P}^1)^S \setminus \text{diag}$.

If we fix $s_0 = \infty$ and let $S \setminus s_0$ vary

then $\dot{\otimes}$ represent the fiber of a local system on

$(\mathbb{A}^1)^S \setminus \text{diag}$ $\rightsquigarrow$ monodromy gives a braiding

on $(\mathcal{O}_\kappa, \dot{\otimes})$.

$\kappa \notin \mathbb{Q}_{\geq 0}$ required for rigidity

3) A Functor from $\mathcal{O}_{\kappa}$ to U_q-mod. $q = e^{-\pi i/\kappa}$

Let V_λ the the irreducible $\mathfrak{g}$-representation of highest wt λ.

Denote $V_{\bar\lambda} \equiv V_\lambda^*$

<u>Idea</u>: Let V a f d $\mathfrak{g}$-rep, then for $\lambda \gg 0$

and fixed ν

$$\mathrm{Hom}_{\mathfrak{g}}(V_{\bar\lambda} \otimes V_{\lambda+\nu}, V) = \mathrm{Hom}_{\mathfrak{g}}(V_{\bar\lambda} \otimes V_{\lambda+\nu} \otimes V^*, \mathbb{C})$$

$\lambda \gg 0$
$$= \mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{\mu} V_{\bar\lambda} \otimes V_{\lambda+\nu+\mu} \otimes (V^*)^{(\mu)}, \mathbb{C}\right).$$

here $W^{(\mu)}$ is the μ wt space of $W \in \mathfrak{g}\text{-mod}^{fd}$.

$$= \bigoplus_{\mu} \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda+\nu+\mu}, V^{(-\mu)} \otimes V_\lambda).$$

$$= V^{(\nu)}$$

We will use this construction to give a graded vector space $X(V)$ for $V \in \mathcal{O}_\kappa$.

and then construct operators E_i, F_i,

Constructing homomorphisms

We will consider $\hat{\mathfrak{g}}_\kappa$ - modules over

R the ring of analytic functions on $\mathbb{C}$ meromorphic at ∞.

Let $\mathcal{G}(V) = V / Q_1^{*} V$

Q_1 = spanned by $(t^{-K_1} x_1) \ldots (t^{-K_r} x_r)$.

Prop a) $V_\lambda^\kappa \otimes V_\mu^\kappa$ has a filtration by Weyl modules.

b) For V a module with a Weyl filtration $/R$

and W $\hat{\mathfrak{g}}_\kappa$ - representation of the same level, then

$\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(V, W)$ is flat $/R$.

c) The functor $\mathcal{G}$ is exact on the subcategory of

$V \in \mathcal{O}_\kappa$ with Weyl filtration.

d) $V \in \mathcal{O}$ has a Weyl filtration iff

$\mathrm{Ext}^1(V, D(V_\lambda^\kappa)) = 0 \qquad \forall \lambda$.