LECTURE 3: REPRESENTATION THEORY OF $\mathrm{SL}_2(\mathbb{C})$ AND $\mathfrak{sl}_2(\mathbb{C})$

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Introduction

We proceed to studying the representation theory of algebraic groups and Lie algebras. Algebraic groups are the groups defined inside $\mathrm{GL}_n(\mathbb{F})$ by polynomial equations, such as $\mathrm{SL}_n(\mathbb{F}), \mathrm{O}_n(\mathbb{F}), \mathrm{Sp}_n(\mathbb{F})$. We are interested in their representations with polynomial matrix coefficients. Often, this problem is reduced to studying representations of Lie algebras (some sort of linearization). The connection between representations of algebraic groups and their Lie algebras is very tight in characteristic 0 but is much more loose in characteristic p, as we will see in the next lecture.

This lecture consists of two parts. First, we define algebraic groups, their Lie algebras, representations of both and connections between those. In the second part we fully classify the representations of $\mathrm{SL}_2(\mathbb{C})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

1. Algebraic groups and their Lie algebras

1.1. **Algebraic groups.** Let $\mathbb{F}$ be an algebraically closed field. By a (linear) *algebraic group* G we mean a subgroup of $\mathrm{GL}_n(\mathbb{F}) = \{A \in \mathrm{Mat}_n(\mathbb{F}) | \det A \neq 0\}$ defined by polynomial equations[1]. Examples of algebraic groups include $\mathrm{GL}_n(\mathbb{F})$ itself or $\mathrm{SL}_n(\mathbb{F}) = \{A \in \mathrm{GL}_n(\mathbb{F}) | \det A = 1\}$. To get further examples, pick a non-degenerate symmetric or skew-symmetric form B on $\mathbb{F}^n$ and consider the subgroup $G_B = \{A \in \mathrm{GL}_n(\mathbb{F}) | B(Au, Av) = B(u, v), \forall u, v \in \mathbb{F}^n\}$. If J is the matrix of this form in some fixed basis, then $G = \{A \in \mathrm{GL}_n(\mathbb{F}) | A^t J A = J\}$. This group is denoted by $\mathrm{O}_n(\mathbb{F})$ if B is symmetric, and by $\mathrm{Sp}_n(\mathbb{F})$ if B is skew-symmetric (note that in this case n is even, because we assume B is non-degenerate).

By a polynomial function on an algebraic group $G \subset \mathrm{GL}_n(\mathbb{F})$ we mean a polynomial in matrix coefficients and $\det^{-1}$. The polynomial functions on G form an algebra to be denoted by $\mathbb{F}[G]$. By a homomorphism of algebraic groups $G \to G'(\subset \mathrm{GL}_n(\mathbb{F}))$, we mean a group homomorphism whose matrix coefficients are polynomial functions. By an isomorphism of algebraic groups, we mean a homomorphism that has an inverse that is also a homomorphism. This allows to consider algebraic groups regardless their embeddings to $\mathrm{GL}_n(\mathbb{F})$ (there is also an internal definition: a linear algebraic group is an affine algebraic variety that is a group such that the group operations are morphisms of algebraic varieties).

We want to study representations $G \to \mathrm{GL}_N(\mathbb{F})$ that are homomorphisms of algebraic groups (they are traditionally called *rational*). Examples are provided by the representation of G in $\mathbb{F}^n$, its dual, their tensor products, subs and quotients of those.

1.2. **Lie algebras.** Let $G \subset \mathrm{GL}_n(\mathbb{F})$ be an algebraic group. Consider $\mathfrak{g} := T_1 G$, the tangent space to G at $1 \in G$, this is a subspace in $T_1 \mathrm{GL}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$. It consists of the tangent vectors to curves $\gamma(t)$ in G with $\gamma(0) = 1$. When $\mathbb{F} = \mathbb{C}$ we can take γ to be a map from a small interval in $\mathbb{R}$ containing 0. In general, we can take some "formal curve", a formal power series $1 + A_1 t + A_2 t^2 + \ldots$ with $A_i \in \mathrm{Mat}_n(\mathbb{F})$ satisfying the defining equations of G.

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[1] We consider algebraic groups with the reduced scheme structure

Such a curve exists for every $A_1 \in \mathfrak{g}$ because every algebraic group is a smooth algebraic variety.

Let us compute the tangent spaces for the groups $\mathrm{SL}_n(\mathbb{F}), \mathrm{O}_n(\mathbb{F}), \mathrm{Sp}_n(\mathbb{F})$. In the case of $\mathrm{SL}_n(\mathbb{F})$, we get $\mathfrak{g} = \{x \in \mathrm{Mat}_n(\mathbb{F}) | \operatorname{tr} x = 0\}$, this space is usually denoted by $\mathfrak{sl}_n(\mathbb{F})$. For $G_B = \mathrm{O}_n(\mathbb{F})$ or $\mathrm{Sp}_n(\mathbb{F})$, we get $\mathfrak{g}_B = \{x \in \mathrm{Mat}_n(\mathbb{F}) | B(xu, v) + B(u, xv) = 0, \forall u, v \in \mathbb{F}^n\}$. When B is orthogonal (resp., symplectic), this space is denoted by $\mathfrak{so}_n(\mathbb{F})$ (resp., $\mathfrak{sp}_n(\mathbb{F})$).

All these spaces have an interesting feature, they are closed with respect to the commutator of matrices, $[x, y] := xy - yx$. This is a general phenomenon: if G is an algebraic group, then $\mathfrak{g}$ is closed with respect to $[\cdot, \cdot]$. The reason is that if $\gamma(t), \eta(s)$ are two curves with $\gamma(t) = 1 + tx + \ldots, \eta(s) = 1 + sy + \ldots$, then the group commutator $\gamma(t)\eta(s)\gamma(t)^{-1}\eta(s)^{-1}$ expands as $1 + ts[x, y] + \ldots$, where "$\ldots$" denotes the terms of order 3 and higher (note that all of them include both t and s).

The commutator on $\mathrm{Mat}_n(\mathbb{F})$ is bilinear and satisfies the following two important identities

$$(1.1) \qquad\qquad\qquad\qquad\qquad [x, x] = 0$$

(this implies that the commutator is skew-symmetric) and the *Jacobi identity*

$$(1.2) \qquad\qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

This motivates the following definition.

Definition 1.1. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (1.1) and (1.2).

As we have seen, for any algebraic group $G \subset \mathrm{GL}_n(\mathbb{F})$, the tangent space $T_1 G$ is a Lie algebra.

The notions of a Lie algebra homomorphism, product of Lie algebras, subalgebras, quotient algebras, ideals are introduced in a standard way.

1.3. **Correspondence between representations.** Now let $\Phi : G \to G'$ be an algebraic group homomorphism. The description of the bracket using the expansion of the commutator in the group shows that the tangent map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras.

We write $\mathfrak{gl}(V)$ for $\mathrm{End}(V)$ when view it as a Lie algebra. By a representation of an arbitrary Lie algebra $\mathfrak{g}$ one means a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. We can also view a representation as a bilinear map $\mathfrak{g} \times V \to V, (x, v) \mapsto x \cdot v$ satisfying $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$. In this case, we call V a $\mathfrak{g}$-module.

Example 1.2. Consider the group $\mathrm{GL}_1(\mathbb{F})$ that coincides with the multiplicative group $\mathbb{F}^\times$ of $\mathbb{F}$. The corresponding Lie algebra is just $\mathbb{F}$ with zero bracket (the one-dimensional abelian Lie algebra). Consider the one-dimensional representations of $\mathbb{F}$ and of $\mathbb{F}^\times$. In the former case, a representation is given by multiplying by an arbitrary $z \in \mathbb{F}$. In the group case, a representation is given by sending $z \in \mathbb{F}^\times$ to $f(z) \in \mathbb{F}^\times$ such that $f(zz') = f(z)f(z'), f(1) = 1$. A function $f(z)$ is a polynomial on $\mathbb{F}^\times$, i.e., a Laurent polynomial. It has no nonzero roots, so it has to be $\alpha z^n, n \in \mathbb{Z}, \alpha \in \mathbb{F}^\times$. Clearly, $\alpha = 1$. The corresponding representation of $\mathbb{F}$ is given by the multiplication by n.

Example 1.3. Let $G \subset \mathrm{GL}_n(\mathbb{F})$. Then $\mathfrak{gl}_n(\mathbb{F})$ carries a representation of G given by $g.x = gxg^{-1}$. The subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ is a subrepresentation. The representation of G in $\mathfrak{g}$ is called the *adjoint representation*. Note that $g.[x, y] = [g.x, g.y]$, in other words, G acts by automorphisms of the Lie algebra $\mathfrak{g}$. The corresponding representation of $\mathfrak{g}$ (also called adjoint and denoted by ad) is given by $\mathrm{ad}(y)x = [y, x]$.

Recall that, for rational representations of G, we can take tensor products and dual representations. This corresponds to taking tensor products and duals of Lie algebra representations. For example, if U, V are representations of G, then the representation of G in $U \otimes V$ is defined by $g \cdot (u \otimes v) = (g \cdot u) \otimes (g \cdot v)$. We plug $\gamma(t) = 1 + tx + \dots$ for g, differentiate, and set $t = 0$ to get $x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v)$. We take this formula for the definition of a $\mathfrak{g}$-action on $U \otimes V$. For similar reasons, we define a $\mathfrak{g}$-module structure on V^* by $(x \cdot \alpha)(v) := -\alpha(x \cdot v)$. Finally, if U, V are $\mathfrak{g}$-modules, then we define a $\mathfrak{g}$-module structure on $\mathrm{Hom}(U, V)$ by $(x \cdot \varphi)(u) = x \cdot \varphi(u) - \varphi(x \cdot u)$. Using these constructions we can produce new representations of Lie algebras from existing ones.

Now let us consider the following questions. Given a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ (e.g., a representation), is there a homomorphism $\Phi : G \to G'$, whose tangent map is φ? In this case we say that Φ integrates φ. If so, is Φ unique? Here is a place, where the answer crucially depends on $\mathrm{char}\,\mathbb{F}$. In the next lecture, we will see that if $\mathrm{char}\,\mathbb{F} > 0$, then the answers to both questions are "no" (we already have seen some of this in Example 1.2).

Proposition 1.4. *Let* $\mathrm{char}\,\mathbb{F} = 0$ *and let* G *be connected. Then the following is true.*
  (1) *If* Φ *exists, then it is unique.*
  (2) *If* G *is simply connected and* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ *(the right had side denotes the linear span of all elements of the form* $[x, y]$, *where* $x, y \in \mathfrak{g}$), *then* Φ *exists.*

Proof/discussion. We will explain how (1) is proved (for $\mathbb{F} = \mathbb{C}$) and give a short discussion of (2) (explaining what "connected" and "simply connected" mean for general $\mathbb{F}$).

Let $\mathbb{F} = \mathbb{C}$. We have a distinguished map $\gamma : \mathbb{R} \to G$ with $\gamma(t) = 1 + tx + \dots$, it is given by $\gamma(t) = \exp(tx)$. In fact, it is the only differentiable group homomorphism with given tangent vector x. By this uniqueness, $\Phi(\exp(tx)) = \exp(t\varphi(x))$. The group G is connected, so elements of the form $\exp(tx)$ generate G. This shows (1).

Now let us discuss (2) for $\mathbb{F} = \mathbb{C}$. Every algebraic group is also a complex Lie group, i.e., a group, which is a complex manifold, so that the group operations are complex analytic. A general result from the theory of Lie groups (based on the existence/uniqueness of solutions of ODE's) shows that there is a complex Lie group homomorphism $\Phi : G \to G'$ integrating φ. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then Φ is a morphism of algebraic varieties, see [OV, Section 3.4].

For an arbitrary linear algebraic group G over an algebraically closed field $\mathbb{F}$, we say that G is connected if it is irreducible as an algebraic variety. We say that G is simply connected if there is no surjective non-bijective algebraic group homomorphism $\tilde{G} \to G$. In this setting (2) is proved using the structure theory of algebraic groups. $\qquad\square$

Now let us explain why conditions in (2) are necessary producing counter-examples when either of the two conditions in (2) fails. Consider $G = \mathrm{SO}(3)$, the special orthogonal group $\mathrm{SO}(n)$, by definition, is $\mathrm{O}(n) \cap \mathrm{SL}(n)$. Then $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$. The 2-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ given by the natural inclusion $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{gl}_2(\mathbb{C})$ does not integrate to $\mathrm{SO}_3(\mathbb{C})$.

On the other hand, one can consider $G = \mathbb{C}$, the additive group of $\mathbb{C}$. The only one dimensional representation that integrates to a rational representation of G is the zero one.

Remark 1.5. Let $\mathrm{char}\,\mathbb{F} = 0$. Let V, V' be two representations of G and $\psi : V \to V'$ a homomorphism of representations of $\mathfrak{g}$. Then ψ is a homomorphism of representations of G provided G is connected. This is proved similarly to part (1) of the previous proposition provided $\mathbb{F} = \mathbb{C}$. In particular, if V is completely reducible over $\mathfrak{g}$, then it is completely reducible over G.

2. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

2.1. **Universal enveloping algebras.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. As with finite groups, we would like to have an associative algebra whose representation theory is the same as that of $\mathfrak{g}$. Such an algebra is called the *universal enveloping algebra*, it is defined as

$$U(\mathfrak{g}) := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x,y] | x, y \in \mathfrak{g}),$$

where we write $T(\mathfrak{g})$ for the tensor algebra of $\mathfrak{g}$ viewed as a vector space. There is a natural Lie algebra homomorphism $\iota : \mathfrak{g} \to U(\mathfrak{g})$ that has the following universal property: if A is an associative unital algebra and $\varphi : \mathfrak{g} \to A$ is a Lie algebra homomorphism, then there is unique associative unital algebra homomorphism $\psi : U(\mathfrak{g}) \to A$ with $\varphi = \psi \circ \iota$. In particular, a $U(\mathfrak{g})$-module is the same thing as a $\mathfrak{g}$-module.

The algebra $U(\mathfrak{g})$ is infinite dimensional if $\mathfrak{g} \neq \{0\}$. We can describe a basis in $U(\mathfrak{g})$ as follows. Choose a basis $x_1, \ldots, x_n$ in $\mathfrak{g}$ (we assume that $\mathfrak{g}$ is finite dimensional, using infinite indexing sets we can cover the case when $\mathfrak{g}$ is infinite dimensional). It is easy to see that the ordered monomials $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n} \in U(\mathfrak{g})$ span $U(\mathfrak{g})$ as a vector space. The following claim is known as the Poincare-Birkhoff-Witt (shortly, PBW) theorem.

Proposition 2.1. *The elements $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$, where $d_1, \ldots, d_n \in \mathbb{Z}_{\geqslant 0}$, form a basis in $U(\mathfrak{g})$.*

For example, when $\mathfrak{g} = \mathfrak{sl}_2$, we have a basis $e^k h^\ell f^m \in U(\mathfrak{g})$.

There is no easy multiplication rule for the basis elements. An easy general observation is that $(x_1^{d_1} \ldots x_n^{d_n}) \cdot (x_1^{e_1} \ldots x_n^{e_n}) = x_1^{d_1+e_1} \ldots x_n^{d_n+e_n} + \ldots$, where "$\ldots$" means a linear combination of monomials of total degree less then $\sum_{i=1}^n (d_i + e_i)$.

Now let $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. The universal property of $U(\mathfrak{h})$ and a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g} \to U(\mathfrak{g})$ give rise to an associative algebra homomorphism $U(\mathfrak{h}) \to U(\mathfrak{g})$. By the PBW theorem, this homomorphism is injective. So we can view $U(\mathfrak{h})$ as a subalgebra in $U(\mathfrak{g})$.

2.2. **Classification of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.** First, we will need a lemma from linear algebra.

Lemma 2.2. *Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. Let $A, B \in \mathrm{End}(V)$ be such that $[A, B] = zB$ for $z \neq 0$. Further, for $a \in \mathbb{F}$, let V_a denote the generalized eigenspace for A in V with eigenvalue a. Then the following is true.*

(1) $B(V_a) \subset V_{a+z}$ *for all a.*
(2) *If $\mathsf{char}(\mathbb{F}) = 0$, then B is nilpotent and there is an eigenvector v for A such that $Bv = 0$.*

Proof. We have $(A - (a+z)\,\mathrm{id})(Bv) = B(A - a\,\mathrm{id})v$ that implies (1). If $\mathsf{char}(\mathbb{F}) = 0$, then all numbers $a + nz, n \in \mathbb{Z}_{\geqslant 0}$, are different. Since we have only finitely many eigenvalues for A in V, (2) follows. □

Let V be a finite dimensional representation of $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$. Taking $A = h$ and $B = e$ so that $[h, e] = 2e$ we get the following corollary.

Corollary 2.3. *There is a nonzero vector $v \in V$ such that $hv = zv, ev = 0$ for some $z \in \mathbb{C}$.*

Now let us introduce Verma modules. Pick $z \in \mathbb{C}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the subspace with basis h, e, this is a subalgebra. Consider the $\mathfrak{b}$-module $\mathbb{C}_z$ that is $\mathbb{C}$ as a vector space, h acts by

z and e acts by 0. We view $\mathbb{C}_z$ as a $U(\mathfrak{b})$-module. As was mentioned before, we can view $U(\mathfrak{b})$ as a subalgebra in $U(\mathfrak{g})$. Then set

$$\Delta(z) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_z = U(\mathfrak{g})/U(\mathfrak{g}) \operatorname{Span}_{\mathbb{C}}(h - z, e).$$

This is a left $U(\mathfrak{sl}_2)$-module. The reason why we need this is that, for any $U(\mathfrak{g})$-module V, we have

$$\begin{aligned}
(2.1) \qquad \operatorname{Hom}_{U(\mathfrak{g})}(\Delta(z), V) &= \operatorname{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_z, V) = \operatorname{Hom}_{U(\mathfrak{b})}(\mathbb{C}_z, V) = \\
&= \{v \in V \mid hv = zv, ev = 0\}.
\end{aligned}$$

So if V is irreducible and there is $v \in V, v \neq 0$ with $hv = zv, ev = 0$, then there is a nonzero homomorphism $\Delta(z) \to V$ and hence V is a quotient of $\Delta(z)$. So we need to understand the structure of $\Delta(z)$.

As a right $U(\mathfrak{b})$-module, $U(\mathfrak{g})$ has basis $f^n, n \in \mathbb{Z}_{\geqslant 0}$. It follows that $\Delta(z)$ has basis $f^n v_z$, where v_z is the image of $1 \in U(\mathfrak{g})$ in $\Delta(z)$. The action of f, h, e on this basis is given as follows

$$(2.2) \qquad f \cdot f^n v_z = f^{n+1} v_z, \quad h f^n v_z = (z - 2n) f^n v_z, \quad e f^n v_z = (z - n + 1) n f^{n-1} v_z.$$

The first equality is clear, the second is easy and the last one follows from the identity

$$(2.3) \qquad e f^n = f^{n-1}(h + 1 - n) + f^n e.$$

Lemma 2.4. *The module $\Delta(z)$ is irreducible if and only if $z \notin \mathbb{Z}_{\geqslant 0}$. If $z \in \mathbb{Z}_{\geqslant 0}$, then $\Delta(z)$ has a unique proper submodule and the quotient by this submodule is finite dimensional.*

Proof. Let U be a submodule of $\Delta(z)$. Since $\Delta(z)$ splits into the direct sum of eigenspaces for h, so does U. So $U = \operatorname{Span}(f^n v_z \mid n \geqslant m)$ for some $m > 0$. From the third equation in (2.2), we conclude that $z = m - 1$. All claims of the lemma follow easily from here. $\qquad \square$

Remark 2.5. Note that the kernel $\Delta(n) \twoheadrightarrow L(n)$ is $\Delta(-2 - n)$.

Proposition 2.6. *There is a bijection $\mathbb{Z}_{\geqslant 0} \xrightarrow{\sim} \operatorname{Irr}_{fin}(\mathfrak{sl}_2)$ (the set of finite dimensional irreducible $\mathfrak{sl}_2$-modules). It sends $n \in \mathbb{Z}_{\geqslant 0}$ to the proper quotient $L(n)$ of $\Delta(n)$. There is a basis $u_0, u_1, \ldots, u_n$ in $L(n)$, where the action of e, h, f is given by*

$$(2.4) \qquad f u_i = (n - i) u_{i+1}, h u_i = (n - 2i) u_i, e u_i = i u_{i-1}.$$

Proof. We already established the existence of $L(n)$ with required properties. Clearly, these modules are pairwise non-isomorphic. Now apply (2.1) to $V \in \operatorname{Irr}_{fin}(\mathfrak{sl}_2)$ and z found in Lemma 2.3. We see that V is a quotient of $\Delta(z)$. By Lemma 2.4, $z \in \mathbb{Z}_{\geqslant 0}$ and $V \cong L(z)$.

We set $u_i := \frac{f^i}{i!} v_n$. Clearly, (2.4) is satisfied. $\qquad \square$

2.3. **Complete reducibility.** Consider an element $C := ef + fe + h^2/2 = 2fe + h^2/2 + h \in U(\mathfrak{g})$ (the *Casimir element*). It is straightforward to check that it is central in $U(\mathfrak{g})$. So C acts by scalar on every irreducible finite dimensional module. To determine this scalar for $L(n)$ compute $C v_n = (h^2/2 + h) v_n = (n^2/2 + n) v_n$. In particular, we see that these scalars distinguish the irreducible modules $L(n)$.

Lemma 2.7. *Any finite dimensional $\mathfrak{sl}_2$-module V is completely reducible.*

Proof. Let us decompose V into the direct sum of generalized eigenspaces for C. We may assume that only one eigenvalue occurs, say $n^2/2 + n$. It remains to show that $L(n)$ has no self-extensions. Indeed, suppose we have an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with $V_i \cong L(n)$ that does not split. Pick a vector $v_0' \notin V_1$ that lies in the generalized eigenspace

for h with eigenvalue n. We see that $(h-n)v$ lies in the eigenspace for h in V with eigenvalue n. Note that $\pm(n+2)$ are not eigenvalues of h in V. Therefore $ev = f^{n+1}v = 0$. By (2.3) applied to $n+1$ rather than to n, $f^n(h-n)v = 0$. Therefore $(h-n)v = 0$. So we get a homomorphism $\Delta(n) \to V$ mapping v_z to v_0'. It must factor through $L(n) \to V$. It follows that our exact sequence splits. $\qquad\square$

2.4. **Representations of** $\mathrm{SL}_2(\mathbb{C})$. We have the following classification result.

Proposition 2.8. *Every rational representation of* $\mathrm{SL}_2(\mathbb{C})$ *is completely reducible. For every* $n \in \mathbb{Z}_{\geqslant 0}$*, there is a unique (up to isomorphism) representation* $L(n)$ *of dimension* $n+1$*.*

Proof. The first claim follows from Remark 1.5. To prove the second one, it only remains to check that every $\mathfrak{sl}_2(\mathbb{C})$-module $L(n)$ integrates to $\mathrm{SL}_2(\mathbb{C})$. Consider the $\mathrm{SL}_2(\mathbb{C})$-module $S^n(\mathbb{C}^2)$. Its basis is $x^n, x^{n-1}y, \ldots, y^n$, where x, y is a natural basis of $\mathbb{C}^2$. The eigenvalues of h on the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module are $n, n-2, \ldots, -n$. It follows that $S^n\mathbb{C}^2 \cong L(n)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module, and so $L(n)$ indeed integrates to $\mathrm{SL}_2(\mathbb{C})$. $\qquad\square$

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