Constructing homomorphisms

Note that we can consider $\hat{\mathfrak{g}}_\kappa$-modules over R the ring of analytic functions on $\mathbb{C}$ meromorphic at ∞

Let $\mathcal{G}(V) = V/Q_1^{\#}V$

<u>Prop</u> a) $V_\lambda^{*} \otimes V_\mu^{\kappa}$ has a filtration by Weyl modules.

b) For V a module with Weyl filtration and W of same level then $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(V, W)$ is flat$/R$

c) The functor $\mathcal{G}$ is exact on the subcategory of $V \in \mathcal{O}$ that have Weyl filtration

d) $V \in \mathcal{O}$ has a Weyl filtration iff

$$\mathrm{Hom}(V, D(V_\lambda^{\kappa})) = 0 \qquad \forall \lambda$$

Pf of b): $\mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(V_\lambda^\kappa, D(N^\kappa)) = R \otimes (V_\lambda \otimes N)^*_{\mathfrak{g}}$ is R flat.

For W general, there is $N^\kappa \twoheadrightarrow D(W)$ from a generalised Weyl module. as W R flat.

$W \hookrightarrow D(N^\kappa)$ and so.

$\mathrm{Hom}(V_\lambda^\kappa, W) \hookrightarrow \mathrm{Hom}(V_\lambda^\kappa, D(N^\kappa))$ flat.

Pf of c): Note that as a $t^{-1}\mathfrak{g}[t^{-1}]$-module Weyl modules are free, hence the result is clear

Pf of d) Follows from proof of c) + the universal property of $D(V_\lambda^\kappa)$.

Lem a) $\mathrm{Hom}(V_{\bar\lambda}^{\kappa} \otimes V_{\lambda}^{\kappa}, V_0^{\kappa})$ is free of rank 1.

b) $\mathrm{Hom}(V_{\lambda+\mu}^{\kappa}, V_{\lambda}^{\kappa} \otimes V_{\mu}^{\kappa})$ " " " " "

Pf: a) $\mathrm{Hom}(V_{\bar\lambda}^{\kappa} \otimes V_{\lambda}^{\kappa}, V_0^{\kappa}) = \mathrm{Hom}(D(V_0^{\kappa}), D(V_{\bar\lambda}^{\kappa} \otimes V_{\lambda}^{\kappa}))$

$$= \mathrm{Hom}(V_0^{\kappa}, D(V_{\bar\lambda}^{\kappa} \otimes V_{\lambda}^{\kappa})).$$

$$= (V_{\bar\lambda} \otimes V_{\lambda})_{\mathfrak{g}} \leftarrow \text{1-dim}$$

using $V_0^{\kappa} \cong D(V_0^{\kappa})$ for generic κ.

$\mathrm{Hom}(V_0^{\kappa}, D(V_0^{\kappa})) = (V_0 \otimes V_0)_{\mathfrak{g}} = V_0$

and the map is an isomorphism as V_0^{κ} irreducible.

as $\mathrm{Hom}(V_{\bar\lambda}^{\kappa} \otimes V_{\lambda}^{\kappa}, V_0^{\kappa})$ free and generically rank 1 it is free of rank 1.

b) For generic κ $V_{\lambda}^{\kappa} \cong D(V_{\bar\lambda}^{\kappa})$ using that V_{λ}^{κ} is irreducible for generic κ. and thus

$$V_{\lambda}^{\kappa} \otimes V_{\mu}^{\kappa} \cong D(V_{\lambda}^{\kappa} \otimes V_{\mu}^{\kappa}).$$

$$\text{thus} \quad \mathrm{Hom}(V_{\lambda+\mu}^{\varkappa}, D(V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa})) = (\overset{\text{1-dim}}{\overline{V_{\lambda+\mu}}\otimes V_\lambda\otimes V_\mu})_{\mathfrak{g}}.$$

and thus just as above.

$$\mathrm{Hom}(V_{\lambda+\mu}^{\varkappa}, V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa}) \text{ free of rank } 1. \ \square$$

We need to choose generators of.

$$\mathrm{Hom}(V_{\bar\lambda}^{\varkappa}\otimes V_\lambda^{\varkappa}, V_0^{\varkappa})$$

and $\quad \mathrm{Hom}(V_{\lambda+\mu}^{\varkappa}, V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa})$.

Define $\quad \mathcal{G}(V) = V/Q_1^{*}V$

$\mathcal{G}(V_\lambda^{\varkappa}) = V_\lambda \quad$ and also it is easy to see.

$\mathcal{G}(V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa}) = V_\lambda\otimes V_\mu$

Further

$$\mathrm{Hom}(V_{\lambda+\mu}^{\varkappa}, V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa}) \hookrightarrow \mathrm{Hom}(\mathcal{G}(V_{\lambda+\mu}^{\varkappa}), \mathcal{G}(V_\lambda^{\varkappa}\otimes V_\mu^{\varkappa}))$$

$\mathcal{G}$ is exact on modules with Weyl filtration, hence the above map is an isomorphism follows from the Proposition

<u>Prop</u> For any generator $t \in \mathrm{Hom}(V^{\mathfrak{X}}_{\lambda+\mu}, V^{\mathfrak{X}}_{\lambda} \otimes V^{\mathfrak{X}}_{\mu})$, the cokernel has Weyl filtration.

<u>Pf</u>: Enough to check

$$\mathrm{Hom}(V^{\mathfrak{X}}_{\lambda} \otimes V^{\mathfrak{X}}_{\mu}, D(V^{\mathfrak{X}}_{\nu})) \to \mathrm{Hom}(V^{\mathfrak{X}}_{\lambda+\mu}, D(V^{\mathfrak{X}}_{\nu}))$$

is surjective. RHS $= 0$ unless $\bar{\lambda} + \bar{\mu} = \nu$.

Ex $T_{\lambda,\mu} \in \mathrm{Hom}(V^{\mathfrak{X}}_{\lambda+\mu}, V^{\mathfrak{X}}_{\lambda} \otimes V^{\mathfrak{X}}_{\mu})$ st $\mathcal{G}(T_{\lambda,\mu})$ is the map $y_{\lambda+\mu} \mapsto y_{\lambda} \otimes y_{\mu}$ for fixed highest wt generators of V_{ν}'s.

Choose generators $S_\lambda \colon \mathrm{Hom}(V_{\bar\lambda}^{\chi_2} \otimes V_\lambda^{\chi}, V_0^{\chi_2})$.

<u>Prop</u>:

$$\begin{tikzcd} V_{\overline{\lambda+\mu}}^{\chi_2} \otimes V_{\lambda+\mu}^{\chi} \arrow[r] \arrow[drr, bend right] & V_{\bar\lambda}^{\chi} \otimes V_{\bar\mu}^{\chi_2} \otimes V_{\mu}^{\chi_2} \otimes V_{\lambda}^{\chi_2} \arrow[r] & V_{\bar\lambda}^{\chi} \otimes V_{\lambda}^{\chi} \arrow[d] \\ & & V_0^{\chi} \end{tikzcd}$$

$$g_{\lambda,\mu} \cdot S_{\lambda+\mu}.$$

For $g_{\lambda,\mu} \in \check{R}$ invertible in $\check{R}$ the ring of analytic functions on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ meromorphic at ∞.

Further $\exists$ choices of S_λ st $g_{\lambda,\mu} = 1 \ \forall \lambda,\mu$.

<u>Pf</u>: By rigidity it's enough to check

$$\begin{tikzcd} V_{\lambda+\mu}^{\chi_1} \arrow[r] \arrow[drr] & V_\lambda^{\chi_2} \otimes V_\mu^{\chi} \arrow[r] & D(V_\lambda^{\chi}) \otimes D(V_\mu^{\chi}) \arrow[d] \\ & & D(V_{\lambda+\mu}^{\chi}) \end{tikzcd}$$

is non-zero at every pt $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

This is clear as mod $\langle L_\nu \mid \nu < \lambda+\mu \rangle$ all maps in this composition are isomorphisms.

Can check we can rescale S_λ by invertible elements of $\check{R}$ to eliminate $g_{\lambda,\mu}$

<u>Prop</u> $\mathrm{Hom}(V^{\kappa}_{\lambda+\mu-\alpha_i}, V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu})$ is a free rank 1 $\check{R}$-module

<u>Pf</u>: Just as above it is free, so it is enough to check at generic κ. Then $D(V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu}) \cong V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu}$ and so $\mathrm{Hom}(V^{\kappa}_{\lambda+\mu-\alpha_i}, V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu})$

$\cong \mathrm{Hom}(V_{\lambda+\mu-\alpha_i}, V_{\lambda} \otimes V_{\mu})$

which has dimension 1. $\square$

Recall that $Q = V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu} \big/ V^{\kappa}_{\lambda+\mu}$ has a Weyl filtration and for a generator of $\mathrm{Hom}(V^{\kappa}_{\lambda+\mu-\alpha_i}, V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu})$. so does $Q/V^{\kappa}_{\lambda+\mu-\alpha_i}$

$$\begin{tikzcd}
0 \arrow[r] & V^{\kappa}_{\lambda+\mu} \arrow[r] \arrow[d, equal] & W \arrow[r, "\dot{\gamma}"] \arrow[d] & V^{\kappa}_{\lambda+\mu-\alpha_i} \arrow[r] \arrow[d] & 0 \quad (*) \\
0 \arrow[r] & V^{\kappa}_{\lambda+\mu} \arrow[r] & V^{\kappa}_{\lambda} \dot{\otimes} V^{\kappa}_{\mu} \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

<u>Lem</u> $\exists\ V^{\varkappa}_{\lambda+\mu-\alpha_i} \xrightarrow{\eta} W$ st $j \circ \eta = [(\lambda+\mu)(\alpha_i^\vee)]_q \,\mathrm{id}$

where $q = e^{-\pi i/\varkappa}$

<u>Sketch of Pf</u>: The SES $(*)$ splits when

$$[(\lambda+\mu)(\alpha_i)] \neq 0 \quad \text{and near } [(\lambda+\mu)(\alpha_i)] = 0$$

the extension can be seen to be of order 1.

<u>Prop</u>: $\exists\ \tau_{\lambda,\mu,i} \in \mathrm{Hom}(V^{\varkappa}_{\lambda+\mu-\alpha_i}, V^{\varkappa}_\lambda \otimes V^{\varkappa}_\mu)$

st $g(\tau_{\lambda,\mu,i}) = \dfrac{\check{\tau}_{\lambda,\mu,i}}{\Gamma(1-\varkappa^{-1}\lambda(\alpha_i^\vee))\,\Gamma(1-\varkappa^{-1}\mu(\alpha_i^\vee))\,\Gamma(1+\varkappa^{-1}(\lambda+\mu)(\alpha_i^\vee))}$

$$\check{\tau}_{\lambda,\mu,i} : V_{\lambda+\mu-\alpha_i} \to V_\lambda \otimes V_\mu$$

$$y_{\lambda+\mu-\alpha_i} \mapsto \lambda(\alpha_i^\vee)\, y_\lambda \otimes f_i y_\mu - \mu(\alpha_i^\vee)\, f_i y_\lambda \otimes y_\mu.$$

<u>Pf</u>: Follows from the above and checking that

$$\frac{1}{\Gamma(1-\varkappa^{-1}\lambda(\alpha_i^\vee))\,\Gamma(1-\varkappa^{-1}\mu(\alpha_i^\vee))\,\Gamma(1+\varkappa^{-1}(\lambda+\mu)(\alpha_i^\vee))\,[(\lambda+\mu)(\alpha_i^\vee)]} \in R^\vee$$

Constructing the functor

We have using the above maps.

$$\begin{tikzcd}
V^{\kappa}_{\overline{\lambda+\nu}} \otimes V^{\kappa}_{\mu+\nu} \arrow[r] \arrow[dr] & V^{\kappa}_{\bar\lambda} \otimes V^{\kappa}_{\bar\nu} \otimes V^{\kappa}_{\nu} \otimes V^{\kappa}_{\mu} \arrow[d] \\
& V^{\kappa}_{\bar\lambda} \otimes V_{\mu}
\end{tikzcd}$$

Thus we get an induced map

$$\mathrm{Hom}(V^{\kappa}_{\bar\lambda} \otimes V^{\kappa}_{\mu}, V) \to \mathrm{Hom}(V^{\kappa}_{\overline{\lambda+\nu}} \otimes V^{\kappa}_{\mu+\nu}, V)$$

Define $X_{\nu}(V) = \lim \mathrm{Hom}(V^{\kappa}_{\bar\lambda} \otimes V^{\kappa}_{\lambda+\nu}, V)$

We will define $X(V) = \bigoplus_{\nu} X_{\nu}(V)$

and construct operators

$$E_i : X_{\nu}(V) \to X_{\nu+\alpha_i}(V).$$
$$F_i : X_{\nu}(V) \to X_{\nu-\alpha_i}(V).$$

using

$$\begin{tikzcd}
V^{\kappa}_{\bar\lambda+\bar\nu-\alpha_i^{\vee}} \otimes V^{\kappa}_{\nu+\mu} \arrow[r, "{[\nu(\alpha_i)]\, \tau \otimes T}"] \arrow[dr] & V^{\kappa}_{\bar\lambda} \otimes V^{\kappa}_{\nu} \otimes V^{\kappa}_{\nu} \otimes V^{\kappa}_{\mu} \arrow[d] \\
& V^{\kappa}_{\bar\lambda} \otimes V^{\kappa}_{\mu}
\end{tikzcd}$$

For E_i and similarly for F_i.

Lemma a) $[E_i, E_j]\, x = [\langle \lambda, \alpha_i^\vee \rangle]\, \delta_{ij}\, x \qquad x \in X_\lambda(V)$

b) $E_i E_j = E_j E_i$ if $a_{ij} = 0$ in Cartan matrix

$$E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0. \qquad a_{ij} = -1$$

Similarly for F_i's.

Very vague idea of Pf:

Use relations of defining maps like

Lemma:

$(Q_1)\quad V^{\kappa}_{\lambda+\mu+\nu-\alpha_i} \longrightarrow V^{\kappa}_{\lambda+\mu} \otimes V^{\kappa}_{\nu} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu} \otimes V^{\kappa}_{\nu}$

$(Q_2)\quad V^{\kappa}_{\lambda+\mu+\nu-\alpha_i} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu+\nu} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu} \otimes V^{\kappa}_{\nu}$

$(Q_3)\quad V^{\kappa}_{\lambda+\mu+\nu-\alpha_i} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu+\nu-\alpha_i} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu} \otimes V^{\kappa}_{\nu}$

$(Q_4)\quad V^{\kappa}_{\lambda+\mu+\nu-\alpha_i} \longrightarrow V^{\kappa}_{\lambda+\mu-\alpha_i} \otimes V^{\kappa}_{\nu} \longrightarrow V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu} \otimes V^{\kappa}_{\nu}$

i) $[(\lambda+\mu)(\alpha_i^\vee)]\, Q_3 + [\nu(\alpha_i^\vee)]\, Q_4 = [\mu(\alpha_i^\vee)]\, Q_1$

ii) $[\lambda(\alpha_i^\vee)]\, Q_3 + [(\mu+\nu)(\alpha_i^\vee)]\, Q_4 = [\mu(\alpha_i^\vee)]\, Q_2$

iii) $[(\mu+\nu)(\alpha_i^\vee)]\, Q_1 - [\nu(\alpha_i^\vee)]\, Q_2 = [(\lambda+\mu+\nu)(\alpha_i^\vee)]\, Q_3$

iv) $-[\lambda(\alpha_i^\vee)]\, Q_1 + [(\lambda+\mu)(\alpha_i^\vee)]\, Q_2 = [(\lambda+\mu+\nu)(\alpha_i^\vee)]\, Q_4$

We have a map $X_\lambda(V) \otimes X_\mu(W) \to X_{\lambda+\mu}(V \dot{\otimes} W)$

using $V^{\kappa}_{\bar\lambda+\bar\lambda'} \otimes V^{\kappa}_{\mu+\mu'} \to V^{\kappa}_{\lambda} \otimes V^{\kappa}_{\mu} \otimes V^{\kappa}_{\bar\lambda'} \otimes V^{\kappa}_{\mu'}$

<u>Prop</u> i) the above maps $X_\lambda(V) \otimes X_\mu(W) \to X_{\lambda+\mu}(V \dot{\otimes} W)$ combine to an isomorphism $X(V) \otimes X(W) \xrightarrow{\sim} X(V \dot{\otimes} W)$

X is a braided monoidal functor.

ii) If $\kappa = -p/q$ $(p,q)=1$ $p \neq 1$, then $\forall\, V \in \mathcal{O}_\kappa$.
$E_i^p = 0$ on $X(V)$

iii) the quantum group relations are satisfied for E_i, F_i and the grading operators K_μ.

Further $E_i^{(n)} = \frac{E_i^n}{[n]!}$ acts on V.

iv) $X(V_\lambda^\kappa) \cong V_\lambda$ the Weyl module for the Lusztig quantum group, for λ a restricted wt.

Theorem $X : \mathcal{O}_{\mathfrak{X}} \longrightarrow \mathrm{Rep}(U_q)$ is an equivalence of categories.

Pf: $V_{\lambda_1}^{\mathfrak{X}} \otimes \cdots \otimes V_{\lambda_k}^{\mathfrak{X}} \otimes V_{\nu}^{\mathfrak{X}} = P_\gamma$ is projective if ν the Steinberg representation.

Using $X(V_\lambda^{\mathfrak{X}}) \cong V_\lambda$ we can check

$$\dim \mathrm{Hom}(P_\gamma, P_{\gamma'}) = \dim \mathrm{Hom}(X(P_\gamma), X(P_{\gamma'}))$$

So enough to show

$$\mathrm{Hom}(P_\gamma, P_{\gamma'}) \hookrightarrow \mathrm{Hom}(X(P_\gamma), X(P_{\gamma'}))$$

The result now follows from the following Lemma.

Lem $X : \mathcal{C} \to \mathcal{D}$ a braided monoidal functor between rigid monoidal categories then

i) X is exact

ii) $X(A) \neq 0$ for $0 \neq A \in \mathcal{C}$

iii) X faithful.