LECTURE 17: DEFORMED PREPROJECTIVE ALGEBRAS

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1. Introduction/recap

In the previous lecture we have stated the Kac theorem and introduced the deformed preprojective algebras. In this lecture we will prove a weaker version of the theorem by studying the representation theory of those algebras.

Theorem 1.1. *Let Q be a quiver and v be a dimension vector. Then the following is true.*

(1) *If there is an indecomposable representation of dimension v, then v is a root.*
(2) *If v is a real root, then there is a unique (up to an isomorphism) indecomposable representation of dimension v.*
(3) *If v is primitive (meaning that $\mathrm{GCD}(v_i) = 1$) and there is an indecomposable representation of dimension v, then p_v, the number of parameters for the isomorphism classes of indecomposable representations, equals*

$$1 - (v, v)/2(= 1 - \dim G_v + \dim \mathrm{Rep}(Q, v)).$$

Remark 1.2. Suppose that there is i such that $v_j = 0$ for all $j \neq i$ and there is no loop at i. Then $\mathrm{Rep}(Q, v) = \{0\}$. The zero representation is indecomposable if and only if $v_i = 1$ (i.e., v corresponds to a simple root).

Also note that if there is an indecomposable representation of dimension v, then the support of v is connected.

Now recall that a deformed preprojective algebra is defined by

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/(\sum_{a\in Q_1}[a, a^*] - \sum_{i\in Q_0}\lambda_i\epsilon_i).$$

The set $\mathrm{Rep}(\Pi^\lambda(Q), v) \subset \mathrm{Rep}(\overline{Q}, v)$ coincides with $\mu^{-1}(\sum_i \lambda_i \,\mathrm{id}_{V_i})$, where $\mu : \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}_v$ is the moment map, $\mu_i(x_a, x_{a^*}) = \sum_{a,h(a)=i} x_a x_{a^*} - \sum_{a,t(a)=i} x_{a^*} x_a$. Being a moment map means that μ is G_v-equivariant and

$$\langle d_x\mu(v), \xi\rangle = \omega(\xi x, v). \tag{1.1}$$

Note that $\sum_i \mathrm{tr}\,\mu_i(x_a, x_{a^*}) = 0$. By $\bar{\mathfrak{g}}_v$, we denote the subalgebra of $\mathfrak{g}_v$ consisting of all elements (y_i) with $\sum_i \mathrm{tr}(y_i) = 0$. So $\mu : \mathrm{Rep}(\bar{Q}, v) \to \bar{\mathfrak{g}}_v$.

2. Connection to indecomposable representations of Q

Let $\pi : \mathrm{Rep}(\Pi^\lambda(Q), v) \to \mathrm{Rep}(Q, v)$ denote the projection, it sends $(x_a, x_{a^*})_{a\in Q_1}$ to $(x_a)_{a\in Q_1}$. Our goal is to describe the pre-image of (x_a).

2.1. **Exact sequence.** A key tool for this is the following lemma. We define a map $c : \mathrm{Rep}(Q^{op}, v) \to \mathfrak{g}_v$ by $c(x_{a^*}) := \mu(x_a, x_{a^*})$ and a map $t : \mathfrak{g}_v \to \mathrm{End}(x_a)^*$ by $\langle t(y_i), (z_i)\rangle = \sum_i \mathrm{tr}(y_i z_i)$. Recall that $\mathrm{End}(x_a)$ denote the endomorphism algebra of the representation x_a, it consists of all Q_0-tuples (z_i) with $z_{h(a)}x_a = x_a z_{t(a)}$.

Lemma 2.1. *The sequence*

$$\mathrm{Rep}(Q^{op}, v) \xrightarrow{c} \mathfrak{g}_v \xrightarrow{t} \mathrm{End}(x_a)^* \to 0$$

is exact.

Proof. The map t is the composition of the identification $\mathfrak{g}_v \cong \mathfrak{g}_v^*$ and the projection $\mathfrak{g}_v^* \twoheadrightarrow \mathrm{End}(x_a)^*$, so t is surjective.

Let us prove that $t \circ c = 0$. This is equivalent to $\sum_{i\in Q_0} \mathrm{tr}(\mu_i(x_a, x_{a^*}), z_i) = 0$. But

$$\sum_{i\in Q_0} \mathrm{tr}(\mu_i(x_a, x_{a^*}), z_i) = \sum_{a\in Q_1} (\mathrm{tr}(x_a x_{a^*} z_{h(a)}) - \mathrm{tr}(x_{a^*} x_a z_{t(a)})) =$$
$$\sum_{a\in Q_1} (\mathrm{tr}(x_{a^*}(z_{h(a)}x_a - x_a z_{t(a)}))) = 0.$$

In order to check that $\ker t = \mathrm{im}\, c$, we will compare the dimensions. We have

$$\ker c = \{(x_{a^*}) | d_{(x_a),0}\mu((0, x_{a^*})) = 0\} = [(1.1)] = \{(x_{a^*}) | \langle (x_{a^*}), \mathfrak{g}_v.(x_a)\rangle = 0\}.$$

The dimension of $\mathfrak{g}_v.(x_a)$ is $\dim \mathfrak{g} - \dim \mathrm{End}(x_a)$ and so $\dim \ker c = \dim \mathrm{Rep}(Q^{op}, v) - \dim \mathfrak{g}_v + \dim \mathrm{End}(x_a)$. We conclude that $\dim \mathrm{im}\, c = \dim \mathfrak{g}_v - \dim \mathrm{End}(x_a) = \dim \ker t$. The second equality holds because t is surjective. $\square$

2.2. **Consequences.** Now let us deduce some corollaries on $\pi^{-1}(x_a)$.

Corollary 2.2. *The following is true.*

1. *We have $\pi^{-1}(x_a) \neq \varnothing$ if and only if $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = 0$ for any $(z_i) \in \mathrm{End}(x_a)$.*
2. *If $\pi^{-1}(x_a)$ is non-empty, then it is an affine space of dimension $\dim \mathrm{Rep}(Q, v) - \dim G_v.(x_a)$.*
3. *Suppose that v is generic with $\lambda \cdot v = 0$ meaning that the equality $\lambda \cdot v' = 0$ with $v' \leqslant v$ (component-wise) implies $v = kv'$ for some $k \in \mathbb{Q}$ (here we write $\lambda \cdot v = \sum_{i\in Q_0} \lambda_i v_i$). Then $\pi^{-1}(x_a) \neq \varnothing$ if and only if the dimensions of all direct summands of (x_a) are proportional to v.*
4. *In addition, suppose v is primitive. Then $\pi^{-1}(x_a) \neq \varnothing$ if and only if (x_a) is indecomposable. Moreover, all representations of $\Pi^\lambda(Q)$ of dimension v are irreducible.*

Proof. (1) is a direct corollary of Lemma 2.1. (2) follows from the proof because $\dim \pi^{-1}(x_a) = \dim \ker c = \dim \mathrm{Rep}(Q^{op}, v) - \dim \mathfrak{g}_v + \dim \mathrm{End}(x_a) = \dim \mathrm{Rep}(Q, v) - \dim G_v.(x_a)$.

Let us prove (3). Let (x'_a) be a direct summand of (x_a) of dimension v'. Let $(z_i) \in \bigoplus_i \mathrm{End}(V_i)$ denote the corresponding projection. Then it is an element of $\mathrm{End}(x_a)$. So $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = \lambda \cdot v' = 0$. Since λ is generic, we see that v' is proportional to v.

Conversely, let $(x_a) = \bigoplus_j (x_a^j)$ be the decomposition into indecomposables. Assume that the dimensions v^i are proportional to v. Let us write an endomorphism (z_i) of (x_a) as a matrix (z^{jk}), with $z^{jk} \in \mathrm{Hom}_Q((x_a^j), (x_a^k))$. Note that since (x_a^j) is indecomposable, the endomorphism z^{jj} acts on the corresponding representation space $V^j = \bigoplus_i V_i^j$ with a single eigenvalue. It follows that the vector $(\mathrm{tr}(z_i^{jj}))_{i\in Q_0}$ is proportional to v^j. We see that $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i^{jj}) = 0$ and so $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = 0$. (3) is fully proved.

Now let us prove (4). The first claim is a direct corollary of (3). To prove the second statement, let (x'_a, x'_{a^*}) be a nonzero sub of $(x_a, x_{a^*}) \in \operatorname{Rep}(\Pi^\lambda(Q), v)$. Then $\pi^{-1}(x'_a) \neq 0$ and hence, by (4), we need to have $\lambda \cdot v' = 0$. Since v is primitive, this is only possible if $v = v'$. □

2.3. **Application to Kac's theorem.** Let us compute d_v under some additional assumptions.

Corollary 2.3. *Assume that v is primitive and λ is generic with $\lambda \cdot v = 0$. If there is an indecomposable representation in* $\operatorname{Rep}(Q, v)$ *or* $\operatorname{Rep}(\Pi^\lambda(Q), v) \neq \varnothing$, *then* $p_v = 1 - (v, v)/2$.

Proof. Let $\operatorname{Rep}^{ind}(Q, v) \subset \operatorname{Rep}(Q, v)$ denote the subset of the indecomposable representations. Then $\operatorname{Rep}(\Pi^\lambda(Q), v)$ admits a morphism π with image $\operatorname{Rep}^{ind}(Q, v)$ whose fiber over (x_a) is an affine space of dimension $\dim \operatorname{Rep}(Q, v) - \dim G_v.(x_a)$. By (4) of the previous corollary all representations in $\operatorname{Rep}(\Pi^\lambda(Q), v)$ are irreducible. By the Schur lemma, all their endomorphisms are constant.

Let $\bar{G}_v$ denote the quotient of G_v by the one-dimensional subgroup of constant matrices. The kernel of $G_v \twoheadrightarrow \bar{G}_v$ acts on $\operatorname{Rep}(Q, v)$ trivially so we get an action of $\bar{G}_v$ on $\operatorname{Rep}(Q, v)$. The Lie algebra of $\bar{G}_v$ is naturally identified with $\bar{\mathfrak{g}}_v$ and $\mu : \operatorname{Rep}(\bar{Q}, v) \to \bar{\mathfrak{g}}_v$ is the moment map. Also note that the action of $\bar{G}_v$ on $\operatorname{Rep}(\Pi^\lambda(Q), v) = \mu^{-1}(\lambda)$ is free. From here and (1.1) one deduces that μ is a submersion at all points of $\operatorname{Rep}(\Pi^\lambda(Q), v)$ and hence $\dim \operatorname{Rep}(\Pi^\lambda(Q), v) = \dim \operatorname{Rep}(\overline{Q}, v) - \dim \mathfrak{g}_v$.

Let us cover $\operatorname{Rep}^{ind}(Q, v)$ with locally closed G_v-stable subvarieties with constant dimensions of orbits, $\operatorname{Rep}^{ind}(Q, v) = \bigsqcup_i X_i$, let d_i denote the dimension of a G_v-orbit in X_i. Let Y_i denote the preimage of X_i in $\mu^{-1}(\lambda)$, it is an affine bundle with rank $\dim \operatorname{Rep}(Q, v) - d_i$ over X_i. So we see that $2 \dim \operatorname{Rep}(Q, v) - \dim \overline{G}_v = \max_i(\dim Y_i) = \max_i(\dim X_i + \dim \operatorname{Rep}(Q, v) - d_i) = \dim \operatorname{Rep}(Q, v) + \max_i(p(X_i))$. It follows that $p_v = \max_i(p(X_i)) = \dim \operatorname{Rep}(Q, v) - \dim G_v + 1 = 1 - (v, v)/2$. □

3. Reflection functors

We will view $\lambda = (\lambda_j)_{j \in Q_0}$ as an element of $\mathfrak{h}$ and a dimension vector v as an element of $\mathfrak{h}^*$ (the pairing is by $\langle \lambda, v \rangle = \lambda \cdot v$). Recall that $W(Q)$ acts on $\mathfrak{h}^*$ as follows: $(s_i v)_j = v_j$ for $j \neq i$ and $(s_i v)_i = \sum_j n_{ij} v_j - v_i$, where n_{ij} is the number of edges between i and j. So $W(Q)$ acts on $\mathfrak{h}$ as follows: $(s_i \lambda)_i = -\lambda_i, (s_i \lambda)_j = \lambda_j + n_{ij} \lambda_i$.

The main result of this section is as follows.

Theorem 3.1. *Pick $i \in Q_0$ such that there are no loops at i. Suppose $\lambda_i \neq 0$. Then is an equivalence* $\Pi^\lambda(Q)\text{-mod} \xrightarrow{\sim} \Pi^{s_i\lambda}(Q)\text{-mod}$ *that maps a representation of dimension v to a representation of dimension $s_i v$.*

Before proving this theorem we will explain how it applies to the Kac theorem.

3.1. **Application to Kac's theorem.**

Corollary 3.2. *Suppose there is an indecomposable representation of dimension vector v. Then v is a root.*

Proof. We can assume that for all $v' \leqslant v, v' \neq v$ (componentwise), the claim is true. We can also assume $(v, v) > 0$, otherwise we are done by Remark 1.2. If $(v, \epsilon_i) \leqslant 0$ for all i, then $(v, v) = \sum_i v_i (v, \epsilon_i) \leqslant 0$. Note that if there is a loop at i, then $(v, \epsilon_i) \leqslant 0$. So it's enough to consider the case when there is i such that there is no loop at i and $(v, \epsilon_i) > 0$ so that $s_i v = v - (v, \epsilon_i)\epsilon_i < v$.

Let us prove that if $\mathrm{Rep}(\Pi^\lambda(Q), v)$ for a Zariski generic λ with $\lambda \cdot v = 0$ contains an indecomposable representation, then v is a real root. We prove it by induction. By Theorem 3.1, $\mathrm{Rep}(\Pi^{s_i\lambda}(Q), s_i v)$ contains an indecomposable representation. This provides an inductive step. The base is given by $v = m\epsilon_i$: there the representation is zero and so $m = 1$.

By (3) of Corollary 2.2, if $\mathrm{Rep}(Q, v)$ contains an indecomposable representation, then so does $\mathrm{Rep}(\Pi^\lambda(Q), v)$. This completes the proof. □

Corollary 3.3. *Let v be a real root. Then there is a unique (up to isomorphism) indecomposable representation of Q with dimension vector v.*

Proof. Let λ be generic with $\lambda \cdot v = 0$. Let us check that there is a unique (up to an isomorphism) representation of $\Pi^\lambda(Q)$ of dimension vector v. If $v = \epsilon_i$, then there is only the zero representation and so we are done. Theorem 3.1 gives the induction step.

Note that, as any real root, v is indecomposable. Now the claim of this corollary follows from (4) of Corollary 2.2. □

3.2. **Construction of equivalence.** Now let us construct the required equivalence. Pick a representation (x_a, x_{a^*}) with dimension vector v. Recall that $\Pi^\lambda(Q)$ does not depend on the orientation of Q up to an isomorphism. So we may assume that i is a sink in Q. Let $W_i := \bigoplus_{a, t(a)=i} V_{h(a)}$. We can write (x_a, x_{a^*}) as $(A, B, \underline{x})$, where $A := \bigoplus_{a,t(a)=i} : V_i \to W_i, B := \bigoplus_{a,t(a)=i} x_{a^*} : W_i \to V_i$ and $\underline{x}$ includes all x_b, x_{b^*} with $t(b) \neq i$. Multiplying the relation of $\Pi^\lambda(Q)$ by ϵ_i, we see that $BA = -\lambda_i \,\mathrm{id}_{V_i}$. Since $\lambda_i \neq 0$, we see that A is injective, B is surjective. Also, we see that $W_i = \operatorname{im} A \oplus \ker B$. Identifying V_i with $\operatorname{im} A$, we can assume that A is the inclusion $V_i \hookrightarrow W$, and $B = -\lambda_i \pi$, where π is the projection along $\ker B$.

Now let us proceed to defining a representation of $\Pi^{s_i\lambda}(Q)$ with dimension vector $s_i v$. The space $V' := \bigoplus V'_i$ is determined as follows: $V'_j := V_j$ if $j \neq i$, and $V'_i := \ker B$. In particular, $v' = s_i v$. The representation is given by $(A', B', \underline{x})$, where A' is the inclusion $V'_i \hookrightarrow W_i$ and B' is $\lambda_i \pi'$, where $\pi' : W_i \twoheadrightarrow V'_i$ is the projection along $\operatorname{im} A$. Note that we have

$$A'B' - AB = \lambda_i \,\mathrm{id}_{W_i}. \tag{3.1}$$

Now let us check that the resulting representation $(A', B', \underline{x})$ factors through $\Pi^{s_i\lambda}(Q)$. For $a \in Q_1$ with $t(a) = i$, let ρ_a, ι_a denote the projection $W_i = \bigoplus_{a,t(a)=i} V_{h(a)} \twoheadrightarrow V_{h(a)}$ and the inclusion $V_{h(a)} \hookrightarrow W_i$ corresponding to this arrow. So we have $x_a = \rho_a \circ A, x_{a^*} = B \circ \iota_a, x'_a = \rho_a \circ A', x'_{a^*} = B' \circ \iota_a$. We have $-\sum_{t(a)=i} x'_{a^*} x'_a = -B'A' = -\lambda_i \,\mathrm{id}_{V'_i}$. So what we need to check is that for $j \neq i$, we have

$$\sum_{a,h(a)=j} x'_a x'_{a^*} - \sum_{a,t(a)=j} x'_{a^*} x'_a = (s_i\lambda)_j \,\mathrm{id}_{V_j} = (\lambda_j + n_{ij}\lambda_i)\,\mathrm{id}_{V_i}.$$

This will follow if we check that

$$\sum_{a,h(a)=j} (x'_a x'_{a^*} - x_a x_{a^*}) - \sum_{a,t(a)=j} (x'_{a^*} x'_a - x_{a^*} x_a) = n_{ij}\lambda_i \,\mathrm{id}_{V_j}.$$

If $t(a) \neq i$, then $x_a = x'_a, x_{a^*} = x'_{a^*}$. So the left hand side is

$$\sum_{t(a)=i,h(a)=j} (x'_a x'_{a^*} - x_a x_{a^*}) = \sum_{t(a)=i,h(a)=j} \rho_a \circ (A'B' - AB) \circ \iota_a =$$
$$= [(3.1)] = \sum_{t(a)=i,h(a)=j} \rho_a \circ (\lambda_i \operatorname{id}_{W_i}) \circ \iota_a = n_{ij} \lambda_i \operatorname{id}_{V_j},$$

as required. So we indeed get a a representation of $\Pi^{s_i \lambda}(Q)$.

Our construction is functorial. Indeed, let $(y_i) : (V_i, x_a, x_{a^*}) \to (\bar{V}_i, \bar{x}_a, \bar{x}_{a^*})$ be a homomorphism of representations. This induces a homomorphism $y : W_i \to \bar{W}_i$ that intertwines A, B with $\bar{A}, \bar{B}$. In particular, y restricts to $\ker B \to \ker \overline{B}$. So it induces a homomorphism $(y'_i) : (V'_i, x'_a, x'_{a^*}) \to (\bar{V}'_i, \bar{x}'_a, \bar{x}'_{a^*})$. We indeed get a functor $\Pi^\lambda(Q)$-mod $\to \Pi^{s_i\lambda}(Q)$-mod that behaves as s_i on the dimension vectors.

We also have a similarly defined functor $\psi : \Pi^{s_i\lambda}(Q)$-mod $\to \Pi^\lambda(Q)$-mod. It sends a representation $(A', B', \underline{x})$ back to $(A, B, \underline{x})$. It is easy to see that $\psi \circ \varphi$ is isomorphic to the identity functor of $\Pi^\lambda(Q)$-mod. Similarly, $\varphi \circ \psi$ is isomorphic to the identity functor. This shows that φ is an equivalence (with quasi-inverse ψ).

4. Further results and applications

4.1. Irreducible representations. A basic question about the representation theory of $\Pi^\lambda(Q)$ is to describe its irreducible representations. Let us state the corresponding result of Crawley-Boevey, we are not going to provide a proof.

For $v \in \mathbb{Z}_{\geqslant 0}^{Q_0}$, set $p(v) = 1 - \frac{1}{2}(v, v)$. Define the set Σ_λ of all positive roots such that $\lambda \cdot v = 0$ $p(v) > \sum_{i=1}^k p(v_i)$ for all proper decompositions of v into the sum $\sum_{i=1}^k v_i$, where all $v_i \in (\mathbb{Z}_{\geqslant 0})^{Q_0} \setminus \{0\}$ such that $\lambda \cdot v_i = 0$. It is not so easy to describe Σ_λ, but this is a combinatorial object.

Theorem 4.1. *The algebra* $\Pi^\lambda(Q)$ *has an irreducible representation of dimension* v *if and only if* $v \in \Sigma_\lambda$. *Moreover,* $\operatorname{Rep}(\Pi^\lambda(Q), v) \subset \operatorname{Rep}(\mathbb{C}\overline{Q}, v)$ *is an irreducible subvariety of dimension* $\dim \operatorname{Rep}(Q, v) + p(v)$ *and a Zariski generic point in* $\operatorname{Rep}(\Pi^\lambda(Q), v)$ *gives an irreducible representation.*

4.2. Application to additive Deligne-Simpson problem. The additive Deligne-Simpson problem (we'll abbreviate this as DS problem) asks about the conditions on the conjugacy classes $C_1, \ldots, C_k$ in $\operatorname{Mat}_n(\mathbb{C})$ such that there are matrices $Y_i \in \operatorname{Mat}_n(\mathbb{C})$ satisfying the following two conditions:

(1) $\sum_{i=1}^k Y_i = 0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

From $C_1, \ldots, C_k$, Crawley-Boevey have constructed a quiver Q, a dimension vector v, and $\lambda \in \mathbb{C}^{Q_0}$ such that there is a bijection between

(a) solutions $(Y_1, \ldots, Y_k)$ of the DS problem (up to $\operatorname{GL}_n(\mathbb{C})$-conjugacy),
(b) irreducible dimension v representations of $\Pi^\lambda(Q)$ (up to an isomorphism).

Then the solution of the DS problem follows from Theorem 4.1 (one needs to use some complicated combinatorics to get the answer explicitly).