# LECTURE 20: KAC-MOODY ALGEBRA ACTIONS ON CATEGORIES, II

IVAN LOSEV

## 1. Introduction

1.1. **Recap.** In the previous lecture we have considered the category $\mathcal{C}_{\mathbb{F}} := \bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. We have equipped it with two endofunctors, $E = \bigoplus_n \operatorname{Res}^n_{n-1}$ and $F = \bigoplus_n \operatorname{Ind}^n_{n+1}$ that are biadjoint. We have decomposed $E$ into the direct sum of eigenfunctors, $E = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} E_i$, for the endomorphism $X$ that is given by $X_M m = L_n m$ for $M \in \mathbb{F}S_n$-mod, where $L_n$ is the Jucys-Murphy element $\sum_{i=1}^{n-1}(in)$. We have also considered the corresponding decomposition $F = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} F_i$.

Besides, we have introduced the decomposition $\mathbb{F}S_n\text{-mod} = \bigoplus_A \mathbb{F}S_n\text{-mod}_A$, where the summation is taken over all cardinality $n$ multi-subsets in $\mathbb{Z}_{\mathbb{F}}$, and $\mathbb{F}S_n\text{-mod}_A$ consists of all $M \in \mathbb{F}S_n$-mod such that $P(L_1,\ldots,L_n)$ acts on $M$ with a single eigenvalue $P(A)$, for every $P \in \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$. This decomposition is related to the functors $E_i, F_i$ as follows. Let $\pi_A$ denote the projection $\mathbb{F}S_n\text{-mod} \twoheadrightarrow \mathbb{F}S_n\text{-mod}_A$. Then, for $M \in \mathbb{F}S_n\text{-mod}_A$, we have $E_iM = \pi_{A\setminus\{i\}}(EM), F_iM = \pi_{A\cup\{i\}}(FM)$. Below, we will write $\mathcal{C}_{\mathbb{F},A} = \mathbb{F}S_{|A|}\text{-mod}_A$. So we get the direct sum decomposition $\mathcal{C}_{\mathbb{F}} = \bigoplus_A \mathcal{C}_{\mathbb{F},A}$, where the sum is taken over all multi-subsets $A$ of $\mathbb{Z}_{\mathbb{F}}$.

Finally, we have also introduced an endomorphism $T$ of $E^2$: $T_M m = (n-1,n)m$ for $m \in M, M \in \mathbb{F}S_n$-mod. We have seen that the assignment $X_i \mapsto 1^{i-1}X1^{d-i}, T_i \mapsto 1^{i-1}T1^{d-i-1}$ extends to an algebra homomorphism $\mathcal{H}(d) \to \operatorname{End}(E^d)$.

1.2. **Goals.** First of all, we will show that $[E_i], [F_i], i \in \mathbb{Z}_{\mathbb{F}}$, together with the decomposition $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A [\mathcal{C}_{\mathbb{F},A}]$ define the structure of a weight representation of $\hat{\mathfrak{sl}}_p$ (if $\operatorname{char}\mathbb{F} = p$) or of $\mathfrak{sl}_\infty$ (if $\operatorname{char}\mathbb{F} = p$). The characteristic 0 case is easy as we can determine $[\mathcal{C}_{\mathbb{F}}], [E_i], [F_i]$ very explicitly. The case when $\operatorname{char}\mathbb{F} = p$ is more tricky because we do not understand the structure of $[\mathcal{C}_{\mathbb{F}}]$ at this point. We will treat this case by reducing to characteristic 0.

After this is done we will give an abstract definition of an action of $\hat{\mathfrak{sl}}_p$ on a category ($\mathcal{C}_{\mathbb{F}}$ for $\operatorname{char}\mathbb{F} = p$ will be the main example). Then we will give an application: modulo some results of Chuang and Rouquier, we will show that $[\mathcal{C}_{\mathbb{F}}]$ is an irreducible $\hat{\mathfrak{sl}}_p$-module.

## 2. $\hat{\mathfrak{sl}}_p$-action on $K_0$

Let $\mathbb{F}$ be a characteristic $p$ field. In this section, we will show that the operators $[E_i], [F_i]$ on $[\mathcal{C}_{\mathbb{F}}]$ give rise to a $\hat{\mathfrak{sl}}_p$-action. Moreover, we will check that $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A [\mathcal{C}_{\mathbb{F},A}]$, where we write $\mathcal{C}_A$ for $\mathbb{F}S_{|A|}\text{-mod}_A$, is a weight decomposition for $\tilde{\mathfrak{sl}}_p$.

2.1. **Comparison of $K_0$'s in characteristics 0 and $p$.** Consider the following situation. Let $R$ be a local Dedekind domain containing $\mathbb{Z}$. Let $\mathbb{K}$ denote the fraction field of $R$ and let $\mathbb{F}$ be the residue field, we will assume that it has characteristic $p$. E.g., we can take $R = \mathbb{Z}_p$, then $\mathbb{K} = \mathbb{Q}_p, \mathbb{F} := \mathbb{F}_p$. Let $A_R$ be an associative unital $R$-algebra that is a free finite rank $R$-module. An example is provided by $RS_n$. Set $A_{\mathbb{K}} = \mathbb{K} \otimes_R A_R, A_{\mathbb{F}} = \mathbb{F} \otimes_R A_R$.

Consider the categories $A_{\mathbb{K}}$-mod and $A_{\mathbb{F}}$-mod of finite dimensional $A_{\mathbb{K}}$- and $A_{\mathbb{F}}$-modules. We are going to produce a group map $K_0(A_{\mathbb{K}}\text{-mod}) \to K_0(A_{\mathbb{F}}\text{-mod})$. Take $M \in A_{\mathbb{K}}$-mod. We can pick an *R-lattice* $M_R \subset M$ meaning a finitely generated $R$-submodule $M_R$ with $\mathbb{K} \otimes_R M_R \xrightarrow{\sim} M$ that is automatically free over $R$. Then we get $M_{\mathbb{F}} := \mathbb{F} \otimes_R M_R \in A_{\mathbb{F}}$-mod. There are different lattices $M_R \subset M$ leading to non-isomorphic modules $M_{\mathbb{F}}$. However, a standard fact (left as an exercise) is that the class of $M_{\mathbb{F}}$ in $K_0$ does not depend on the choice of $M_R$. So we do get a well-defined map $K_0(A_{\mathbb{K}}\text{-mod}) \to K_0(A_{\mathbb{F}}\text{-mod})$.

**Lemma 2.1.** *This map is additive.*

*Proof.* Let $M' \subset M$ be an $A_{\mathbb{K}}$-submodule with the projection $\pi : M \to M/M'$. Then $M'_R := M' \cap M_R$ is a lattice in $M'$, while $\pi(M_R)$ is a lattice in $M/M'$ so that we have an exact sequence $0 \to M'_R \to M_R \to \pi(M_R) \to 0$. Since $\pi(M_R)$ is free over $R$, we see that the sequence $0 \to M'_{\mathbb{F}} \to M_{\mathbb{F}} \to \mathbb{F} \otimes_R \pi(M_R) \to 0$ is exact. This completes the proof. □

The following result is much more interesting.

**Proposition 2.2.** *The map $K_0(\mathbb{K}S_n\text{-mod}) \to K_0(\mathbb{F}S_n\text{-mod})$ is surjective.*

We will discuss why this is true in the next lecture.

2.2. **Fock space.** Let $\mathcal{C}_{\mathbb{K}} := \bigoplus_{n \geqslant 0} \mathbb{K}S_n\text{-mod}$. The $\mathbb{C}$-vector space $[\mathcal{C}_{\mathbb{K}}]$ has basis $[M_\lambda]$ labeled by all partitions $\lambda$. It is customary to write $|\lambda\rangle$ for $[M_\lambda]$. The space $\mathcal{C}_{\mathbb{K}}$ is known as the Fock space. We will denote it by $\mathcal{F}$.

Let us produce an action of $\mathfrak{sl}_\infty$ on $\mathcal{F}$. We set $e_i^\infty|\lambda\rangle = |\mu\rangle$, where $\mu$ is obtained from $\lambda$ by deleting a box of content $i$ if such $\mu$ exists, and $e_i^\infty|\lambda\rangle = 0$, else. Similarly, set $f_i^\infty|\lambda\rangle = |\nu\rangle$ if $\nu$ is obtained from $\lambda$ by adding a box of content $i$ if such $\nu$ exists, and $f_i^\infty|\lambda\rangle = 0$, else. Finally, set $h_i^\infty|\lambda\rangle = (a_i^\infty(\lambda) - r_i^\infty(\lambda))|\lambda\rangle$, where $a_i^\infty(\lambda)$ is the number of addable boxes of content $i$ in $\lambda$ and $r_i^\infty(\lambda)$ is the number of removable boxes of content $i$ in $\lambda$.

**Lemma 2.3.** *The operators $e_i^\infty, f_i^\infty$ give rise to a weight representation of $\mathfrak{sl}_\infty$ in $\mathcal{F}$ (with $h_i^\infty$ as specified above).*

The proof is left as an exercise.

We have seen in Section 2.1 of Lecture 19 that $e_i^\infty = [E_i^{\mathbb{K}}]$ (we write $E_i^{\mathbb{K}}$ for the functor $E_i$ for $\mathcal{C}_{\mathbb{K}}$). From the adjointness of $E_i^{\mathbb{K}}, F_i^{\mathbb{K}}$, we conclude that $\operatorname{Hom}_{\mathcal{C}_{\mathbb{K}}}(F_i^{\mathbb{K}}M_\lambda, M_\nu) = \operatorname{Hom}_{\mathcal{C}_{\mathbb{K}}}(M_\lambda, E_i^{\mathbb{K}}M_\nu)$ and therefore $F_i^{\mathbb{K}}M_\lambda = M_\nu$ if $\nu$ is obtained from $\lambda$ by adding a box of content $i$ if such $\nu$ exists, and $F_i^{\mathbb{K}}M_\lambda = 0$, else. So $[F_i^{\mathbb{K}}] = f_i^\infty$.

Let us proceed to an action of $\tilde{\mathfrak{sl}}_p$ on $\mathcal{F}$. For $j \in \mathbb{Z}/p\mathbb{Z}$, by a $j$-box we mean a box whose content is congruent to $j$ modulo $i$. Let $a_j(\lambda), r_j(\lambda)$ denote the number of addable and removable $j$-boxes in $\lambda$. We set

$$e_j = \sum_{i \equiv j \mod p} e_i^\infty, f_j = \sum_{i \equiv j \mod p} e_i^\infty, h_j|\lambda\rangle = (a_j(\lambda) - r_j(\lambda))|\lambda\rangle, d|\lambda\rangle = |\lambda||\lambda\rangle.$$

The next lemma follows mostly from Lemma 2.3.

**Lemma 2.4.** *The operators $e_j, f_j$ define a weight representation of $\tilde{\mathfrak{sl}}_p$ in $\mathcal{F}$ (with $h_j, d$ acting as specified).*

**Example 2.5.** Take the diagram $\lambda = (3, 1, 1, 1)$ and assume $p = 3$. This diagram has two removable boxes: $(3, 1), (1, 4)$ and three addable boxes: $(4, 1), (2, 2), (1, 5)$. The boxes $(4, 1), (2, 2), (1, 4)$ are 0-boxes, while $(3, 1), (1, 5)$ are 2-boxes and there are no 1-boxes. So we have $e_0|\lambda\rangle = |\mu_2\rangle, e_1|\lambda\rangle = 0, e_2|\lambda\rangle = |\mu_1\rangle$, where $\mu_1 = (2, 1, 1, 1), \mu_2 = (3, 1, 1)$. Further, we have $f_0|\lambda\rangle = |\nu_1\rangle + |\nu_2\rangle, f_1|\lambda\rangle = 0, f_2|\lambda\rangle = |\nu_3\rangle$, where $\nu_1 = (4,1,1,1), \nu_2 = (3,2,1,1), \nu_3 = (3,1,1,1,1)$. So $h_0|\lambda\rangle = |\lambda\rangle, h_1|\lambda\rangle = h_2|\lambda\rangle = 0$ and $d|\lambda\rangle = 6|\lambda\rangle$.

Now let us discuss the weight spaces for $\tilde{\mathfrak{sl}}_p$ in $\mathcal{F}$.

**Lemma 2.6.** *For diagrams $\lambda, \lambda'$ the following are equivalent.*

(1) $c(\lambda) \mod p = c(\lambda') \mod p$ *(the equality of multisubsets of $\mathbb{Z}_{\mathbb{F}}$).*
(2) $a_j(\lambda) - r_j(\lambda) = a_j(\lambda') - r_j(\lambda')$ *for all $j$ and $|\lambda| = |\lambda'|$.*

*Proof.* Let $n_j$ denote the number of $j$-boxes in $\lambda$ so that (1) means $n_j(\lambda) = n_j(\lambda')$ for all $j$. Adding a $j$-box, we increase $a_{j\pm 1} - r_{j\pm 1}$ by 1 (if $p > 2$; for $p = 2$ we increase it by 2) and decrease $a_j - r_j$ by 2. We also increase $|\lambda|$ by 1. It follows that $a_j(\lambda) - r_j(\lambda) = n_{j+1}(\lambda) + n_{j-1}(\lambda) - 2n_j(\lambda) + \delta_{j0}$. Clearly, $|\lambda| = \sum_j n_j(\lambda)$. These equalities easily imply that (1) and (2) are equivalent. □

For a multisubset $A \subset \mathbb{Z}_{\mathbb{F}}$ define the subspace $\mathcal{F}_A$ as the span of all $|\lambda\rangle$ with $c(\lambda) = A$. So $\mathcal{F} = \bigoplus_A \mathcal{F}_A$ is the weight decomposition for the action of $\tilde{\mathfrak{sl}}_p$.

2.3. **Action of $\tilde{\mathfrak{sl}}_p$ on $[\mathcal{C}_{\mathbb{F}}]$.** Now we are ready to prove the following theorem.

**Theorem 2.7.** *The surjection $[\mathcal{C}_{\mathbb{K}}] \twoheadrightarrow [\mathcal{C}_{\mathbb{F}}]$ intertwines the operator $e_j$ with $[E_j^{\mathbb{F}}]$, the operator $f_j$ with $[F_j^{\mathbb{F}}]$, and maps $\mathcal{F}_A$ onto $[\mathbb{F}S_n\text{-mod}_A]$, where $n = |A|$. In particular, $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A [\mathcal{C}_{\mathbb{F},A}]$ is a weight representation of $\tilde{\mathfrak{sl}}_p$.*

*Proof.* The proof is in several steps. Let $\rho : \mathcal{F} \twoheadrightarrow [\mathcal{C}_{\mathbb{F}}]$ denote the surjection.

*Step 1.* Let us show that $\rho(\mathcal{F}_A) = [\mathcal{C}_{\mathbb{F},A}]$. Since $\rho$ is a surjection, it is enough to show that $\rho(\mathcal{F}_A) \subset [\mathcal{C}_{\mathbb{F},A}]$. Pick $\lambda$ with $c(\lambda) = \tilde{A}$, where $\tilde{A} \mod p = A$. Then $\rho(|\lambda\rangle) = [M_{\lambda,\mathbb{F}}]$, where $M_{\lambda,R} \subset M_{\lambda,\mathbb{K}}$ is an $R$-form. For $P \in \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$, the polynomial $P(L_1,\ldots,L_n)$ acts on $M_{\lambda,\mathbb{K}}$ with the single eigenvalue $P(\tilde{A})$. So the same is true for $M_{\lambda,R}$ and hence for $M_{\lambda,\mathbb{F}}$. It follows that $M_{\lambda,\mathbb{F}} \in \mathbb{F}S_n\text{-mod}_A$.

*Step 2.* Set $f = \sum_j f_j, e = \sum_j e_j$ and let us show that $\rho \circ e = [E] \circ \rho, \rho \circ f = [F] \circ \rho$. To prove the former, note that, tautologically, $\operatorname{Res}^n_{n-1} M_R$ is an $R$-lattice in $\operatorname{Res}^n_{n-1} M_{\mathbb{K}}$ and hence $(\operatorname{Res}^n_{n-1} M)_{\mathbb{F}} = \operatorname{Res}^n_{n-1}(M_{\mathbb{F}})$. To prove $\rho \circ f = [F] \circ \rho$ note that $\operatorname{Ind}^{n-1}_n M_R$ is an $R$-lattice in $\operatorname{Ind}^{n-1}_n M_{\mathbb{K}}$.

*Step 3.* Let us prove that $\rho \circ e_i = [E_i] \circ \rho$. It is enough to prove that $\rho(e_i|\lambda\rangle) = [E_i](\rho|\lambda\rangle)$. Note that $e_i|\lambda\rangle$ coincides with the projection of $e|\lambda\rangle$ to $\mathcal{F}_{c(\lambda)\setminus\{i\}}$ (here we consider $c(\lambda)$ modulo $p$). From Step 1, it follows that $\rho(e_i|\lambda\rangle)$ coincides with the projection to $[\mathbb{F}S_n\text{-mod}_A]$ of $\rho(e|\lambda\rangle)$. By Step 2, $\rho(e|\lambda\rangle)$ equals the projection to $[\mathbb{F}S_n\text{-mod}_A]$ of $[E] \circ \rho(|\lambda\rangle)$. As we have seen above, the last projection coincides with $[E_i](\rho(|\lambda\rangle))$.

The proof of $\rho \circ f_i = [F_i] \circ \rho$ is similar. □

## 3. Action of $\hat{\mathfrak{sl}}_p$ on a category

Let $\mathbb{F}$ be a characteristic $p$ field and let $\mathcal{C}$ be an $\mathbb{F}$-linear abelian category. We suppose that all objects in $\mathcal{C}$ have finite length. The category $\mathcal{C} = \bigoplus_n \mathbb{F}S_n\text{-mod}$ is of this kind.

An action of $\hat{\mathfrak{sl}}_p$ on $\mathcal{C}$ is a collection of data together with four axioms. For us, the data is a pair of functors $E, F$ with fixed adjointness – $F$ is left adjoint to $E$ – as well as endomorphisms $X \in \operatorname{End}(E), T \in \operatorname{End}(E^2)$. The axioms are as follows:

(1) $F$ is isomorphic to a right adjoint of $E$ (and hence both $E, F$ are exact).

(2) $E = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} E_i$, where $E_i$ is the generalized eigenfunctor with eigenvalue $i$ for the action of $X$ on $E$. By the fixed adjointness, we get the decomposition $F = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} F_i$ so that $F_i$ is left adjoint to $E_i$.

(3) We have a weight decomposition $\mathcal{C} = \bigoplus_\nu \mathcal{C}_\nu$ such that the decomposition $[\mathcal{C}] = \bigoplus_\nu [\mathcal{C}_\nu]$ and the maps $[E_i], [F_i]$ define an integrable representation of $\hat{\mathfrak{sl}}_p$ on $[\mathcal{C}]$. Recall that a representation of $\hat{\mathfrak{sl}}_p$ is called *integrable* if the operators $e_i, f_i$ are locally nilpotent. Also note that, thanks to the weight decomposition of $\mathcal{C}$, $F_i$ is isomorphic to the right adjoint of $E_i$.

(4) The assignment $X_i \mapsto 1^{i-1}X1^{d-i}, T_i \mapsto 1^{i-1}T1^{d-1-i}$ lifts to an algebra homomorphism $\mathcal{H}(d) \to \operatorname{End}(E^d)$, where we write $\mathcal{H}(d)$ for the degenerate affine Hecke algebra.

We have already seen that we have a categorical $\hat{\mathfrak{sl}}_p$-action on $\bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod.

Let us make a couple of remark regarding this definition. First, it has a multiplicative version that will work for the categories of modules over the type A Hecke algebra (an interesting case is when $q$ is a root of 1). Second, we can extend this definition to other Lie algebras of type A. For example, to get a categorical action of $\mathfrak{sl}_2$ we need to require that $X$ acts on $E$ with a single eigenvalue and to modify (3) in an obvious way. In this way, a categorical action of $\hat{\mathfrak{sl}}_p$ gives rise to $p$ categorical actions of $\mathfrak{sl}_2$. It is possible to define categorical actions of Kac-Moody algebras outside of type A but this requires essentially new ideas. Finally, let us note that the functors $E, F$ are symmetric, i.e., we can have another categorical action with these functors swapped (for this we need, in particular, that the algebra $\mathcal{H}(d)$ is naturally identified with its opposite, which is left as an exercise).

## 4. Application: crystals

**4.1. $E_i$ and $F_i$ on irreducible objects.** We would like to understand the structure of $E_iL, F_iL$, where $L$ is a simple object in $\mathcal{C}$. Here we have the following result due to Chuang and Rouquier (who have introduced the notion of a categorical $\mathfrak{sl}_2$-action).

**Proposition 4.1.** *The following is true.*

(1) *Suppose $E_iL \neq \{0\}$. Then the head (the maximal semisimple quotient) and the socle (the maximal semisimple sub) of $E_iL$ are simple and isomorphic (let's denote this simple object by $\tilde{e}_iL$).*

(2) *Let $d$ be the maximal number such $E_i^dL \neq 0$. Then $e_i[L] = d[\tilde{e}_iL] + \sum_{L_0}[L_0]$, where the sum is taken over simples $L_0$ with $E_i^{d-1}L_0 = 0$.*

*The similar results also hold for $F_iL$ (in particular, we get the simple/head socle of $F_iL$ to be denoted by $\tilde{f}_iL$).*

If $E_iL = 0$ (resp., $F_iL = 0$), then we set $\tilde{e}_iL = 0$ (resp., $\tilde{f}_iL$). So we get a collection of maps $\tilde{e}_i, \tilde{f}_i : \operatorname{Irr}(\mathcal{C}) \to \operatorname{Irr}(\mathcal{C}) \sqcup \{0\}$. A nice and very useful exercise is to check that if $\tilde{e}_iL \neq 0$, then $\tilde{f}_i\tilde{e}_iL = L$.

Proposition 4.1 implies, in particular, that the classes $[L], L \in \operatorname{Irr}(\mathcal{C})$, form a so called *perfect basis* (as defined by Berenstein and Kazhdan). This implies that the maps $\tilde{e}_i, \tilde{f}_i$ endow $\operatorname{Irr}(\mathcal{C})$ with a crystal structure (a crystal is a combinatorial shadow of a Lie algebra action first constructed by Kashiwara using quantum groups).

**4.2. Irreducibility of $\bigoplus_n[\mathbb{F}S_n$-mod].** Using Proposition 4.1, we will show that the $\hat{\mathfrak{sl}}_p$-module $\bigoplus_n[\mathbb{F}S_n$-mod] is irreducible (and hence it is the irreducible highest weight module of weight $\omega_0$, a.k.a., the basic representation of $\hat{\mathfrak{sl}}_p$).

**Theorem 4.2.** *The $\hat{\mathfrak{sl}}_p$-module $V := \bigoplus_n [\mathbb{F}S_n\text{-mod}]$ is irreducible.*

*Proof.* The module $V$ is a quotient of $\mathcal{F}$ and so is an integrable highest weight representation. Such a representation is irreducible if and only if it has a unique *singular* (=annihilated by all $e_i$) vector $v$. One such vector is $[\mathbb{C}] \in [\mathbb{F}S_0\text{-mod}]$. Moreover, the space $V^0$ of singular vectors does not contain any other vector of the form $[L]$. Indeed, $\sum_i e_i[L] = [EL]$, but $EL = \mathrm{Res}^n_{n-1} L$ is nonzero if $L \not\in [\mathbb{F}S_0\text{-mod}]$. The following lemma combined with Proposition 4.1 implies that $V^0$ is spanned by vectors of the form $[L], L \in \mathrm{Irr}(\mathcal{C})$. This completes the proof. □

**Lemma 4.3.** *Let $V$ be an $\hat{\mathfrak{sl}}_p$-module and let $\mathcal{B}$ be a basis with the following property. Let $d_i(b)$ denote the maximal number $d$ such that $e_i^d b \neq 0$. For any $b \in \mathcal{B}, i \in \mathbb{Z}/p\mathbb{Z}$, we have that either $e_i b = 0$ or $e_i b = \alpha \tilde{e}_i b + \sum_{b_0} n_{b_0} b_0$, where $\alpha \neq 0$, if $n_{b_0} \neq 0$, then $e_i^{d_i(b)-1} b_0 = 0$, and $\tilde{e}_i b \in \mathcal{B}$. Assume further that $\tilde{e}_i b_1 = \tilde{e}_i b_2 \neq 0$ implies $b_1 = b_2$. Then the space of singular vectors $V^0$ is spanned by $V^0 \cap \mathcal{B}$.*

*Proof.* Pick $v \in V^0$ and expand it in the basis $\mathcal{B}$, $v = \sum_{b \in \mathcal{B}'} n_b b$, where $n_b \neq 0$ for all $b \in \mathcal{B}'$. Let $d := \max\{d_i(b) | b \in \mathcal{B}'\}$. Let $\mathcal{B}'_i := \{b \in \mathcal{B}' | d_i(b) = d\}$. Assume $d > 0$. Then

$$0 = e_i^d v = \sum_{b \in \mathcal{B}'_i} m_b \tilde{e}_i^d b,$$

where all $m_b$'s are nonzero and all $\tilde{e}_i^d b$ are distinct. We get a contradiction that finishes the proof. □