LECTURE 19: KAC-MOODY ALGEBRA ACTIONS ON CATEGORIES, I

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1. Introduction

We have started this class by studying the representation theory of the symmetric group S_n over the complex numbers. We finish by giving a brief introduction to the representation theory of S_n over a field $\mathbb{F}$ of positive characteristic p. We will also establish a connection between the representations of $\hat{\mathfrak{sl}}_p$ and those of $\mathbb{F}S_n$. This connection was one of motivations to consider Kac-Moody algebra actions on categories. We would like to point out that while the representation theory of S_n in characteristic 0 is a classical and very well understood subject (all representations are completely reducible, the irreducible ones are classified by the Young diagrams, and character formulas are known in some way, at least), the representation theory in characteristic p is very complicated (representations are no longer completely reducible, and, although the classification of the irreducible representations is known, currently, there is not even a conjecture on their characters).

1.1. **Kac-Moody algebras and their representations.** Let us give a reminder regarding Kac-Moody algebras. We are interested only in symmetric Kac-Moody algebras. Those are associated to unoriented graphs I without loops. From I we can define its Cartan matrix $A = (a_{ij})_{i,j\in I}$ by $a_{ii} = 2$ and $a_{ij} = -n_{ij}$, where n_{ij} is the number of edges between i and j. Then we can define the Kac-Moody algebra $\mathfrak{g}(I)$ by generators $e_i, h_i, f_i, i \in I$, and the following relations:

- (R1) $[h_i, h_j] = 0, [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$.
- (R2) $[e_i, f_j] = \delta_{ij}h_j$.
- (R3) $\operatorname{ad}(e_i)^{1-a_{ij}}e_j = \operatorname{ad}(f_i)^{1-a_{ij}}f_j = 0$.

Example 1.1. First, let us recall that, if I is a Dynkin diagram of type A_ℓ, then $\mathfrak{g}(I) = \mathfrak{sl}_{\ell+1}$. We will need an infinite version of this, the algebra $\mathfrak{sl}_\infty$ that consists of infinite (in both directions) matrices with finitely many nonzero entries and trace 0. It corresponds to the graph I, where the vertices are the integers and we connect vertices whose difference is ± 1.

Now let I be a cycle with ℓ vertices. We can view vertices as elements of $\mathbb{Z}/\ell\mathbb{Z}$ connected if the difference is ± 1 (for $\ell = 2$ we have two edges). The corresponding Kac-Moody algebra is $\hat{\mathfrak{sl}}_\ell$. It can be defined as $\hat{\mathfrak{sl}}_\ell = \mathfrak{sl}_\ell \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c$ with commutation relations $[x\otimes t^p, y\otimes t^q] = [x, y]\otimes t^{p+q} + p\delta_{p+q,0}\operatorname{tr}(xy)c, [c, x\otimes t^p] = 0$. Note that $h_i = E_{ii} - E_{i+1,i+1}, i = 1, \ldots, \ell-1, h_0 = c + E_{\ell\ell} - E_{11}$. We have a Cartan subalgebra $\mathfrak{h} := \operatorname{Span}(h_i|i \in I) \subset \mathfrak{g}(I)$, the elements h_i form a basis in $\mathfrak{h}$.

Often one considers a slightly bigger algebra, $\tilde{\mathfrak{sl}}_\ell = \hat{\mathfrak{sl}}_\ell \oplus \mathbb{C}d$, where the additional commutation relations are $[d, c] = 0, [d, x \otimes t^k] = kx \otimes t^k$. In this case, one includes d as one more basis element in $\mathfrak{h}$.

To give a representation of $\mathfrak{g}(I)$ in a vector space V, we need to equip V with operators e_i, f_i, h_i satisfying the relations above. We only care about so called weight representations. We say that V is a weight representation if $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V|xv =$

$\langle\mu, x\rangle v, \forall x \in \mathfrak{h}\}$. The relations (R1) are equivalent to $e_i V_\mu \subset V_{\mu+\alpha_i}, f_i V_\mu \subset V_{\mu-\alpha_i}$, where $\alpha_i \in \mathfrak{h}^*$ are the simple roots defined by $[x, e_i] = \langle\alpha_i, x\rangle e_i, \forall x \in \mathfrak{h}$.

1.2. **Actions on categories.** Now let us discuss what one should mean by an action of $\mathfrak{g}(I)$ on a category. We are going to work with abelian categories $\mathcal{C}$ that are linear over some field $\mathbb{F}$ and where all objects have finite length. $\mathbb{F}S_n$-mod is an example of such a category. To $\mathcal{C}$ we can assign its complexified Grothendieck group $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$. For an exact functor $F : \mathcal{C} \to \mathcal{C}$, we get a linear map $[\mathcal{F}] : [\mathcal{C}] \to [\mathcal{C}], [M] \mapsto [\mathcal{F}M]$.

So here is a rough idea of what an action of $\mathfrak{g}(I)$ on $\mathcal{C}$ should mean. We want a collection of exact functors, $E_i, F_i : \mathcal{C} \to \mathcal{C}$ and a weight decomposition $\mathcal{C} = \bigoplus_{\mu\in\mathfrak{h}^*} \mathcal{C}_\mu$ such that $e_i := [E_i], f_i := [F_i]$ and $[\mathcal{C}] = \bigoplus_\mu [\mathcal{C}_\mu]$ gives a weight representation of $\mathfrak{g}(I)$. This occurs in many examples but is not powerful enough to produce an interesting theory. A crucial idea due to Chuang and Rouquier was to additionally include some functor morphisms that are also present in examples.

The category $\mathcal{C}$ we are interested in is $\mathcal{C} = \bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. We will see that if $\operatorname{char}\mathbb{F} = 0$, then $\mathcal{C}$ carries a categorical action of $\mathfrak{sl}_\infty$ (relatively boring case), while, for $\operatorname{char}\mathbb{F} = p$, $\mathcal{C}$ carries a categorical action of $\hat{\mathfrak{sl}}_p$ that makes $[\mathcal{C}]$ into the irreducible module $V(\omega_0)$ (and, in particular, computes the number of the irreducible $\mathbb{F}S_n$-modules for any n). The functors E_i come from restrictions (from S_n to S_{n-1}), while functors F_i come from inductions (from S_{n-1} to S_n).

We'll proceed as follows. First, we produce the functors E_i. Then we decompose $\mathcal{C}$ into the direct sum of subcategories (that later will be shown to be weight subcategories). Next, we will define the functors F_i. Finally, we will discuss functor morphisms we need. In the next lecture, we will start by showing that $[E_i], [F_i]$ define a weight Kac-Moody representation on $[\mathcal{C}]$.

2. Restriction and induction functors

Let $\mathbb{Z}_{\mathbb{F}}$ denote the ring of integers inside $\mathbb{F}$ (i.e., $\mathbb{Z}$ if $\operatorname{char}\mathbb{F} = 0$ and $\mathbb{Z}/p\mathbb{Z}$ if $\operatorname{char}\mathbb{F} = p$). We write $\mathcal{C}$ (or $\mathcal{C}_{\mathbb{F}}$ if we want to indicate the dependence on the base field $\mathbb{F}$) for $\bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. Here $S_0 = S_1 = \{1\}$.

2.1. **Functors E_i.** As in Lectures 1,2, to study the representations of $\mathbb{F}S_n$ we use "induction" based on the chain of inclusions $S_0 \subset S_1 \subset \ldots \subset S_{n-1} \subset S_n \subset \ldots$ (where, recall, S_{n-1} consists of all permutations in S_n that fix n). We can restrict an $\mathbb{F}S_n$-module to $\mathbb{F}S_{n-1}$ getting the *restriction functor* $\operatorname{Res}^n_{n-1} : \mathbb{F}S_n\text{-mod} \to \mathbb{F}S_{n-1}\text{-mod}$. Our goal now is to decompose $\operatorname{Res}^n_{n-1}(M)$ into a direct sum in a functorial (in M) way.

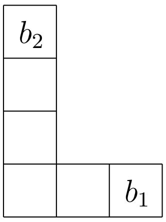
For this, recall the Jucys-Murphy element $L_n := \sum_{i=1}^{n-1}(in) \in \mathbb{F}S_n$. It commutes with S_{n-1} and hence the map $X_M : \operatorname{Res}^n_{n-1}(M) \to \operatorname{Res}^n_{n-1}(M)$ given by $X_M(m) = L_n m$ is an endomorphism of an $\mathbb{F}S_{n-1}$-module. This endomorphism is functorial in M because any S_n-linear homomorphism $M \to M'$ commutes with L_n. So the endomorphisms X_M form an endomorphism of the functor $\operatorname{Res}^n_{n-1}$.

The decomposition of $\operatorname{Res}^n_{n-1}(M)$ we need is that into the generalized eigenspaces for X_M. Let us recall how this works when $\operatorname{char}\mathbb{F} = 0$. Independently of $\mathbb{F}$, the irreducible S_n-modules are parameterized by the Young diagrams λ with n boxes, we write M_λ for the irreducible module corresponding to λ. We have

$$\operatorname{Res}^n_{n-1}(M_\lambda) = \bigoplus_\mu M_\mu,$$

where the sum is taken over all diagrams μ obtained from λ by removing a single box. Moreover, the summands M_μ are precisely the eigenspaces for L_n. More precisely, let b be the box in $\lambda \setminus \mu$. We define its *content* $c(b)$ as $x - y$, where x, y are the coordinates of b, see the example below.

Example 2.1. Let $\lambda = (3, 1, 1, 1)$, the diagram is as follows



We have two removable boxes in λ denoted by b_1, b_2. Note that $c(b_1) = 3 - 1 = 2$ and $c(b_2) = 1 - 4 = -3$. So $\operatorname{Res}_5^6(M) = M_{\mu_1} \oplus M_{\mu_2}$, where $\mu_1 = (2, 1, 1, 1)$ and $\mu_2 = (3, 1, 1)$. The eigenvalues of L_6 on M_{μ_1}, M_{μ_2} are 2 and -3, respectively.

Lemma 2.2. *Let* $\operatorname{char} \mathbb{F} = p$. *Then the eigenvalues of* X_M *are in* $\mathbb{Z}/p\mathbb{Z}$.

Proof. For $\mathbb{F} = \mathbb{C}$, the eigenvalues of L_n in any module, in particular, in $\mathbb{C}S_n$ are integral. So there are integers $a_1, \ldots, a_m$ such that $\prod_{i=1}^m (L_n - a_i) = 0$. This equality holds in $\mathbb{C}S_m$ and hence also in $\mathbb{Z}S_n$. Therefore it also holds in $\mathbb{F}S_n$ and we are done. $\square$

For $i \in \mathbb{Z}_\mathbb{F}$, let $\operatorname{Res}_{n-1}^n(M)_i$ denote the generalized eigenspace for X_M in M with eigenvalue i. The assignment $M \mapsto \operatorname{Res}_{n-1}^n(M)_i$ is a functor $\mathbb{F}S_n\text{-mod} \to \mathbb{F}S_{n-1}\text{-mod}$ (again, for the reason that any S_n-linear homomorphism is also L_n-linear). We have $\operatorname{Res}_{n-1}^n = \bigoplus_{i \in \mathbb{Z}_\mathbb{F}} \operatorname{Res}_{n-1}^n(\bullet)_i$.

We write E for $\bigoplus_{i=0}^\infty \operatorname{Res}_{n-1}^n(\bullet)$ so that $EM = \operatorname{Res}_{n-1}^n(M)$ for $M \in \mathbb{F}S_n\text{-mod}$. Then E is an endofunctor of $\mathcal{C}$. Here we set $\operatorname{Res}_{-1}^0 = 0$. Further, we set $E_i := \bigoplus_{n=0}^\infty \operatorname{Res}_{n-1}^n(\bullet)_i$. In other words, E_i is the generalized eigen-subfunctor of E for the endomorphism X with eigenvalue i. We have $E = \bigoplus_{i \in \mathbb{Z}_\mathbb{F}} E_i$.

2.2. **Direct sum decomposition for $\mathcal{C}$.** Let $\mathcal{H}(n)$ denote the degenerate affine Hecke algebra with generators $X_1, \ldots, X_n, T_1, \ldots, T_{n-1}$ to be recalled later. Recall that we have an algebra isomorphism $\mathcal{H}(n)/(X_1 = 0) \xrightarrow{\sim} \mathbb{F}S_n$ sending X_i to the Jucys-Murphy element $L_i = \sum_{j=1}^{i-1} (ji)$. Recall, Problem 3 in Homework 1, that the symmetric polynomials in the elements X_i are central in $\mathcal{H}(n)$. We get the following corollary.

Lemma 2.3. *Symmetric polynomials in* $L_1, \ldots, L_n$ *are central in* $\mathbb{F}S_n$.

Let us take an unordered n-tuple A of elements of $\mathbb{Z}_\mathbb{F}$. For $M \in \mathbb{F}S_n\text{-mod}$, define the generalized eigenspace for $\mathbb{Z}[L_1, \ldots, L_n]^{S_n}$ with eigenvalue A by

$$M_A := \{m \in M | (P(L_1, \ldots, L_n) - P(A))^k m = 0, \forall k \gg 0, P \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}\}.$$

Then $M = \bigoplus_A M_A$. Define $\mathbb{F}S_n\text{-mod}_A$ as the full subcategory of $\mathbb{F}S_n\text{-mod}$ consisting of all modules M that coincide with M_A. We have $\mathbb{F}S_n\text{-mod} = \bigoplus_A \mathbb{F}S_n\text{-mod}_A$ (there are no homomorphisms/extensions between modules belonging to different categories $\mathbb{F}S_n\text{-mod}_A$).

Example 2.4. Let $\operatorname{char} \mathbb{F} = 0$. Recall that M_λ has a basis v_T labelled by the standard Young tableaux T of shape λ. We have $L_i v_T = c(b_i) v_T$, where b_i is the box labelled by i in T. Hence $M_\lambda = (M_\lambda)_{c(\lambda)}$. So if A coincides with $c(\lambda)$ (the unordered collection of the contents of the boxes in λ), then $\mathbb{F}S_n$-mod$_A$ is spanned by M_λ and, otherwise, $\mathbb{F}S_n$-mod$_A$ is zero.

We view A as a multiset. We will write $A\setminus\{i\}$ (resp., $A\cup\{i\}$) for the multiset, where the multiplicity of i is decreased (resp., increased) by 1.

Lemma 2.5. *Let $M\in\mathbb{F}S_n$-mod$_A$. If $i\notin A$, then $E_iM=0$. Otherwise $E_iM\in\mathbb{F}S_n$-mod$_{A\setminus\{i\}}$.*

Proof. A is a collection of simultaneous eigenvalues of $(L_1,\ldots,L_n)$. So if $i\notin A$, then $E_iM=0$. If $i\in A$, and B is a collection of simultaneous eigenvalues of $(L_1,\ldots,L_{n-1})$ in E_iM, then $B\cup\{i\}$ is a collection of simultaneous eigenvalues of $L_1,\ldots,L_n$ in M. So $A=B\cup\{i\}$ and $B=A\setminus\{i\}$. □

Let π_A denote the projection functor $\mathbb{F}S_n\text{-mod}\twoheadrightarrow\mathbb{F}S_n\text{-mod}_A$, $\pi_A(M):=M_A$. Then, for $M\in\mathbb{F}S_n$-mod$_A$, we get $E_iM=\pi_{A\setminus\{i\}}\circ EM$, where we assume that $\pi_{A\setminus\{i\}}:=0$ if $i\notin A$. Note that π_A is both left and right adjoint for the inclusion functor $\mathbb{F}S_n\text{-mod}_A\hookrightarrow\mathbb{F}S_n\text{-mod}$.

2.3. **Functors F_i.** We have a left adjoint $\operatorname{Ind}_n^{n-1}$ and a right adjoint $\operatorname{Coind}_n^{n-1}$ functors to $\operatorname{Res}^n_{n-1}:\mathbb{F}S_n\text{-mod}\to\mathbb{F}S_{n-1}\text{-mod}$. The former is given by $N\mapsto\mathbb{F}S_n\otimes_{\mathbb{F}S_{n-1}}N$, while the latter is given by $N\mapsto\operatorname{Hom}_{S_{n-1}}(\mathbb{F}S_n,N)$.

Lemma 2.6. *We have a functor isomorphism $\operatorname{Ind}_n^{n-1}\cong\operatorname{Coind}_n^{n-1}$.*

Proof. Note that $\operatorname{Coind}_n^{n-1}(N)=(\mathbb{F}S_n)^*\otimes_{\mathbb{F}S_{n-1}}N$, where $(\mathbb{F}S_n)^*$ is equipped with a bimodule structure given by $\langle a\alpha b,c\rangle:=\langle\alpha,bca\rangle$. The claim about the isomorphism of functors will follow if we check that $\mathbb{F}S_n\cong(\mathbb{F}S_n)^*$ as an $\mathbb{F}S_n$-$\mathbb{F}S_{n-1}$-bimodule. In fact, we have an isomorphism of $\mathbb{F}S_n$-bimodules. Namely, consider the bilinear form $(\cdot,\cdot)$ on $\mathbb{F}S_n$ given by $(g,h)=\delta_{gh,1}$. It is a direct check that the identification $\mathbb{F}S_n\cong(\mathbb{F}S_n)^*$ with respect to this form is an isomorphism of $\mathbb{F}S_n$-bimodules. □

Our goal now is to produce a left adjoint of the functor $E_i, i\in\mathbb{Z}_{\mathbb{F}}$, to be denoted by F_i. This can be done in two equivalent ways. We can define the functors F_i on all categories $\mathbb{F}S_{n-1}$-mod$_B$ and then extend them to $\mathbb{F}S_{n-1}$-mod by additivity. On $\mathbb{F}S_{n-1}$-mod$_B$, the functor F_i is defined by $\pi_{B\cup\{i\}}\circ F$ so that $F=\bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}}F_i$.

Lemma 2.7. *The functor F_i is biadjoint to E_i.*

Proof. Let us show that F_i is left adjoint to E_i, the other adjunction is similar. It is enough to establish a bi-functorial isomorphism $\operatorname{Hom}_{S_n}(F_iN,M)\cong\operatorname{Hom}_{S_{n-1}}(N,E_iM)$ for $N\in\mathbb{F}S_{n-1}$-mod$_B$, $M\in\mathbb{F}S_n$-mod$_A$. Both sides are zero if $A\neq B\cup\{i\}$. If $A=B\cup\{i\}$, then the l.h.s. is $\operatorname{Hom}_{S_n}(FN,M)$ (because $\operatorname{Hom}_{S_{n-1}}(F_jN,M)=0$ for $j\neq i$) and similarly the r.h.s. is $\operatorname{Hom}_{S_{n-1}}(N,EM)$. Since F is left adjoint to E, we are done. □

Here is an equivalent way to produce F_i. Since F is left adjoint to E, the algebra $\operatorname{End}(E)$ gets identified with $\operatorname{End}(F)^{opp}$ (where the superscript means that the multiplication is taken in the opposite order). This is a consequence of the Yoneda lemma and the adjointness. So we get an endomorphism $X\in\operatorname{End}(F)^{opp}$. Then F_i is the generalized eigenfunctor for X with eigenvalue i.

2.4. **Functor morphisms.** As we have mentioned before, we also need to consider some functor morphisms. We have already seen some of those: we had an endomorphism X of the functor E. We also had morphisms $1\to EF, FE\to 1$, where 1's denote the identity functors (from the adjointness: F is left adjoint to E). But we actually need more morphisms. Those will be endomorphisms of $E^d=\bigoplus\operatorname{Res}^n_{n-d}$.

A recipe to construct these endomorphisms is similar to what was done for X: we will get them from elements of $(\mathbb{F}S_n)^{S_{n-d}}$. The elements that we are going to use are $L_{n-d+i} = \sum_{j=1}^{n-d+i-1}(j, n+d-i), i = 1,\ldots,d$, and $(n-d+i, n-d+i+1), i = 1,\ldots,i-1$. Recall from Lecture 2 that these elements satisfy the relations of the degenerate affine Hecke algebra $\mathcal{H}(d)$ that is generated by the elements $X_1,\ldots,X_d,T_1,\ldots,T_{d-1}$ subject to the relations:

$$X_iX_j = X_jX_i,$$
$$T_iT_j = T_jT_i, \text{ for } |i-j|>1, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_i^2 = 1,$$
$$T_iX_j = X_jT_i, \text{ for } j-i \neq 0,1, \quad T_iX_i = X_{i+1}T_i - 1.$$

As we have seen, there is an algebra homomorphism $\mathcal{H}(d) \to (\mathbb{F}S_n)^{S_{n-d}}$ mapping X_i to L_{n-d+i} and T_i to $(n-d+i, n-d+i+1)$. This yields an algebra homomorphism $\mathcal{H}(d) \to \mathrm{End}(E^d)$.

Let us make a remark that will become useful later. We can recover the images of X_i in $\mathrm{End}(E^d)$ from $X \in \mathrm{End}(E)$. For this we need the following general construction. Let $\mathcal{F},\mathcal{G}$ be endofunctors of a category $\mathcal{C}$ and let $X \in \mathrm{End}(\mathcal{F}), Y \in \mathrm{End}(\mathcal{G})$. Then we get the endomorphisms of $\mathcal{F}\mathcal{G}$ given by $(X1)_M := X_{\mathcal{G}(M)}, (1Y)_M := \mathcal{F}(Y_M)$. With this notation, X_i goes to the endomorphism $1^{i-1}X1^{d-i}$.

Similarly, we can recover the image of T_i from $T \in \mathrm{End}(E^2)$ that is the image of T_1, i.e., $T_M(m) = (n-1,n)m$ for $M \in \mathbb{F}S_n$-mod$, m \in M$. Namely, T_i maps to $1^{i-1}T1^{d-i-1}$.

Note that this description makes some of the relations between the images of X_i, T_i in $\mathrm{End}(E^d)$ automatic (e.g., the relation that $X_iX_j = X_jX_i$), while others only need to be checked for small d (for example, it is enough to check that $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $d = 3$).