# LECTURE 10: KAZHDAN-LUSZTIG BASIS AND CATEGORIES $\mathcal{O}$

IVAN LOSEV

## Introduction

In this and the next lecture we will describe an entirely different application of Hecke algebras, now to the category $\mathcal{O}$. In the first section we will define the Kazhdan-Lusztig basis in the Hecke algebra of $W$ and explain how to read the multiplicities in the category $\mathcal{O}$ from this basis (the Kazhdan-Lusztig conjecture proved independently by Beilinson-Bernstein and Brylinski-Kashiwara).

In the remainder of this lecture and in the next one, we will explain some steps towards a proof of this conjecture based on works of Soergel and of Elias-Williamson. We will start by defining projective functors between different infinitesimal blocks of category $\mathcal{O}$. As an application, we will show how the computation of multiplicities in $\mathcal{O}_\lambda$ for $\lambda \in P$ reduces to $\lambda = 0$.

## 1. Kazhdan-Lusztig basis and conjecture

1.1. **Recap on category $\mathcal{O}$.** Pick a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let $W$ denote the Weyl group. Let $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_i \omega_i$ (where $\omega_i$ denote the fundamental weight corresponding to a simple root $\alpha_i$). Define the shifted action of $W$ on $\mathfrak{h}$ by $w\cdot\lambda := w(\lambda+\rho)-\rho$.

Recall that in Lecture 7 we have introduced the BGG category $\mathcal{O}$ consisting of all finitely generated $U(\mathfrak{g})$-modules with diagonalizable action of $\mathfrak{h}$ and locally nilpotent action of $\mathfrak{n}$. Also we have identified the center $Z$ of $U(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{h}]^{W\cdot} = \{f \in \mathbb{C}[\mathfrak{h}] | f(w\cdot\lambda) = f(\lambda), \forall w \in W, \lambda \in \mathfrak{h}\}$, where we send $z \in Z$ to the polynomial $\tilde{f}_z$ such that $z$ acts by $\tilde{f}_z(\lambda)$ on the Verma module $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. This allowed to split $\mathcal{O}$ into the direct sum of infinitesimal blocks $\mathcal{O}_\lambda$ consisting of all modules $M$ in $\mathcal{O}$, where $z$ acts with generalized eigenvalue $\tilde{f}_z(\lambda)$. We are going to be interested in $\mathcal{O}_\lambda$, where $\lambda \in P$ (the weight lattice) and mainly in $\mathcal{O}_0$ (we will see that the study of $\mathcal{O}_\lambda$ with $\lambda \in P$ basically reduces to the study of $\mathcal{O}_0$). The simple objects in $\mathcal{O}_0$ are $L(w\cdot 0), w \in W$, all of these objects are different because $W_\rho = \{1\}$. We have seen in Lecture 7 that any $L(w\cdot 0)$ appears in the composition series of $\Delta(w\cdot 0)$ once, and all other composition are $L(w'\cdot 0)$, where $w'\cdot 0 < w\cdot 0$ meaning that $w\cdot 0 - w'\cdot 0$ is a sum of positive roots. In fact, we can take a weaker *Bruhat* order (getting a stronger result).

**Definition 1.1.** We say that $u \prec w$ (in the Bruhat order) if $w = s_{\beta_k}\ldots s_{\beta_1}u$, where $\beta_k,\ldots,\beta_1$ are roots (not necessarily simple) and $\ell(s_{\beta_i}\ldots s_{\beta_1}u) > \ell(s_{\beta_{i-1}}\ldots s_{\beta_1}u)$ for all $i$.

In this order, the minimal element in $W$ is 1, while the maximal element is the longest (with respect to the length $\ell(w)$) element $w_0 \in W$. It is uniquely characterized by the property that it maps the positive Weyl chamber $C$ to $-C$. For $W = S_n$, we have $w_0(i) = n+1-i$ for all $i$.

Here are properties of $\prec$ to be used below.

**Lemma 1.2.** *The following is true:*

(1) *If* $u \prec w$, *then* $u \cdot 0 > w \cdot 0$.
(2) *If* $u$ *is obtained from* $w$ *by deleting some elements in the reduced expression of* $w$, *then* $u \prec w$.
(3) $u \preceq w$ *if and only if* $w_0 w \preceq w_0 u$.

The proof is left as an exercise.

We will write $L_w$ for $L(w_0 w \cdot 0)$ and $\Delta_w = \Delta(w_0 w \cdot 0)$. One can show that if $L_u$ is a composition factor of $\Delta_w$, then $u \preceq w$. What we want to compute is the character of $L_w$. Let $m_w^u$ denote the multiplicity of $L_u$ in $\Delta_w$. Consider the multiplicity matrix $M$, it is unitriangular and hence invertible. Let $M^{-1} = (n_w^u)$. So $\mathsf{ch} L_w = \sum_{u \preceq w} n_w^u \mathsf{ch}\Delta_u$. So what we need to compute is the numbers $n_w^u$.

It is convenient to reformulate this problem. The category $\mathcal{O}_0$ is abelian. So we can consider its *Grothendieck group* $K_0(\mathcal{O}_0)$. It is defined as the quotient of the free group generated by the isomorphism classes of the objects $M \in \mathcal{O}_0$ modulo the relation $M = M' \oplus M''$ if there is an exact sequence $0 \to M' \to M \to M'' \to 0$. We denote the image of $M$ in $K_0(\mathcal{O}_0)$ by $[M]$. Since the objects in $\mathcal{O}_0$ have finite length, the classes $[L_w]$ form a basis in $K_0(\mathcal{O}_0)$. Since the matrix $M$ is uni-triangular, the same is true for $[\Delta_w]$. We identify $K_0(\mathcal{O}_0)$ with the group ring $\mathbb{Z}W$ in such a way that $[\Delta_w]$ corresponds to $w$. So we need to compute the basis $[L_w] = \sum_{u \preceq w} n_w^u u$.

**Example 1.3.** It is easy to compute two basis elements $[L_w]$. Namely, we have $n_w^w = 1$ and $n_w^u \neq 0 \Rightarrow u \preceq w$. This immediately implies $[L_1] = 1$. The proof of the Weyl character formula in Lecture 7 says $[L_{w_0}] = \sum_{w \in W} \mathrm{sgn}(w_0 w) w$.

In general, however, we cannot even describe the basis $[L_w]$ staying inside $\mathbb{Z}W$. This is where the Hecke algebra comes into play.

1.2. **Kazhdan-Lusztig basis.** First, it will be convenient to modify the Hecke algebra slightly. Let us recall the previous definition (in the specialization $v_s = v$ for all $s \in S$, where $S$ denotes the set of simple reflections in $W$). The Hecke algebra $\mathcal{H}_v(W)$ is generated by elements that we will now denote by $T'_s$ with relations $T'_s T'_t T'_s \ldots = T'_t T'_s T'_t \ldots$ ($m_{st}$ times) and $(T'_s - v)(T'_s + 1) = 0$. Now let $q$ be another independent variable (that has nothing to do with a prime power). Define the $\mathbb{Z}[q^{\pm 1}]$-algebra $\mathcal{H}_q(W)$ by generators $T_s$ with relations $T_s T_t T_s \ldots = T_t T_s T_t \ldots$ and $(T_s - q)(T_s + q^{-1}) = 0$. Clearly, $\mathcal{H}_q(W) = \mathcal{H}_v(W)[q^{\pm 1}]/(v - q^2)$ with $T'_s \mapsto q T_s$. We see that $\mathcal{H}_q(W)$ has basis $T_w$ such that

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1, \\ T_{sw} + (q - q^{-1}) T_w, & \text{if } \ell(sw) = \ell(w) - 1. \end{cases} \tag{1.1}$$

We have a ring involution of $\mathcal{H}_q(W)$ (called the bar involution and denoted by $\bar{\bullet}$), given on generators by $\bar{q} := q^{-1}, \bar{T}_s := T_s^{-1} (= T_s + q^{-1} - q)$. Since $\bar{\bullet}$ preserves the relations, we see that $\bar{\bullet}$ is indeed a well-defined ring involution. Note that $\bar{T}_w = (T_{w^{-1}})^{-1}$.

The following fundamental result is due to Kazhdan and Lusztig, [KL].

**Theorem 1.4.** *For any* $w \in W$, *there is a unique element* $C_w \in \mathcal{H}_q(W)$ *such that* $C_w = T_w + \sum_{u \prec w} P_w^u(q) T_u$, *where* $P_w^u(q) \in q\mathbb{Z}[q]$, *and* $\bar{C}_w = C_w$.

Since the matrix of expressing $C_w$'s in terms of $T_w$'s is uni-triangular, we see that the elements $C_w, w \in W$, form a basis in $\mathcal{H}_q(W)$. This is a so called *Kazhdan-Lusztig* basis.

*Proof.* The proof is by induction with respect to the Bruhat order: we assume that $C_u$ exits and is unique for all $u \prec w$. Let $w = s_{i_1} \ldots s_{i_\ell}$ be a reduced expression. We have

$\bar{T}_w = \bar{T}_{i_1}\dots\bar{T}_{i_\ell} = (T_{i_1}+q^{-1}-q)\dots(T_{i_\ell}+q^{-1}-q)$. Decompose $\bar{T}_w$ in the basis $T_u$. We have $\bar{T}_w = T_w + \sum_{u\prec w} R^u_w(q)T_u$ (all $u$'s are obtained by removing some simple reflections from the reduced decomposition of $w$ and so $u \prec w$ by (2) of Lemma 1.2). By the existence of $C_u$, what we need to show that there is a unique $\tilde{P}^u_w(q) \in q\mathbb{Z}[q]$ such that $C_w = T_w + \sum_{u\prec w}\tilde{P}^u_w(q)C_u$ and $\bar{C}_w = C_w$. We also have $\bar{T}_w - T_w = \sum_{u\prec w} Q^u_w(q)C_u$. Applying $\bar{\bullet}$ to the equation, we get $T_w - \bar{T}_w = \sum_{u\prec w} Q^u_w(q^{-1})C_u$ and so $\bar{Q}^u_w(q^{-1}) = -\bar{Q}^u_w(q)$. But we have

$$\bar{C}_w = \bar{T}_w + \sum_{u\prec w}\tilde{P}^u_w(q^{-1})C_u = T_w + \sum_{u\prec w} Q^u_w(q)C_u + \sum_{u\prec w}\tilde{P}^u_w(q^{-1})C_u$$

So we need to prove that there is a unique $\tilde{P}^u_w(q) \in q\mathbb{Z}[q]$ such that $\tilde{P}^u_w(q^{-1}) - \tilde{P}^u_w(q) = Q^u_w(q)$. This follows from $Q^u_w(q^{-1}) = -Q^u_w(q)$. □

**Example 1.5.** We have $C_1 = 1$ and $C_s = T_s - q$, where $s \in S$.

Let us consider a more interesting example: $W = S_3$. Let $s,t$ denote the simple reflections. The Bruhat order is that $1 < s,t < st, ts < sts = tst$ (elements in the same group are not comparable). We have

$$C_{st} = T_{st} - q(T_s+T_t)+q^2, C_{ts} = T_{ts} - q(T_s+T_t)+q^2, C_{sts} = T_{sts} - q(T_{st}+T_{ts}) + q^2(T_s+T_t) - q^3.$$

More generally, $C_{w_0} = \sum_{w\in W}(-q)^{\ell(w_0)-\ell(w)}T_w$. To check these equalities is a part of the homework.

1.3. **Kazhdan-Lusztig conjecture.** We have a surjection $\mathcal{H}_q(W) \twoheadrightarrow \mathbb{Z}W$ given by setting $q = 1$. The following theorem was conjectured by Kazhdan-Lusztig and proved by Beilinson-Bernstein, [BB], and Brylinski-Kashiwara, [BK].

**Theorem 1.6.** *We have* $[L_w] = C_w|_{q=1}$.

By Example 1.5, this agrees with the Weyl character formula: $[L_{w_0}] = \sum_{w\in W_0}\mathrm{sgn}(w_0 w)w$.

This is a difficult theorem whose proof found in the 80's required a heavy machinery and is one of the greatest achievements of Geometric Representation theory. Recently, a more elementary (but also difficult) proof was found, see [EW]. Starting the next section, we will outline some ideas relevant for that proof.

1.4. **Stronger version.** Now we are going to explain how to recover $C_w$ itself (not just its specialization to 1) from the structure of Verma modules. This description was found in [BGS].

Let $M$ be an object of $\mathcal{O}$. By $\mathsf{head}(M)$ we mean the maximal semisimple quotient of $M$ and by the radical $\mathsf{Rad}(M)$ we mean the kernel $M \twoheadrightarrow \mathsf{head}(M)$. Now define the *radical filtration* $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \dots$ by setting $M_i := \mathsf{Rad}(M_{i-1})$. Now take $M := \Delta_w$ and for $u \preceq w$ define $m^u_w(q) := \sum[M_i/M_{i+1} : L_u]q^i$, where the square bracket denotes the multiplicity of $L_u$ in the composition series of $M_i/M_{i+1}$. For example, $m^w_w(q) = 1$.

**Theorem 1.7.** *We have* $T_w = \sum_{u\preceq w} m^u_w(q)C_u$.

**Example 1.8.** For $\mathfrak{g} = \mathfrak{sl}_2$, the module $\Delta(1)$ has simple radical, $\Delta(-2) = L(-2)$. For $s \in S_2 \setminus \{1\}$, we get $m^s_s(q) = 1, m^1_s(q) = q$. Indeed, $T_s = C_s + qC_1$.

## 2. Projective functors, I

We are going to explain how to reduce the study of $\mathcal{O}_\lambda$ with $\lambda \in P$ to $\lambda = 0$.

2.1. **Tensor products with finite dimensional modules.** Recall that if $V$ is a finite dimensional $\mathfrak{g}$-module and $M \in \mathcal{O}$, then $V \otimes M \in \mathcal{O}$. So we get the functor $V \otimes \bullet : \mathcal{O} \to \mathcal{O}$. This functor is exact (preserves exact sequences), it has both left and right adjoints, both are given by $V^* \otimes \bullet$.

Now we are going to get a partial description of $V \otimes \Delta(\lambda)$. Pick a weight basis $v_1, \ldots, v_m$ of $V$ and let $\nu_1, \ldots, \nu_m \in \mathfrak{h}^*$ be the corresponding weights. We may assume that they are ordered compatibly with the order on $\mathfrak{h}^*$, i.e., if $\nu_i \geqslant \nu_j$, then $i \geqslant j$.

**Proposition 2.1.** *There is a filtration $V \otimes M = M_0 \supset M_1 \supset \ldots \supset M_m = \{0\}$ such that $M_{i-1}/M_i \cong \Delta(\lambda + \nu_i)$.*

*Proof.* Recall that $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. We claim that $V \otimes \Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C})$. This follows from

$$\begin{aligned}&\operatorname{Hom}_{\mathfrak{g}}(V \otimes \Delta(\lambda), M) = \operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda), V^* \otimes M) = \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V^* \otimes M) = \\ &= \operatorname{Hom}_{\mathfrak{b}}(V \otimes \mathbb{C}_\lambda, M) = \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C}_\lambda), M).\end{aligned}$$

Consider the filtration $V \otimes \mathbb{C}_\lambda = N_0 \supset \ldots \supset N_m = \{0\}$, where $N_i := \operatorname{Span}_{\mathbb{C}}(v_{i+1}, \ldots, v_m)$. This is $\mathfrak{b}$-module filtration (because $\mathfrak{n}$ increases weights) with $N_{i-1}/N_i = \mathbb{C}_{\lambda+\nu_i}$. Set $M_i := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N_i$. Recall that $U(\mathfrak{g})$ is a free right $U(\mathfrak{b})$-module. So the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bullet$ is exact and we have $M_{i-1}/M_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_{i-1}/N_i) = \Delta(\lambda + \nu_i)$. □

Let $\mathsf{pr}_\mu$ denote the functor $\mathcal{O} \twoheadrightarrow \mathcal{O}_\mu$ that sends $M \in \mathcal{O}$ to the generalized eigenspace of $Z$ in $M$ with eigenvalue $\mu$. The functors of the form $\mathsf{pr}_\mu(V \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ (and their compositions) are known as *projective functors*. They are tremendously useful in the study of $\mathcal{O}$.

**Corollary 2.2.** *The object $\mathsf{pr}_\mu(V \otimes \Delta(w \cdot \lambda))$ admits a filtration by $\Delta(\lambda + \nu_i)$ with $w \cdot \lambda + \nu_i \in W \cdot \mu$.*

2.2. **Application: translation functors.** We are going to consider a special case of Corollary 2.2, where it is especially easy to describe what weights $\nu_i$ appear.

**Proposition 2.3.** *Assume that $\lambda, \mu \in P$ are such that $\lambda, \lambda - \mu, \mu + \rho$ are dominant. Let $V$ be the irreducible finite dimensional module with highest weight $\lambda - \mu$. Then $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ and $\mathsf{pr}_\lambda(V \otimes \Delta(w \cdot \mu))$ is filtered with $\Delta(wu \cdot \lambda), u \in W_{\mu+\rho}$.*

Note that $W_{\mu+\rho}$ is generated by the simple reflections $s_i$ such that $\langle \mu + \rho, \alpha_i^\vee \rangle = 0$.

*Proof.* To prove the claim about $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda))$ we need to find all weights $\nu$ of $V^*$ such that $w \cdot \lambda + \nu \in W \cdot \mu$, equivalently, $\lambda + \rho + w^{-1}\nu = u(\mu + \rho)$ for some $u \in W$. We have $u(\mu + \rho) \leqslant \mu + \rho$ for any $u \in W$ and $w^{-1}\nu \geqslant \mu - \lambda$ (the lowest weight of $V^*$) with equality if and only if $w = w_0$. So $\lambda + \rho + w^{-1}\nu \geqslant \lambda + \rho + \mu - \lambda = \mu + \rho \geqslant u(\mu + \rho)$. The equality $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ follows from Corollary 2.2.

The claim about $\mathsf{pr}_\lambda(V \otimes \Delta(w \cdot \mu))$ follows similarly using the observation that $\lambda - w \cdot \mu \geqslant \lambda - \mu$ for any $w \in W$, and the equality is equivalent to $w \in W_{\mu+\rho}$. □

This proposition has several important corollaries.

**Corollary 2.4.** *Let $\lambda_1, \lambda_2$ be dominant. Then there is an equivalence $\mathcal{O}_{\lambda_1} \xrightarrow{\sim} \mathcal{O}_{\lambda_2}$ sending $\Delta(w \cdot \lambda_1)$ to $\Delta(w \cdot \lambda_2)$.*

*Proof.* We may assume that $\lambda_1 - \lambda_2$ is dominant. Otherwise, we replace $\lambda_1$ with $\lambda_1 + \lambda_2$ and take a composed equivalence $\mathcal{O}_{\lambda_1} \xrightarrow{\sim} \mathcal{O}_{\lambda_1+\lambda_2} \xrightarrow{\sim} \mathcal{O}_{\lambda_2}$.

Apply Proposition 2.3 to $\lambda = \lambda_1 + \lambda_2$ and $\mu = \lambda_2$. We get functors $\varphi : \mathcal{O}_\mu \to \mathcal{O}_\lambda, \varphi(M) := \mathsf{pr}_\lambda(V \otimes M)$ and $\varphi^* := \mathsf{pr}_\mu(V^* \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$. The notation $\varphi^*$ is justified by the observation that $\varphi^*$ is left and right adjoint to $\varphi$. We are going to prove that $\varphi^*, \varphi$ are mutually inverse (quasi-inverse, if we want to be precise).

Note that $\varphi(\Delta(w \cdot \mu)) = \Delta(w \cdot \lambda)$ and $\varphi^*(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ (we have $W_{\mu+\rho} = \{1\}$). Also we have an adjointness homomorphism $\varphi^* \circ \varphi(M) \to M$ (induced by $\varphi^* \circ \varphi(M) \hookrightarrow V^* \otimes V \otimes M \to M$). This homomorphism is zero if and only if $\varphi(M)$ is zero. Now apply this to $M = \Delta(w \cdot \lambda)$, we get a nonzero homomorphism $\Delta(w \cdot \mu) = \varphi^* \circ \varphi(\Delta(w \cdot \mu)) \to \Delta(w \cdot \mu)$. But any Verma module is generated by its highest weight vector and any endomorphism maps that vector to its multiple. We deduce that any nonzero endomorphism of a Verma module is an isomorphism. So $\varphi^* \circ \varphi(M) \xrightarrow{\sim} M$ when $M$ is a Verma module. Since any object in $\mathcal{O}_\mu$ is filtered by quotients of Verma modules, we see that $\varphi^* \circ \varphi(M) \xrightarrow{\sim} M$ for any $M \in \mathcal{O}_\lambda$. So $\varphi^*$ is left inverse of $\varphi$. Similarly, we see that $\varphi^*$ is a right inverse of $\varphi$. $\square$

Now let us consider the case when $\mu + \rho$ is dominant, but $W_{\mu+\rho}$ is non-trivial. The simples in $\mathcal{O}_\mu$ are naturally labelled by $W/W_{\mu+\rho}$. There is a distinguished representative in each right coset $wW_{\mu+\rho}$ – it is known that such a coset contains a unique longest element (w.r.t. the length function $\ell$; it also contains a unique shortest element, but we do not need this). So it is natural to label the simples in $\mathcal{O}_{\mu+\rho}$ with longest elements of right $W_{\mu+\rho}$-cosets.

**Corollary 2.5.** *Let $\lambda, \mu$ be such as in Proposition 2.3. Then $\mathsf{pr}_\mu(V^* \otimes L(w \cdot \lambda)) = L(w \cdot \mu)$ if $w$ is longest in its right $W_{\mu+\rho}$-coset $wW_{\mu+\rho}$ and is zero else.*

*Sketch of proof.* The proof is again based on using adjoint functors $\varphi := \mathsf{pr}_\mu(V^* \otimes \bullet)$ and $\varphi^* := \mathsf{pr}_\lambda(V \otimes \bullet)$.

*Step 1.* We need to show that $\varphi(L(w \cdot \lambda)) = 0$ when $w$ is not longest in its right $W_{\mu+\rho}$-coset. In other words, we can find a simple reflection $s_i \in W_{\mu+\rho}$ such that $\ell(ws_i) > \ell(w)$. In this case, we have a nonzero homomorphism $\eta : \Delta(ws_i \cdot \lambda) \to \Delta(w \cdot \lambda)$. One can show that $\varphi(\eta) \neq 0$. So $\varphi(\eta)$ is an isomorphism. In particular, $\varphi(\operatorname{coker} \eta) = 0$ and hence $\varphi(L(w \cdot \lambda)) = 0$.

*Step 2.* Now let $w$ be longest in its right $W_{\mu+\rho}$-coset. The object $\varphi(L(w \cdot \lambda))$ is a quotient of $\varphi(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$. So we need to show that $\varphi(L(w \cdot \lambda)) \neq 0$ and that $\operatorname{Hom}(\Delta(w' \cdot \mu), \varphi(L(w \cdot \lambda))) = 0$ if $w' \cdot \mu < w \cdot \mu$ (this will show that $\varphi(L(w \cdot \lambda))$ is simple). The equality follows because $\operatorname{Hom}(\Delta(w' \cdot \mu), \varphi(L(w \cdot \lambda))) = \operatorname{Hom}(\varphi^*(\Delta(w' \cdot \mu)), L(w \cdot \lambda))$ and $\Delta(w \cdot \lambda)$ does not appear in the filtration of $\varphi^*(\Delta(w' \cdot \mu))$. On the other hand, if $\varphi(L(w \cdot \lambda)) = 0$, then the class $[\varphi(\Delta(w \cdot \lambda))]$ is a linear combination of $[\varphi(L(w' \cdot \lambda))]$ with $w' \cdot \mu < w \cdot \mu$ and hence of $[\varphi(\Delta(w' \cdot \lambda))]$ with $w' \cdot \mu < w \cdot \mu$. But $\varphi(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ and $\varphi(\Delta(w' \cdot \lambda)) = \Delta(w' \cdot \mu)$, contradiction. $\square$

Let us give a corollary of the previous two corollaries that reduces the question about the multiplicities in the categories $\mathcal{O}_\mu$ for $\lambda \in P$ to $\lambda = 0$.

**Corollary 2.6.** *Let $\mu$ be such that $\mu + \rho$ is dominant. Pick $w$ that is longest in its right $W_{\lambda+\rho}$-coset. Then $[\Delta(u \cdot \mu) : L(w \cdot \mu)] = [\Delta(u \cdot 0) : L(w \cdot 0)]$.*

## References

[BB] A. Beilinson, J. Bernstein. Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 1518.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel. *Koszul duality patterns in representation theory*. J. Amer. Math. Soc. 9 (1996), no. 2, 473527.

[BK] J.-L. Brylinski, M. Kashiwara. *Kazhdan-Lusztig conjecture and holonomic systems*. Invent. Math. 64 (1981), no. 3, 387410.

[EW] B. Elias, G. Williamson. *The Hodge theory of Soergel bimodules*. Ann. of Math. (2) 180 (2014), no. 3, 10891136. arXiv:1212.0791.

[KL] D. Kazhdan, G. Lusztig. *Representations of Coxeter groups and Hecke algebras*. Invent. Math. 53 (1979), no. 2, 165-184.