LECTURE 2: REPRESENTATIONS OF SYMMETRIC GROUPS, II

IVAN LOSEV

1. Introduction

Recall that last time we have introduced the set of weights $\mathsf{Wt}(n) \subset \mathbb{C}^n$ and an equivalence relation $\sim$ on $\mathsf{Wt}(n)$. Our goal is to describe this set and this equivalence relation – the equivalence classes are in a natural bijection with $\mathrm{Irr}(S_n)$. The first step here is as follows. Pick a path $P = V^1 \to V^2 \to \ldots \to V^n$ in the branching graph and an integer k with $1 < k < n$. We fix all vertices but V^k and vary V^k. The questions that we are going to answer: how many paths do we get? What is the relation between their weights? Let $\mathsf{Path}(P,k)$ be the resulting set of paths and $V^n_{P,k} := \mathrm{Span}(v_{P'} | P' \in \mathsf{Path}(P,k))$. We will see that $V^n_{P,k}$ is an irreducible $Z_{k-1}(k+1)$-submodule in V^n, where, recall, $Z_{k-1}(k+1)$ stands for the centralizer of $\mathbb{C}S_{k-1}$ inside $\mathbb{C}S_{k+1}$. Next, we will construct the degenerate affine Hecke algebra $\mathcal{H}(2)$ and its homomorphism to $Z_{k-1}(k+1)$ so that $V^n_{P,k}$ becomes an irreducible $\mathcal{H}(2)$-module. We will obtain a complete classification of the finite dimensional irreducible $\mathcal{H}(2)$-modules. This will ultimately allow us to describe the set $\mathsf{Wt}(n)$ and $\sim$.

2. Degenerate affine Hecke algebras

2.1. Comparing $V^n_{P,k}$ to $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$.

Recall that the space $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$ has a basis $\varphi_{\underline{P}}$, where $\underline{P}$ runs over $\mathsf{Path}(V^{k-1}, V^{k+1})$. The space $V^n_{P,k}$ has a basis indexed by the same set, $v_{P_1\underline{P}P_2}$, where $P_1 = V^1 \to \ldots \to V^{k-1}, P_2 = V^{k+1} \to \ldots \to V^n$ are the fixed parts of the paths in $\mathsf{Path}(P,k)$.

Lemma 2.1. *The map $\psi : \mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1}) \to V^n$ given by $\varphi \mapsto \varphi_{P_2} \circ \varphi(v_{P_1})$ is a $Z_{k-1}(k+1)$-equivariant embedding whose image coincides with $V^n_{P,k}$.*

Proof. By Remark 3.2 in Lecture 1 (concatenation of paths gives the composition of homomorphisms), $v_{P_1\underline{P}P_2}$ is proportional to $\varphi_{P_2}(\varphi_{\underline{P}}(v_{P^1}))$. This shows that ψ is an embedding whose image coincides with $V^n_{P,k}$. It remains to check that it is $Z_{k-1}(k+1)$-equivariant. Note that φ_{P_2} is $\mathbb{C}S_{k+1}$-equivariant and hence is $Z_{k-1}(k+1)$-equivariant. By Remark 2.3 in Lecture 1, we have $[z \cdot \varphi](u) = z \cdot \varphi(u)$ for $z \in Z_{k-1}(k+1), \varphi \in \mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$ and $u \in V^{k-1}$. This implies the $Z_{k-1}(k+1)$-equivariance of ψ. □

Recall the Jucys-Murphy elements $L_m = \sum_{i=1}^{m-1}(im)$. We have $L_k, L_{k+1}, (k, k+1) \in Z_{k-1}(k+1)$.

Corollary 2.2. *The subspace $V^n_{P,k}$ is stable under $L_k, L_{k+1}, (k, k+1)$ (the latter stands for the transposition of $k, k+1$) and has no proper stable subspaces.*

Proof. Recall, Theorem 2.4 from Lecture 1, that $Z_{k-1}(k+1)$ is generated by $Z_{k-1}(k-1), L_k, L_{k+1}, (k, k+1)$. By (1) of Corollary 2.5 of Lecture 1, $Z_{k-1}(k-1)$ is in the center of $Z_{k-1}(k+1)$. So elements of $Z_{k-1}(k-1)$ act by scalars on any irreducible $Z_{k-1}(k+1)$-module U. So U is irreducible with respect to $L_k, L_{k+1}, (k, k+1)$. Since $V^n_{P,k}$ is an irreducible $Z_{k-1}(k+1)$-module, we are done. □

2.2. **Degenerate affine Hecke algebra $\mathcal{H}(2)$.** Corollary 2.2 motivates us to find relations between $L_k, L_{k+1}, (k, k+1)$. Then we can form an associative algebra with three generators corresponding to $L_k, L_{k+1}, (k, k+1)$ and relations we have found, the space $V^n_{P,k}$ will be an irreducible module over this algebra.

Lemma 2.3. *We have the following relations*

$$L_k L_{k+1} = L_{k+1} L_k, \quad (k, k+1)^2 = 1, \quad (k, k+1) L_k = L_{k+1}(k, k+1) - 1. \tag{2.1}$$

Proof. The element L_{k+1} commutes with $\mathbb{C}S_k$ and hence with $L_k \in \mathbb{C}S_k$. This gives the first relation. The second relation is obvious. The left hand side of the third relation is

$$(k, k+1)\sum_{i=1}^{k-1}(ik) = \sum_{i=1}^{k-1}(k, k+1)(ik) = \sum_{i=1}^{k-1}(i, k, k+1).$$

The right hand side is

$$\left(\sum_{i=1}^{k}(i, k+1)\right)(k, k+1) - 1 = \sum_{i=1}^{k+1}(i, k+1)(k, k+1) - 1 =$$
$$= \sum_{i=1}^{k-1}(i, k, k+1) + (k, k+1)^2 - 1 = \sum_{i=1}^{k-1}(i, k, k+1).$$

These computations prove the third relation. $\square$

Define the *degenerate affine Hecke algebra* $\mathcal{H}(2)$ by generators X_1, X_2, T and relations that mirror those found in Lemma 2.1.

$$X_1 X_2 = X_2 X_1, T^2 = 1, T X_1 = X_2 T - 1.$$

There is a consequence of these relations: $X_1 T = T X_2 - 1$ (multiply the third relation by T both from the left and from the right).

Our conclusion is that we have a unique homomorphism $\mathcal{H}(2) \to Z_{k-1}(k+1)$ given on generators by $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$.

Corollary 2.4. *The space $V^n_{P,k}$ is an irreducible $\mathcal{H}(2)$-module.*

2.3. **Classification of irreducible $\mathcal{H}(2)$-modules.** Let us classify the finite dimensional irreducible $\mathcal{H}(2)$-modules M.

Since X_1, X_2 commute, they have a common eigenvector $m \in M$. Let $X_1 m = am, X_2 m = bm$, where $a, b \in \mathbb{C}$.

Let us consider two cases:

1) Tm is proportional to m. Since $T^2 = 1$, we have two options:

1.1) $Tm = m$. Let us apply the third relation to m. The left hand side gives $TX_1 m = am$, while the right hand side gives $(X_2 T - 1)m = (b-1)m$, so here $b = a + 1$.

1.2) $Tm = -m$. Similarly to the previous case, we get $b = a - 1$.

2) m and Tm are linearly independent. Let us see how X_1, X_2 act on Tm:

$$X_1(Tm) = [X_1 T = T X_2 - 1] = T X_2 m - m = b(Tm) - m,$$
$$X_2(Tm) = [X_2 T = T X_1 + 1] = T X_1 m + m = a(Tm) + m.$$

In particular, we see that $\mathrm{Span}(m, Tm)$ is stable under $\mathcal{H}(2)$. Since M is irreducible, we see that m and Tm form a basis in M. In this basis, we have

$$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, X_2 \mapsto \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}. \tag{2.2}$$

In particular, we see that, in case 2), two modules M, M' that give the same pair of eigenvalues a, b are isomorphic. Note that if $b = a \pm 1$, then (2.2) give a reducible module. Indeed, assume in the sake of being definite, that $b = a+1$. Then the line $\mathbb{C}(m+Tm)$ is a submodule, and our 2-dimensional module is a non-split extension of the 1-dimensional modules in 1.1) and 1.2). In particular, the pair (a, b) of eigenvalues for X_1, X_2 determines any irreducible $\mathcal{H}(2)$-module uniquely up to an isomorphism. Let us denote the corresponding module by $M(a, b)$.

Note that if $a \neq b, b \pm 1$, then $\dim M(a, b) = 2$ and the action of X_1, X_2 on $M(a, b)$ is diagonalizable, as X_1 has distinct eigenvalues and X_2 commutes with X_1. The pairs of eigenvalues that appear are (a, b) and (b, a). It follows that $M(a, b) \cong M(b, a)$. Moreover, if $M(a, b) = M(a', b')$ and $(a, b) \neq (a', b')$, then $b \neq a \pm 1$, and $a' = b, b' = a$.

We arrive at the following classification result.

Proposition 2.5. *The finite dimensional irreducible $\mathcal{H}(2)$-modules are classified by pairs of complex numbers, $(a, b) \mapsto M(a, b)$, with $M(a, b) \cong M(b, a)$ if $b \neq a, a \pm 1$. The pair (a, b) is a pair of simultaneous eigenvalues of X_1, X_2 in $M(a, b)$. Moreover, the following is true.*

1. *If $b = a + 1$, then $M(a, b) = \mathbb{C}$ with $T \mapsto 1, X_1 \mapsto a, X_2 \mapsto b$.*
2. *If $b = a - 1$, then $M(a, b) = \mathbb{C}$ with $T \mapsto -1, X_1 \mapsto a, X_2 \mapsto b$.*
3. *If $b \neq a \pm 1$, then formulas (2.2) define an irreducible representation, and this is $M(a, b)$.*
4. *The action of X_1, X_2 on $M(a, b)$ is diagonalizable if and only if $a \neq b$.*

2.4. **Algebras $\mathcal{H}(d)$.** One can define the algebra $\mathcal{H}(d)$ for all $d \geqslant 1$. It is generated by generators $X_1, \ldots, X_d, T_1, \ldots, T_{d-1}$ with relations:

$$\begin{aligned}
&X_i X_j = X_j X_i, \\
&T_i^2 = 1, \quad T_i T_j = T_j T_i, \text{ for } |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\
&X_i T_j = T_j X_i, \text{ for } i - j \neq 0, 1, \quad T_i X_i = X_{i+1} T_i - 1.
\end{aligned}$$

Note that in the second line we have precisely the relations for the transpositions $(i, i+1) \in \mathbb{C}S_n, i = 1, \ldots, n-1$. So the map $(i, i+1) \mapsto T_i$ extends to an algebra homomorphism $\mathbb{C}S_d \to \mathcal{H}(d)$. We also have an algebra homomorphism $\mathbb{C}[X_1, \ldots, X_d] \to \mathcal{H}(d)$ given in an obvious way.

Also note that we have a homomorphism $\mathcal{H}(d) \to Z_{n-d}(n)$ with $X_i \mapsto L_{n-d+i}, T_i \mapsto (n - d + i, n - d + i + 1)$. In particular, $\mathbb{C}S_n$ is a quotient of $\mathcal{H}(n)$ by the two-sided ideal generated by X_1, the element X_i gets mapped to L_i. The algebra $\mathcal{H}(d)$ first appeared in connection with Yangians, [D], and then was used to study the representations of the usual affine Hecke algebra, [L], that, in its turn, arises in the study of the representations of $\mathrm{GL}(\mathbb{Q}_p)$.

A classification of the finite dimensional irreducible $\mathcal{H}(d)$-modules and a computation of their dimensions is known but isn't easy, formulas will be in terms of so called Kazhdan-Lusztig polynomials, see [BK].

3. Completion of classification

3.1. **Consequences for weights.** Here we are going to get some restrictions for $w_{P'}$, where $P' \in \mathsf{Path}(P,k)$. Let $w_P = (w_1, \ldots, w_n)$ and $w_{P'} = (w'_1, \ldots, w'_n)$. Recall (Lemma 3.3 of Lecture 1) that w'_i is completely determined by V'_i and V'_{i-1}. Since $V'_i = V_i$ when $i \neq k$, we see that $w'_i = w_i$ if $i \neq k, k+1$.

Below we are going to use the following notation. We write s_i for $(i, i+1) \in S_n$. For $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$, we write $s_i x$ for $(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$.

Proposition 3.1. *Let P be a path of length n. The following is true:*

(a) *If $w_k \neq w_{k+1} \pm 1$, then $s_k w_P$ is a weight equivalent to w_P.*
(b1) $w_1 = 0$.
(b2) $w_k \neq w_{k+1}$ *for any* k.
(b3) *If $w_k = w_{k+2}$, then $w_{k+1} \neq w_k \pm 1$.*

Proof. The proof is based on the observation that if $P' \in \mathsf{Path}(P,k)$, then $V^n_{P,k} \cong M(w'_k, w'_{k+1})$, an isomorphism of $\mathcal{H}(2)$-modules, where on the left hand side the structure of the module is given by $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$. So we can use Proposition 2.5.

In the situation of (a), the module $M(w_k, w_{k+1})$ is two-dimensional. So $\mathsf{Path}(P,k)$ consists of two elements, P and $P' \neq P$. Since $M(w'_k, w'_{k+1}) \cong V^n_{P,k} \cong M(w_k, w_{k+1})$, we use (3) of Proposition 2.5 to see that $w'_k = w_{k+1}, w'_{k+1} = w_k$. This proves (a).

(b1) follows from $L_1 = 0$.

To prove (b2), we note that the action of X_1, X_2 on $V^n_{P,k} \cong M(w_k, w_{k+1})$ is diagonalizable. So $w_k \neq w_{k+1}$ by (4) of Proposition 2.5.

Let us prove (b3). Assume the converse, say, $w_k = w_{k+2} = w_{k+1} - 1$. Then $\mathbb{C}v_P \cong M(w_k, w_k + 1)$ via $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$. In particular, $(k, k+1)v_P = v_P$. Similarly, $(k+1, k+2)v_P = -v_P$. But $(k+1, k)(k+1, k+2)(k, k+1) = (k+1, k+2)(k, k+1)(k+1, k+2)$ in S_n. Applying the two sides to v_P, we arrive at $-v_P = v_P$. Contradiction. □

3.2. **Combinatorial weights and combinatorial equivalence.** Motivated by (a), define an equivalence relation $\sim_c$ (combinatorial equivalence) on $\mathbb{C}^n$ by $x \sim y$ if y is obtained from x by a sequence of permutations of adjacent elements whose difference is not ± 1 (we call such a permutation *admissible*). We define the subset $\mathsf{CWt}(n) \subset \mathbb{C}^n$ (combinatorial weights) as the set of all x such that any y with $y \sim_c x$ satisfies (b1)-(b3). The next lemma follows from Proposition 2.5.

Lemma 3.2. $\mathsf{Wt}(n)$ *is the union of some equivalence classes for $\sim_c$ in $\mathsf{CWt}(n)$. Moreover, for $w, w' \in \mathsf{Wt}(n)$, we have $w \sim_c w' \Rightarrow w \sim w'$.*

Note that $\mathsf{CWt}(n) \subset \mathbb{Z}^n$.

We are going to embed the set of equivalence classes $\mathsf{CWt}(n)/\sim_c$ into $\mathcal{P}(n)$, the set of partitions on n. This is done by reducing an element of $\mathsf{CWt}(n)$ to a "normal form". We say that an element $x \in \mathsf{CWt}(n)$ is *normal* if it has the form

$$(0, 1, \ldots, n_1, -1, 0, \ldots, n_2, -2, \ldots, n_3, \ldots),$$

where $n_1 + 1 \geqslant n_2 + 2 \geqslant \ldots \geqslant 0$.

Proposition 3.3. *For any $y \in \mathsf{CWt}(n)$, there exists a unique normal $x \in \mathsf{CWt}(n)$ with $y \sim_c x$.*

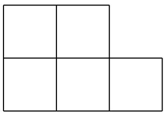
Proof. Let n_1 be the maximal entry in y. We can find $y' \sim_c y$ such that $y'_k = n_1$ and $y'_{k+1}, \dots, y'_n < n_1$, while there is no other $z \sim_c y$ with $z_k, \dots, z_n < n_1$ (the maximal entries in y' are as far to the left as possible). We claim that $k = n_1 + 1$ and $y_i = i - 1$ for $i \leqslant k$. Indeed, if $y'_{k-1} \neq n_1 - 1$, then we can switch y'_{k-1} and y'_k (here we use (b2)), etc.

Then we freeze the first $n_1 + 1$ entries in y' (we no longer permute them) and pick the maximal unfrozen entry n_2 in y'. Again, consider y'', where the entries n_2 are as far to the left as possible (with the first $n_1 + 1$ entries frozen). Similarly to the previous paragraph, y'' starts with $0, 1, \dots, n_1, a, a + 1, \dots, n_2$. If $a < -1$, then we can move a all the way to the left and get a contradiction with (b1). If $a > -1$, then we can move it the left until we get a segment $a, a + 1, a$, which contradicts (b3). So $a = -1$.

Then we freeze the first $(n_1 + 1) + (n_2 + 2)$ entries and repeat the argument. This shows the existence of x.

Uniqueness follows from the observation that n_1 is the maximal element of $y_1, \dots, y_n$, n_2 is maximal after removing $0, 1, \dots, n_1$, etc. $\square$

3.3. **Young diagrams and tableaux, finally.** Partitions of n that are often depicted as Young diagrams, the following diagram corresponds to the partition $5 = 3 + 2$.



Define a map $\mathsf{CWt}(n)/\sim_c \to \mathcal{P}(n)$ by sending x to the partition with parts $(n_1+1), (n_2+2)$, where n_1, n_2, etc., are as in Proposition 3.3. By the uniqueness part of that proposition, our map is an embedding. The following theorem completes the classification of $\mathrm{Irr}(S_n) = \mathsf{Wt}(n)/\sim$.

Theorem 3.4. *We have* $\mathsf{Wt}(n) = \mathsf{CWt}(n), \sim = \sim_c$ *and* $\mathsf{CWt}(n)/\sim_c \xrightarrow{\sim} \mathcal{P}(n)$.

Proof. We have a surjection $\mathsf{Wt}(n)/\sim_c \twoheadrightarrow \mathsf{Wt}(n)/\sim$ and embeddings

$$(\mathsf{Wt}(n)/\sim_c) \hookrightarrow (\mathsf{CWt}(n)/\sim_c) \hookrightarrow \mathcal{P}(n).$$

Therefore we get the following chain of inequalities:

$$|\mathsf{Wt}(n)/\sim| \leqslant |\mathsf{Wt}(n)/\sim_c| \leqslant |\mathsf{CWt}(n)/\sim_c| \leqslant |\mathcal{P}(n)|$$

By Introduction to Lecture 1, $|\mathrm{Irr}(S_n)| = |\mathsf{Wt}(n)/\sim| = |\mathcal{P}(n)|$. So all these embeddings and a surjection are bijections. $\square$

Now let us relate $\mathsf{Wt}(n)$ to the set of *standard Young tableaux* $\mathsf{SYT}(n)$. Recall that a standard Young tableaux on a Young diagram λ with n boxes is a filling of λ with numbers from 1 to n that strictly increase bottom to top and left to right. For example, these two fillings are examples of Young tableaux of shape $(3, 2)$.

4	5	
1	2	3

3	5	
1	2	4

To a Young tableaux T we assign its *content* as follows. Let (x_i, y_i) be the coordinate of the box numbered by i. Then the content $c(T)$ of T is, by definition, $(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$. The following two collections are contents of the tableaux in the previous example: $(0, 1, 2, -1, 0)$ and $(0, 1, -1, 2, 0)$.

Proposition 3.5. *The map* $T \mapsto c(T)$ *is a biijection* $\mathsf{SYT}(n) \to \mathsf{Wt}(n)$ *that intertwines the surjections* $\mathsf{SYT}(n) \twoheadrightarrow \mathcal{P}(n)$ *(taking the shape) and* $\mathsf{Wt}(n) \twoheadrightarrow \mathcal{P}(n)$.

Proof. We can define an admissible permutation of k and $k+1$ in a tableaux T: it permutes k and $k+1$ if the result is still a tableau. For example, the two tableaux above are obtained from one another by permuting 3 and 4.

The admissible permutations give rise to an equivalence relation $\sim_c$ on $\mathsf{SYT}(n)$. Clearly, an admissible permutation of $k, k+1$ in T corresponds to the admissible permutation s_k of $c(T)$. It is not hard to show that $c(T)$ satisfies conditions (b1)-(b3). So $c(T)$ is indeed an element of $\mathsf{CWt}(n)$ and the image of c is the union of equivalence classes for $\sim_c$.

We can define normal Young tableaux, where we fill the first row by numbers from 1 to some n_1, then the second row by the numbers from n_1+1 to n_1+n_2, etc., for example, the first tableau above is normal. Clearly, if T is normal, then so is $c(T)$. From Proposition 3.3, it follows that c is surjective. On the other hand, it is easy to check that c is injective. $\square$

References

[BK] J. Brundan, A. Kleshchev. *The degenerate analogue of Ariki's categorification theorem.* Math. Z. 266 (2010), no. 4, 877-919. arXiv:0901.0057.

[D] V. Drinfeld. *Degenerate affine Hecke algebra and Yangians.* Funktsional. Anal. i Prilozh., 20 (1986), 69-70.

[L] G. Lusztig. *Affine Hecke algebras and their graded version.* J. Amer. Math. Soc. 2 (1989), no. 3, 599635.