Lecture $18\frac{1}{2}$

Deformations of Kleinian singularities

1) Kleinian groups & McKay correspondence
2) Kleinian singularities & their deformations
3) Construction via DPA.

1) Kleinian gr-p = finite subgroup in $SL_2(\mathbb{C})$. ~~Those~~ These subgroups up to $SL_2(\mathbb{C})$-conjugacy are parameterized by affine Dynkin diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_\ell$, $\ell = 6,7,8$

Ex 1: $\Gamma \simeq \mathbb{Z}/_{n+1} = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \epsilon^{n+1} = 1 \right\} \rightsquigarrow \tilde{A}_n$: 

$\Gamma = [\text{dihedral gr-p}] = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} \middle| \epsilon^{2n} = 1 \right\} \rightsquigarrow$  $(\tilde{D}_{n+2}), n \geq 2$

Exceptional groups $\rightsquigarrow \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

Construction of diagram from the group (McKay): $V = \mathbb{C}^2$

$\Gamma \subset SL_2(\mathbb{C}) \rightsquigarrow N_0, \dots N_r$ - the irreps of Γ, $N_0 = triv$

Quiver $\vec{Q}$: $Q_0 = \{0, 1, \dots r\}$

$\#\{a : i \to j\} = \dim \operatorname{Hom}_\Gamma(V \otimes N_i, N_j)$ $(= \dim \operatorname{Hom}_\Gamma(N_j \otimes V, N_i)$ b/c V is self-dual)

$\rightsquigarrow \bar{Q}$ = double of Q b/c have no loops (check). if $\Gamma \neq \{1\}$

Total # edges between i, j in Q is $\dim \operatorname{Hom}_\Gamma(N_i \otimes V, N_j)$

Ex: $\Gamma \simeq \mathbb{Z}/_{n+1} \rightsquigarrow$ irred char-s $\chi_k : z \mapsto \exp(\frac{2\pi\sqrt{-1}}{n+1} z)$, have edges from k to $k \pm 1$ b/c $V = \chi_1 \oplus \chi_{-1}$

Thm: Q is an affine Dynkin quiver. Moreover, 0 is the extending vertex (one added to the Dynkin quiver). Finally, $(\dim N_i)_{i=0}^r$ is the indecomposable imaginary root for Q (to be denoted by δ)

Rem: The last claim can be proved conceptually: note that $V \otimes \mathbb{C}\Gamma \simeq \mathbb{C}\Gamma \oplus \mathbb{C}\Gamma$

2) Kleinian singularity: $\mathbb{C}^2/\Gamma$ - affine alg-c variety w. algebra of functions $\mathbb{C}[x,y]^\Gamma$. This algebra has 3 generators, say a, b, c, and one relation, $F(a,b,c)$

Ex: $\Gamma = \mathbb{Z}_{n+1}$: $a = x^{n+1}$, $b = y^{n+1}$, $c = xy$. $F = ab - c^{n+1}$

Others: $\tilde{D}_r$: $a^{r-1} + ab^2 + c^2 = 0$

$\tilde{E}_6$: $a^4 + b^3 + c^2 = 0$

$\tilde{E}_7$: $a^3 b + b^3 + c^2 = 0$

$\tilde{E}_8$: $a^5 + b^3 + c^2 = 0$

$A = \mathbb{C}[x,y]^\Gamma$ is graded (as a subalgebra of $\mathbb{C}[x,y]$). Our goal is to produce filtered deformations, i.e filtered associative algebras $\mathcal{A}$ s.t. $\mathrm{gr}\,\mathcal{A} = A$ (isomorphism of graded algebras)

Now $\Pi^\lambda(Q)$ has a filtration induced from $\mathbb{C}\bar{Q}$ (by length of the path) and $\Pi^0(Q)$ is actually graded (the relation is homogeneous of degree 2). Consider the subspace $\epsilon_0 \Pi^\lambda(Q)\epsilon_0 \subset \Pi^\lambda(Q)$. It's closed under the product and ϵ_0 is the unit. So $\epsilon_0 \Pi^\lambda(Q)\epsilon_0$ is a filtered associative algebra w. 1 (the filtration is restricted from $\Pi^\lambda(Q)$). We note that $\Pi^0(Q) \twoheadrightarrow \mathrm{gr}\,\Pi^\lambda(Q)$ (the relations for $\Pi^0(Q)$ ~~are~~ is top degree component of that for $\Pi^\lambda(Q)$)

Thm (Crawley-Boevey & Holland)

- $\mathrm{gr}\,\Pi^\lambda(Q) \simeq \Pi^0(Q)$ ($\Rightarrow \mathrm{gr}\,\epsilon_0 \Pi^\lambda(Q)\epsilon_0 = \epsilon_0 \Pi^0(Q)\epsilon_0$)
- $\epsilon_0 \Pi^0(Q)\epsilon_0 = \mathbb{C}[x,y]^\Gamma$

So the algebras $\epsilon_0 \Pi^\lambda(Q)\epsilon_0$ are filtered deformations of $\mathbb{C}[x,y]^\Gamma$. In fact, all filtered deformations can be obtained in this way.

3) We need an alternative description of $\Pi^\lambda(Q)$. 1st step is as follows.

3.1) Semi-direct tensor products.

Let A be an associative unital algebra with an action of a finite group Γ (e.g $A = \mathbb{C}[x,y]$, $\Gamma \subset SL_2(\mathbb{C})$). We define an algebra $A \# \mathbb{C}\Gamma$ as follows: it is $A \otimes \mathbb{C}\Gamma$ as a vector space and the product of monomials is defined as follows:

$a_1 \otimes \gamma_1 \cdot a_2 \otimes \gamma_2 = a_1 \gamma_1(a_2) \otimes \gamma_1\gamma_2$ (where $\gamma_1(a_2)$ stands for the image of a_2 under the action of γ_1)

We get an associative algebra with unit $1\otimes 1$.

Let us explain a connection between $A\otimes\mathbb{C}\Gamma$ and the invariant subalgebra A^Γ. Consider the trivial idempotent $e=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma\in\mathbb{C}\Gamma\subset A\otimes\mathbb{C}\Gamma$. Consider the subalgebra $e(A\otimes\mathbb{C}\Gamma)e$ (with unit e)

Lem: The map $a\mapsto ea(=ae)$, $A^\Gamma\to e(A\otimes\mathbb{C}\Gamma)e$ is an algebra isomorphism. Proof is an exercise

Main examples of A: $A=\mathbb{C}[x,y]$, $A'=\mathbb{C}\langle x,y\rangle$ - free algebra in 2 generators ($\Gamma\subset SL_2(\mathbb{C})$)

3.2) Deformations of $A\otimes\mathbb{C}\Gamma$ & connection to DPA

$c:\Gamma\to\mathbb{C}$ - function constant on conjugacy classes

$\rightsquigarrow C=\sum_{\gamma\in\Gamma}c(\gamma)\gamma$ - central element

$\rightsquigarrow H_c=A'\otimes\mathbb{C}\Gamma/([x,y]=C)$ $(H_0=A\otimes\mathbb{C}\Gamma)$

Thm 1 (CB & H) $\operatorname{gr}H_c=H_0$

The algebra H_c is related to $\Pi^\lambda(Q)$ as follows. Recall that $N_0\ldots N_r$ denote the irreducible representations of Γ. Pick a primitive idempotent $e_i\in\operatorname{End}(N_i)\subset\mathbb{C}\Gamma\subset H_c$. Set $\tilde e=\sum_{i=0}^r e_i$, $\lambda_i=\operatorname{tr}_{N_i}C$

Thm 2 (CB & H) $\Pi^\lambda(Q)\simeq\tilde e H_c\tilde e$ (w. $e_i\leftrightarrow\epsilon_i$)

Cor: $\epsilon_0\Pi^\lambda(Q)\epsilon_0=eH_ce$ $(e=e_0)$

The proof is in 2 steps. First, we'll check that $\tilde e(A'\otimes\mathbb{C}\Gamma)\tilde e\simeq\mathbb{C}\bar Q$ (easy and pretty formal part). A more difficult part is that $\tilde e(xy-yx-C)_{A'\otimes\mathbb{C}\Gamma}\,\bar e=\left(\sum_a[a,a^*]-\sum_i\lambda_i\epsilon_i\right)_{\mathbb{C}\bar Q}$

This will prove Thm 2

3.3) Tensor algebras

We are going to show that $\tilde e(A'\otimes\mathbb{C}\Gamma)\tilde e\simeq\mathbb{C}\bar Q$. The main point is that both $A'\otimes\mathbb{C}\Gamma$, $\mathbb{C}\bar Q$ are tensor algebras (of bimodules over finite dimensional algebras.

The general construction is as follows. Let A_0 be an algebra and A_1 is its bimodule. We can form the tensor power $A_1^{\otimes k} = A_1 \otimes_{A_0} A_1 \otimes_{A_0} \dots \otimes_{A_0} A_1$

Example: 1) $A_0 = \mathbb{C}Q_0$, $A_1 = \mathbb{C}\bar{Q}_1 \rightsquigarrow T_{A_0}(A) = \mathbb{C}\bar{Q}$.

2) $A_0 = \mathbb{C}\Gamma$, $A_1 = \mathbb{C}^2 \otimes \mathbb{C}\Gamma$, $\gamma_1(v \otimes a)\gamma_2 = \gamma_1 v \otimes \gamma_1 a \gamma_2$

$\rightsquigarrow T_{A_0}(A_1) = \mathbb{C}\langle x,y\rangle \otimes \mathbb{C}\Gamma$

Now assume that A_0 is s/simple, $A_0 = \bigoplus_i \mathrm{End}_{\mathbb{C}}(N_i)$, $e_i \in \mathrm{End}_{\mathbb{C}}(N_i)$ primitive idempotent, $\tilde{e} = \sum_i e_i$.

Lem: $\tilde{e}\, T_{A_0}(A_1)\tilde{e} = T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e})$

Proof: The algebra $T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e})$ has a univ. property: let A be an algebra with an embedding $\tilde{e}A_0\tilde{e} \hookrightarrow A$ and $\tilde{e}A_0\tilde{e}$-bimodule map $\tilde{e}A_1\tilde{e} \to A$. Then they extend to a unique algebra homomorphism $T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e}) \longrightarrow \tilde{e}\,T_{A_0}(A_1)\tilde{e}$. Now $\tilde{e}$ gives a Morita equivalence A_0-bimod $\xrightarrow{\sim} \tilde{e}A_0\tilde{e}$-bimod, $B \to \tilde{e}B\tilde{e}$. This shows that the homomorphism is an iso $\square$

Cor: $\tilde{e}(\mathbb{C}\langle x,y\rangle \# \Gamma)\tilde{e} \xrightarrow{\sim} \mathbb{C}\bar{Q}$

Proof: $\tilde{e}\mathbb{C}\Gamma\tilde{e} \simeq \mathbb{C}Q_0$ $(e_i \leftrightarrow \epsilon_i)$. Now we need to check that $\tilde{e}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)\tilde{e} = \mathbb{C}\bar{Q}_1$. But $e_i(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)e_j = \bigoplus_k e_i(\mathbb{C}^2 \otimes N_k) \otimes N_k^* e_j$ $= e_i(\mathbb{C}^2 \otimes N_j) = \mathrm{Hom}(N_i, \mathbb{C}^2 \otimes N_j) = \epsilon_i \mathbb{C}\bar{Q}_1 \epsilon_j$, this implies $\tilde{e}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)\tilde{e}$ $= \mathbb{C}\bar{Q}_1$ $\square$

3.4) Equality $\tilde{e}(xy - yx - c)_{A' \otimes \mathbb{C}\Gamma}\tilde{e} = \left(\sum_{a \in Q_1}[a, a^*] - \sum_{i \in Q_0} \lambda_i \epsilon_i\right)_{\mathbb{C}\bar{Q}}$

First note that $\tilde{e}$ commutes w. $xy - yx - c$ b/c $xy - yx - c$ is Γ-invariant. So l.h.s. is gen-d by $(xy - yx - c)e_i$.

Prop: $(xy - yx - c)e_i = \sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a - \lambda_i \epsilon_i$ after a suitable choice of generators a, a^* (under the isomorphism of 3.3)

This is quite technical. (see Lemma 4.2 in Lec 5 of the SRA class)

Let's do an example instead of proof.

Ex: $\Gamma = \mathbb{Z}/(n+1)\mathbb{Z}$ in which case $\tilde{e} = 1$ so we have $A' \otimes \mathbb{C}\Gamma =$

$\mathbb{C}\bar{Q}$. Then we have $C = \sum_i \lambda_i \epsilon_i$. ~~We take the counter clockwise arrows for~~

~~[illegible]~~ Let $a_i : i \to i+1$ and $a_i^* : i+1 \to i$. We set $a_i = x\epsilon_i$, $a_i^* = \epsilon_i y$.

The DPA relation becomes $[x, y] = C$.

35) Further results

A natural questions ~~[illegible]~~ is when $eH_c e$ is commutative.

Thm: $eH_c e$ is commutative $\Longleftrightarrow \lambda \cdot \delta = 0$