LECTURE 13: REPRESENTATIONS OF $U_q(\mathfrak{g})$ AND R-MATRICES

IVAN LOSEV

Introduction

In this lecture we study the representation theory of $U_q(\mathfrak{g})$ when q is not a root of 1. In Section 1, we classify the finite dimensional irreducible representations of $U := U_q(\mathfrak{sl}_2)$, sketch the proof of complete reducibility and explain what happens for a general $\mathfrak{g}$.

As we have seen in the previous lecture, the obvious isomorphism $\sigma : v_1 \otimes v_2 \to v_2 \otimes v_1 : V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ is not U-linear. However, one can find an element R in a suitable completion of $U \otimes U$ such that $R \circ \sigma : V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ is a U-module isomorphism. This will be done in Section 2. This construction is of importance for knot invariants, as will be explained in the next lecture.

1. Representation theory of $U_q(\mathfrak{g})$, I

Recall that, as an algebra, U is given by generators $E, F, K^{\pm 1}$ and relations

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

1.1. **Classification of the irreducibles.** Let us start by producing some examples of the irreducible representations of U.

Example 1.1. Let us classify the one-dimensional representations. We have $KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F$. Since $q \neq \pm 1$, it follows that E, F act by 0. So $K - K^{-1} = (q - q^{-1})(EF - FE)$ acts by 0. We deduce that K acts by ± 1. Both choice give representations. Of course, the representation, where K acts by 1, E, F act by 0, is the trivial representation, one that is given by the counit η.

Example 1.2. The assignment $E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ gives rise to a two-dimensional representation of U. This is a so called tautological representation.

Lemma 1.3. *Suppose q is not a root of* 1. *Let V be a finite dimensional U-module. Then the elements E, F act on V nilpotently.*

Proof. For $\alpha \in \mathbb{C}^\times$, let V_α denote the generalized eigenspace for K with eigenvalue α. It is easy to show that $EV_\alpha \subset V_{q^2\alpha}$. Since q is not a root of 1, we see that all numbers $q^{2n}\alpha$ are different. It follows that E acts nilpotently. For the same reasons, F acts nilpotently. $\square$

Now the classification of $\mathrm{Irr}_{fin}(U)$ works in the same way as for $U(\mathfrak{sl}_2)$. Namely, we have the subalgebra $U_q(\mathfrak{b}) \subset U$ spanned by K, E, it has a basis $K^\ell E^m$ for $\ell \in \mathbb{Z}, m \in \mathbb{Z}_{\geqslant 0}$. The algebra U is a free right module over $U_q(\mathfrak{b})$ with basis $F^k, k \in \mathbb{Z}_{\geqslant 0}$. Then, for $\alpha \in \mathbb{C}^\times$, we can form the Verma module $\Delta_q(\alpha) := U \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\alpha$, where K acts on $\mathbb{C}_\alpha$ by α and E acts by

0. The U-module $\Delta_q(\alpha)$ has basis $v_k := F^k v_\alpha, k \in \mathbb{Z}_{\geqslant 0}$. The action of F, K, E in this basis is given by

$$Fv_k = v_{k+1}, \quad Kv_k = q^{-2k}\alpha v_k, \quad Ev_k = [k]_q \frac{\alpha q^{1-k} - \alpha^{-1}q^{k-1}}{q - q^{-1}} v_{k-1}.$$

The third equation follows from

$$[E, F^n] = [n]_q F^{n-1} \frac{Kq^{1-n} - K^{-1}q^{n-1}}{q - q^{-1}}. \tag{1.1}$$

Theorem 1.4. *Suppose q is not a root of 1. Then the finite dimensional irreducible U-modules are in one-to-one correspondence with the set $\{\pm q^n\}_{n \in \mathbb{Z}_{\geqslant 0}}$. The module $L(\pm q^n)$, the irreducible quotient of $\Delta_q(\pm q^n)$, has basis $u_0, \ldots, u_n$, where the action of the generators is given by*

$$Ku_i = \pm q^{n-2i} u_i, Fu_i = [n-i]_q u_{i+1}, Eu_i = \pm [i]_q u_{i-1}.$$

Proof. As in the proof for U, we need to understand when $\Delta_q(\alpha)$ has a proper quotient. This is only possible when $Ev_k = 0$. The number $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ is never zero thanks to our assumption that q is not a root of 1. So $Ev_k = 0$ if and only if

$$\frac{\alpha q^{1-k} - \alpha^{-1}q^{k-1}}{q - q^{-1}} = 0 \Leftrightarrow \alpha^2 = q^{2(k-1)} \Leftrightarrow \alpha = \pm q^{k-1}.$$

In this case, we have a k-dimensional quotient of $\Delta_q(\alpha)$. Thanks to the standard universal property of $\Delta_q(\alpha)$, every simple module is a quotient of one of $\Delta_q(\alpha)$ (and it is easy to see that α is recovered uniquely from the simple module). We set $u_k := v_k/[n-k]_q!$. □

Example 1.5. The modules from Example 1.1 are $L(\pm 1)$. The module from Example 1.2 is $L(q)$. Note that $L(-q^n) = L(-1) \otimes L(q^n)$. Thanks to this, one usually only studies the modules $L(q^n)$.

1.2. **Complete reducibility.** We can introduce the quantum Casimir element

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} \in U.$$

One can show (this is a part of the homework) that this element is central in U.

Theorem 1.6. *Let q be not a root of 1. Then any finite dimensional U-module is completely reducible.*

Proof. For $v_\alpha \in \Delta_q(\alpha)$, we get $Cv_\alpha = \frac{\alpha q + \alpha^{-1}q^{-1}}{(q-q^{-1})^2} v_\alpha$. In particular, we see that all scalars of the action of C on $L(\pm q^n)$ are distinct. As in the case of $U(\mathfrak{sl}_2)$, we only need to prove that $L(\pm q^n)$ has no self-extensions. This is done similarly to that case. □

1.3. **General $U_q(\mathfrak{g})$.** We assume that q is not a root of 1. Let $\mathfrak{g}$ be a semisimple Lie algebra with generators $e_i, h_i, f_i, i = 1, \ldots, n$. Let us explain the classification of finite dimensional irreducible $U_q(\mathfrak{g})$-modules.

Theorem 1.7. *Any finite dimensional representation of $U_q(\mathfrak{g})$ is completely reducible.*

Let us explain how to classify the finite dimensional irreducible representations. First, let us consider the one-dimensional representations. On such a representation, all elements E_i, F_i act by 0, and the elements K_i act by ± 1. So for $\kappa \in \{\pm 1\}^n$, we have the one-dimensional module $L(\kappa)$, where K_i acts by κ_i.

Modulo these 2^n modules, the finite dimensional representation theory of $U_q(\mathfrak{g})$ looks just like the representation theory of $\mathfrak{g}$. Recall that P^+ denotes the set of dominant weights.

Theorem 1.8. *There is a bijection $\{\pm 1\}^n \times P^+ \xrightarrow{\sim} \mathrm{Irr}_{fin}(U_q(\mathfrak{g}))$ that sends (κ, λ) to a unique finite dimensional irreducible module $L(\kappa q^\lambda)$ that has a highest vector v_λ such that $E_i v_\lambda = 0, K_i v_\lambda = \kappa_i q^{\lambda(\alpha_i^\vee)} v_\lambda$.*

We do not provide a proof. We note that this description implies $L(\kappa q^\lambda) = L(\kappa) \otimes L(q^\lambda)$.

To finish let us point out that one can define the notion of a character of $L(q^\lambda)$ in a natural way. The character is given by the Weyl character formula.

2. Universal R-matrix

2.1. **Three coproducts.** Recall the coproduct $\Delta : U \to U^{\otimes 2}$ given on the generators by

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

The opposite coproduct $\Delta^{op} := \sigma \circ \Delta$, where σ denotes the permutation of the tensor factors is then given by

$$\Delta^{op}(K) = K \otimes K, \Delta^{op}(E) = E \otimes K + 1 \otimes E, \Delta^{op}(F) = F \otimes 1 + K^{-1} \otimes F.$$

We want to find an element $R \in U^{\otimes 2}$ (in fact, we will have to use a completion) such that $R\Delta^{op}(u) = \Delta(u)R$. If V_1, V_2 are U-modules, then the map $R_{V_1,V_2} \circ \sigma : V_1 \otimes V_2 \to V_2 \otimes V_1, v_1 \otimes v_2 \mapsto R_{V_1,V_2}(v_2 \otimes v_1)$ is an isomorphism of U-modules.

In order to produce R, we will need the third coproduct, Δ', given by

$$\Delta'(K) = K \otimes K, \Delta'(E) = E \otimes 1 + K^{-1} \otimes E, \Delta'(F) = F \otimes K + 1 \otimes F.$$

This coproduct is obtained from Δ by a twist with an anti-involution τ of U_q given by $\tau(K) = K^{-1}, \tau(E) = E, \tau(F) = F$. Then $\Delta'(u) = \tau \otimes \tau(\Delta(\tau(u)))$. This equality implies that Δ' also gives a coassociative coproduct on U (or we can check this directly). We will produce R as a product $\Theta\Psi$ with $\Theta^{-1}\Delta(u)\Theta = \Delta'(u)$ and $\Psi^{-1}\Delta'(u)\Psi = \Delta^{op}(u)$.

2.2. **Construction of Θ.** Let us construct Θ. This will be an infinite sum of the form $\sum_{n=0}^\infty a_n F^n \otimes E^n$, where we will find the coefficients a_n from $\Theta\Delta'(E) = \Delta(E)\Theta$.

$$\begin{aligned}
&\Theta\Delta'(E) = \Delta(E)\Theta \\
\Leftrightarrow &\Big(\sum_{n=0}^\infty a_n F^n \otimes E^n\Big)(E \otimes 1 + K^{-1} \otimes E) = (E \otimes 1 + K \otimes E)\Big(\sum_{n=0}^\infty a_n F^n \otimes E^n\Big) \\
\Leftrightarrow &\sum_{n=0}^\infty a_n(F^n E \otimes E^n + F^n K^{-1} \otimes E^{n+1}) = \sum_{n=0}^\infty a_n(EF^n \otimes E^n + KF^n \otimes E^{n+1}) \\
\Leftrightarrow &\sum_{n=0}^\infty a_n [E, F^n] \otimes E^n = \sum_{n=0}^\infty a_n(F^n K^{-1} - KF^n) \otimes E^{n+1} \\
\Leftrightarrow &a_{n+1}[E, F^{n+1}] = a_n(F^n K^{-1} - KF^n).
\end{aligned}$$

By (1.1), we get $[E, F^{n+1}] = [n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}}$. On the other hand $F^n K^{-1} - KF^n = F^n(K^{-1} - q^{-2n}K)$. Therefore

$$a_{n+1}[n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}} = a_n F^n(K^{-1} - q^{-2n}K) \Leftrightarrow a_{n+1} = \frac{(q^{-1} - q)q^{-n}}{[n+1]_q} a_n.$$

We conclude that

$$\Theta = \sum_{n=0}^{\infty} \frac{(q^{-1}-q)^n q^{-n(n-1)/2}}{[n]_q!} F^n \otimes E^n. \tag{2.1}$$

One can show that $\Theta\Delta'(K) = \Delta(K)\Theta$ (this is almost immediate) and that $\Theta\Delta'(F) = \Delta(F)\Theta$ (this is a computation very similar to what was done above).

Example 2.1. Let us compute $\Theta_{V\otimes V}$, where $V = L(q)$. We have $\Theta_{V\otimes V} = 1+(q^{-1}-q)F\otimes E$. Let us write this operator as a matrix. Let v_1, v_2 (resp., v_1', v_2') be the natural basis in the first and in the second factor. Then in the basis $v_1\otimes v_1', v_1\otimes v_2', v_2\otimes v_1', v_2\otimes v_2'$ we get the following matrix of Θ: $\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q^{-1}-q & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

2.3. **Construction of Ψ.** An element Ψ satisfying $\Psi\Delta^{op}(u) = \Delta'(u)\Psi$ will depend only on K. But it is not expressed in terms of algebraic functions in q, K (we have to use log's). So we will just define $\Psi_{V_2\otimes V_1}: V_2\otimes V_1 \to V_2\otimes V_1$, where K acts on V_2, V_1 with powers of q (we can also define Ψ as an element of the *idempotent completion* of $U\otimes U$, where we add infinite sums $\sum_{\lambda,\mu} a_{\lambda\mu}\pi_\lambda\otimes\pi_\mu$, where $\lambda,\mu\in\{\pm q^n\}$ and π_λ acts on a U-module V as the projection to V_λ). Let $\Psi_{V_2\otimes V_1}(v_n\otimes u_m) = \psi(n,m)v_n\otimes u_m$, where $v_n\in V_2, u_m\in V_1$ are K-eigenvectors with eigenvalues q^n, q^m, and ψ is a function we need to determine. We have

$$\Psi\Delta^{op}(E)(v_n\otimes u_m) = \Psi(E\otimes K + 1\otimes E)(v_n\otimes u_m) = \Psi(q^m Ev_n\otimes u_m + v_n\otimes Eu_m) =$$
$$= q^m\psi(n+2,m)Ev_n\otimes u_m + \psi(n,m+2)v_n\otimes Eu_m.$$
$$\Delta'(E)\Psi(v_n\otimes u_m) = \psi(n,m)(K^{-1}\otimes E + E\otimes 1)(v_n\otimes u_m) =$$
$$= \psi(n,m)(Ev_n\otimes u_m + q^{-n}v_n\otimes Eu_m).$$
$$\Psi\Delta^{op}(E) = \Delta'(E)\Psi \Leftrightarrow \psi(n+2,m) = q^{-m}\psi(n,m) \text{ and } \psi(n,m+2) = q^{-n}\psi(n,m).$$

Conversely, for any ψ satisfying the conditions above, we have $\Psi\Delta^{op}(u) = \Delta'(u)\Psi$. Indeed, for $u = K$ this holds for any ψ and for $u = F$ the conditions on ψ are equivalent to what we had above. Note that to recover ψ we just need to specify the values $\psi(\alpha,\beta)$ when $\alpha,\beta\in\{-1,0\}$.

Example 2.2. Let us consider the case when $V_1 = V_2 = V = L(q)$. We set $\psi(-1,-1) = q$. Then $\psi(1,-1) = \psi(-1,1) = 1, \psi(1,1) = q^{-1}$. In the same basis as in Example 2.1, we get $\Psi_{V\otimes V} = \operatorname{diag}(q^{-1})$. So

$$R_{V\otimes V} = \Theta_{V\otimes V}\Psi_{V\otimes V} = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q^{-1}-q & 1 & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}, \quad R_{V\otimes V}\circ\sigma = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & q^{-1}-q & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}.$$

Of course, it is easy to check directly that $R\circ\sigma: V\otimes V\to V\otimes V$ is U-linear. This is the case that we will mostly need. But for higher dimensional V_1, V_2 constructing an isomorphism by hand is very hard.

2.4. **Yang-Baxter equation.** We have constructed an isomorphism $R_{V_2\otimes V_1}\circ\sigma: V_1\otimes V_2\to V_2\otimes V_1$. Now pick three U-modules, V_1, V_2, V_3. We can produce two isomorphisms $V_1\otimes V_2\otimes V_3\to V_3\otimes V_2\otimes V_1$ by using the isomorphisms of the form $R_{?\otimes ?}\circ\sigma$ (note that applying this to V_1 and V_3 and inserting id_{V_2} in the middle does not give a U-linear map). Let us write $\tau_{?,?}$ for $R_{?\otimes ?}\circ\sigma$.

$$\begin{tikzcd}
 & V_2\otimes V_1\otimes V_3 \arrow[r, "\mathrm{id}_{V_2}\otimes\tau_{V_1,V_3}"] & V_2\otimes V_3\otimes V_1 \arrow[dr, "\tau_{V_2,V_3}\otimes\mathrm{id}_{V_1}"] & \\
V_1\otimes V_2\otimes V_3 \arrow[ur, "\tau_{V_1,V_2}\otimes\mathrm{id}_{V_3}"] \arrow[dr, "\mathrm{id}_{V_1}\otimes\tau_{V_2,V_3}"'] & & & V_3\otimes V_2\otimes V_1 \\
 & V_1\otimes V_3\otimes V_2 \arrow[r, "\tau_{v_1,V_3}\otimes\mathrm{id}_{V_2}"] & V_3\otimes V_1\otimes V_2 \arrow[ur, "\mathrm{id}_{V_3}\otimes\tau_{V_1,V_2}"'] &
\end{tikzcd}$$

We want this diagram to commute (the hexagon axiom). Note that the top isomorphism is $(R_{23}R_{13}R_{12})_{V_3\otimes V_2\otimes V_1}\circ\sigma_{13}$, while the bottom isomorphism is $(R_{12}R_{13}R_{23})_{V_3\otimes V_2\otimes V_1}\circ\sigma_{13}$. Here the notation is as follows. We write R_{12} for $R\otimes 1\in U^{\tilde{\otimes}3}$ (where we put $\tilde{\otimes}$ to indicate that we take a completion) and R^{23} for $1\otimes R$. The notation R_{13} means $\sum_i R_i^1\otimes 1\otimes R_i^2$, where $R=\sum_i R_i^1\otimes R_i^2$. We write σ_{13} for the permutation $u\otimes v\otimes w\mapsto w\otimes v\otimes u$. So the hexagon diagram is commutative provided

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{2.2}$$

This is the quantum Yang-Baxter equation (shortly, QYBE) that first appeared in Statistical Physics and was an initial motivation for introducing quantum groups.

Theorem 2.3. *QYBE holds for $R\in U\tilde{\otimes}U$.*

We omit the proof.

2.5. **Braid group representation.** There is an alternative way to view QYBE when we are dealing with the n-fold tensor product of a single U-module V. Define the U-module automorphism $\tau_{i,i+1}$ of $V^{\otimes n}$ as $\mathrm{id}_V^{\otimes i-1}\otimes\tau_{V,V}\otimes\mathrm{id}^{n-1-i}$ (we permute the ith and $i+1$th copies). The hexagon axiom gives $\tau_{i,i+1}\tau_{i+1,i+2}\tau_{i,i+1}=\tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2}$ for all $i\in\{1,\dots,n-1\}$. Clearly, $\tau_{i,i+1}\tau_{j,j+1}=\tau_{j,j+1}\tau_{i,i+1}$ if $|i-j|>1$.

Definition 2.4. The braid group B_n is the group with generators $T_1,\dots,T_{n-1}$ and relations $T_iT_j=T_jT_i$ when $|i-j|>1$ and $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$.

So we have a representation of B_n in $V^{\otimes n}$ given by $T_i\mapsto\tau_{i,i+1}$. When $V=L(q)$, one can verify this directly without referring to Theorem 2.3. Now note that $\tau=\tau_{1,2}\in\mathrm{End}(V\otimes V)$ satisfies $\tau^2=1+(q^{-1}-q)\tau$. In other words, the action of $\mathbb{C}B_n$ on $V^{\otimes n}$ factors through the Hecke algebra $\mathcal{H}_{q^{-1}}(n)$.

Remark 2.5. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra so that we can form the quantum group $U_q(\mathfrak{g})$. We still have the universal R-matrix $R\in U_q(\mathfrak{g})\tilde{\otimes}U_q(\mathfrak{g})$ satisfying QYBE and such that $R_{V_2\otimes V_1}\circ\sigma: V_1\otimes V_2\to V_2\otimes V_1$ is an isomorphism of $U_q(\mathfrak{g})$-modules. It was constructed by Drinfeld.