Categorical actions, III

1) Reminder: cat-y $\mathcal{O}$ for $\mathfrak{gl}_n$.

2) Categorical action

3) Parabolic versions

1) $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{C})$. Consider "integral part" of category $\mathcal{O}$. (before we were dealing w. s/simple $\mathfrak{g}$ but $\mathfrak{g}=\mathfrak{gl}_n$ is not very different)

$\mathcal{O}=\{M\in U(\mathfrak{g})\text{-mod}:$ fin gen
$\mathfrak{h}$ acts diagon. w. integral eigenvalues
$\mathfrak{n}$ acts locally nilpotently$\}$

Ex: $\lambda=(\lambda_1,\dots\lambda_n)\in\mathbb{Z}^n \rightsquigarrow \mathbb{C}_\lambda\in U(\mathfrak{b})\text{-mod}$, $\mathfrak{b}=\left\{\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\right\}$

$\rightsquigarrow \Delta(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_\lambda$

$\exists!$ irred $L(\lambda)\twoheadleftarrow\Delta(\lambda)$ & $\mathbb{Z}^n\xrightarrow{\sim}\mathrm{Irr}(\mathcal{O})$, $\lambda\mapsto L(\lambda)$

$[\mathcal{O}]=(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$, where $\mathbb{C}^{\mathbb{Z}}$ is a vector space w. basis $v_i, i\in\mathbb{Z}$

$[\Delta(\lambda)]\longmapsto v_{\lambda_1}\otimes v_{\lambda_2-1}\otimes\dots\otimes v_{\lambda_n+1-n}$

Reason for shift: $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha=\frac{1}{2}\sum_{i<j}(\epsilon_i-\epsilon_j)=(\frac{n-1}{2},\frac{n-3}{2},\dots,\frac{1-n}{2})$ - up to $-\frac{n-1}{2}(\epsilon_1+\dots+\epsilon_n)$ get $(0,-1,\dots,1-n)$ - same values on roots as ρ. So we redefine ρ as $(0,-1,\dots,1-n)$

Blocks: Define equiv. on $\mathbb{Z}^n$ by $\lambda\sim\mu$ if $\lambda+\rho\sim_{S_n}\mu+\rho$. Then $\mathcal{O}=\bigoplus_{\alpha\in\mathbb{Z}^n/\sim}\mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is the Serre span of $L(\lambda)$, $\lambda\in\alpha$

$e_i v_j=\delta_{ij}v_{j+1}$, $f_i v_j=\delta_{i,j+1}v_j$

Observation: $\mathfrak{sl}_\infty\curvearrowright\mathbb{C}^{\mathbb{Z}}$ (tautological representation) $\rightsquigarrow\mathfrak{sl}_\infty\curvearrowright(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$

Want to show that $\mathcal{O}$ carries a categorical $\mathfrak{sl}_\infty$-action categorifying that on $(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$.

Rmk: All objects in $\mathcal{O}$ have finite length.

2) Recall that we need: functors $E,F\curvearrowright\mathcal{O}$ w. fixed one-sided adjunction and endomorphisms $X\in\mathrm{End}(E)$, $T\in\mathrm{End}(E^2)$ subject to:

(1) E,F are biadjoint

(2) E-decomp-n of E w.r.t. X looks like $E=\bigoplus_{i\in\mathbb{Z}}E_i$

by fixed adjointness, have $F = \bigoplus_{i\in\mathbb{Z}} F_i$

(3) There's decomp-n $\mathcal{O} = \bigoplus_\nu \mathcal{O}_\nu$ s.t. $[\mathcal{O}] = \bigoplus_\nu [\mathcal{O}_\nu]$ is the weight decomp-n of $(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$ for the $\mathfrak{sl}_\infty$-action, and $[E_i], [F_i]$ coincide w. e_i, f_i (defined by $e_i . v_{i_1}\otimes\ldots\otimes v_{i_n} = \sum_{j=1}^n \delta_{i i_j} v_{i_1}\otimes\ldots\otimes v_{i+1}\otimes\ldots\otimes v_{i_n}$, $f_i . v_{i_1}\otimes\ldots\otimes v_n = \sum_{j=1}^n \delta_{i, i_j+1} v_{i_1}\otimes\ldots\otimes v_{i-1}\otimes\ldots\otimes v_{i_n}$).

(4) The assignment $X_i \mapsto 1^{i-1}X1^{d-i}$, $T_i \mapsto 1^{i-1}T1^{d-i-1}$ extends to an algebra homomorphism $\mathcal{H}(d) \longrightarrow \mathrm{End}(E^d)$

2.1) Data: $E(M) = \mathbb{C}^n \otimes M$ ($\mathbb{C}^n$ - tautol. $\mathfrak{g}$-module), $F(M) = (\mathbb{C}^n)^* \otimes M$

$X_M(v\otimes m) = \sum_{i,j=1}^n E_{ij}v \otimes E_{ji}m$, $T_M(v\otimes v'\otimes m) = v'\otimes v\otimes m$

tensor Casimir

Axiom 1 is clear, Axiom 4 is Problem 4 in HW2

2.2) Objects $E\Delta(\lambda)$, $F\Delta(\lambda)$

To check axioms 2, 3 we compute $E\Delta(\lambda)$, $F\Delta(\lambda)$ and how X acts on these objects.

Let V be a finite dimensional GL_n-module w. weight basis $v_1 \ldots v_m$ w. weights $\nu_1 \ldots \nu_m$ ordered in non-decreasing way. Then recall (Prop 2.1 in Lecture 10) that $V\otimes\Delta(\lambda)$ is filtered w. successive quotients $\Delta(\lambda+\nu_i)$ (ordered bottom to top, e.g. $\Delta(\lambda+\nu_1)$ is a sub and $\Delta(\lambda+\nu_m)$ is a quotient)

Ex: $V = \mathbb{C}^n$, then get weights $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_n$ and filtr-n is by $\Delta(\lambda+\epsilon_1), \Delta(\lambda+\epsilon_2), \ldots \Delta(\lambda+\epsilon_n)$ $(\epsilon_i = (\delta_{1i}, \ldots \delta_{ni}))$

$V = (\mathbb{C}^n)^* \rightsquigarrow -\epsilon_n > -\epsilon_{n-1} > \ldots > -\epsilon_1$, filtr-n by $\Delta(\lambda-\epsilon_n), \ldots \Delta(\lambda-\epsilon_1)$

Prop: X_M preserves the filtr-n on $E\Delta(\lambda)$ and acts on the subquotient $\Delta(\lambda+\epsilon_i)$ by $\lambda_i + 1 - i$

Proof: The first claim follows from $\mathrm{Hom}(\Delta(\lambda+\epsilon_i), \Delta(\lambda+\epsilon_j)) = 0$ (this is because $\lambda+\epsilon_i > \lambda+\epsilon_j$). To deduce it is left as an exercise

Any endomorphism of a Verma module acts on it by scalar. To determine the scalar, we need to compare the tensor Casimir $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} \in U(\mathfrak{g})\otimes U(\mathfrak{g})$ with the usual Casimir $\sum_{i,j=1}^{n} E_{ij}E_{ji} \, (=C) \in U(\mathfrak{g})$. Let $\delta: U(\mathfrak{g}) \to U(\mathfrak{g})\otimes U(\mathfrak{g})$ be the coproduct. Then $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} = \frac{1}{2}(\delta(C) - C\otimes 1 - 1\otimes C)$. ~~The element C acts on C~~ We can rewrite C as $\sum_{j<i} 2E_{ij}E_{ji} + \sum_{j<i}[E_{ji}, E_{ij}] + \sum_{i=1}^{n} E_{ii}^2$

So $C|_{\Delta(\mu)} = \sum_{i=1}^{n}(n+1-2i)\mu_i + \sum_{i=1}^{n}\mu_i^2$

The element $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} = \frac{1}{2}(\delta(C) - C\otimes 1 - 1\otimes C)$ acts on the subquotient $\Delta(\lambda+\epsilon_i)$ by $\frac{1}{2}(C|_{\Delta(\lambda+\epsilon_i)} - C|_{L(\epsilon_1)} - C|_{\Delta(\lambda)}) =$

$= \frac{1}{2}(n+1-2i+2\lambda_i+1-n) = \lambda_i + i - 1$ □

Cor: Axiom 2 holds

Proof: We only need to check that eigenvalues of X_M on EM are integral for every simple M. Since every simple is a quotient of some $E\Delta(\lambda)$. □

Cor: Axiom 3 holds

Proof: Proposition shows that $[E_j\Delta(\lambda)] = e_j[\Delta(\lambda)]$. Let's show the claim about F_j's. Let α be the equivalence class of λ and α' be the class where one entry j is replaced with $j+1$. For $M \in \mathcal{O}_\alpha$ we have $E_j M = \pi_{\alpha'} \circ EM$. Using adjointness we deduce that, for $N \in \mathcal{O}_{\alpha'}$, we have $F_j M = \pi_\alpha \circ FM$. The claim that $[F_j\Delta(\lambda)] = f_j\Delta(\lambda)$ easily follows from here □

2.3*) Crystals. Here we determine operators $\tilde{e}_j, \tilde{f}_j$ for $Irr(\mathcal{O})$. Take $\lambda \in \mathbb{Z}^n$ and write $\lambda+\rho = (\lambda_1, \lambda_2-1, \dots, \lambda_n+1-n)$. To each entry $j+1$ we assign bracket (, and to each j we assign). For example, $j=3$, $\lambda+\rho = (3,4,4,5,3,2,4)$. Then we cancel all brackets that

) (( ) (← are correct

or ()(()))($\rightsquigarrow$)(. It's the standard fact that the result doesn't depend on the order of cancellations. We end up with a sequence like)...)(...(. To define $\tilde{e}_j L(\lambda)$ we switch the rightmost) to (and set $\tilde{e}_j L(\lambda) = L(\lambda')$, where $\lambda' = \lambda + \epsilon_k$, k is the position where the switch occurred. If there are no ")", then we set $\tilde{e}_j L(\lambda) = 0$. To compute $\tilde{f}_j L(\lambda)$ we switch the left-most (to) and set $\tilde{f}_j L(\lambda) = L(\lambda'')$, $\lambda'' = \lambda - \epsilon_\ell$ (or $\tilde{f}_j L(\lambda) = 0$ if there is no (. In the example above, $\lambda' + \rho = (4,4,4,5,3,2,4)$, $\lambda'' + \rho = (3,3,4,5,3,2,4)$

3) Parabolic categories

This is a generalization of $\mathcal{O}$. Pick positive integers $n_1 \dots n_k$ w. $n_1 + \dots + n_k = n$ and denote $(n_1 \dots n_k) = \underline{n}$. We introduce some notations: Let L denote the subgroup of GL_n consisting of all block diagonal matrices, where blocks have sizes $n_1 \dots n_k$. Let $\mathfrak{m}$ (resp $\mathfrak{m}^-$) denote the subalgebra of all strictly upper triangular (resp. lower triangular) block matrices. For example, take $\underline{n} = (1,2,1)$.

Then $L = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$, $\mathfrak{m} = \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$ $\mathfrak{m}^- = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \right\}$.

~~Consider~~ Let $\mathfrak{l}$ denote the Lie algebra of L and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$, this is a Lie subalgebra.

Now consider the category $\mathcal{O}^{\underline{n}} = \{ M \in U(\mathfrak{g})\text{-mod} \mid$ fin. gen.

$\mathfrak{l}$-action is loc. finite and integrates to L

$\mathfrak{m}$ acts locally nilpotently$\}$

An example of an object in $\mathcal{O}^{\underline{n}}$ is provided by a parabolic Verma module. Namely pick an irreducible representation of L, say V and set $\Delta^{\underline{n}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$. Note that the irreducibles L are labeled by highest weights $(\lambda_1 \dots \lambda_n)$ subject to $\lambda_1 \geq \dots \geq \lambda_{n_1}, \lambda_{n_1+1} \geq \dots \geq \lambda_{n_2}, \dots, \lambda_{n_1 + \dots + n_{k-1}+1} \geq \dots \geq \lambda_n$. For such λ we write $\Delta^{\underline{n}}(\lambda)$ instead of $\Delta^{\underline{n}}(V)$.

Note that $\mathcal{O}^{\underline{n}} \subset \mathcal{O}$. Let's determine $[\mathcal{O}^{\underline{n}}] \subset [\mathcal{O}] = (\mathbb{C}^{\mathbb{Z}})^{\otimes n}$. It's easy to see that the classes $[\Delta^{\underline{n}}(\lambda)]$ constitute a basis in $[\mathcal{O}^{\underline{n}}]$ (compare to the analogous claim for $\mathcal{O}$). ~~The Weyl character formula for the an irreducible module $V(\lambda)$ of L together shows that~~ ~~$[V(\lambda)] =$~~ The Weyl character formula (for L) implies the following (exercise)

$$[\Delta^{\underline{n}}(\lambda)] = (v_{\lambda_1} \wedge \ldots \wedge v_{\lambda_{n_1}+1-n_1}) \otimes (v_{\lambda_{n_1+1}-n_1} \wedge \ldots \wedge v_{\lambda_{n_2}+1-n_2}) \otimes \ldots$$

Thm: $\mathcal{O}^{\underline{n}} \subset \mathcal{O}$ is a categorical $\mathfrak{sl}_\infty$-representation with
$[\mathcal{O}^{\underline{n}}] = \Lambda^{n_1}\mathbb{C}^{\mathbb{Z}} \otimes \Lambda^{n_2}\mathbb{C}^{\mathbb{Z}} \otimes \ldots \otimes \Lambda^{n_k}\mathbb{C}^{\mathbb{Z}}$.

We don't provide the proof.