LECTURE 4: REPRESENTATION THEORY OF $\mathrm{SL}_2(\mathbb{F})$ AND $\mathfrak{sl}_2(\mathbb{F})$

IVAN LOSEV

In this lecture we will discuss the representation theory of the algebraic group $\mathrm{SL}_2(\mathbb{F})$ and of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of positive characteristic, say p.

1. Representations of $\mathfrak{sl}_2(\mathbb{F})$

Here we assume that $p > 2$. The reason is that $\mathfrak{sl}_2(\mathbb{F})$ is not simple, i.e., contains proper ideals when $p = 2$.

1.1. Center of $U(\mathfrak{sl}_2)$.

The first thing we want to understand is the structure of the center. It is useful because the central elements act by scalars on any finite dimensional irreducible module.

An important feature of the positive characteristic story is that the center becomes bigger. Roughly speaking, in addition to elements that were central in characteristic 0, we get a large central subalgebra called the p-center. A general principle is that one should look at the pth powers of elements.

We start with $\mathfrak{gl}_n(\mathbb{F})$. For $x \in \mathfrak{gl}_n(\mathbb{F})$, we can take $x^p \in U(\mathfrak{gl}_n(\mathbb{F}))$. On the other hand, $\mathfrak{gl}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$ is itself an associative algebra and we can take the pth power of x there (known as the *restricted pth power*). In order to distinguish between these two situations, we write $x^{[p]}$ for the pth power of x in $\mathrm{Mat}_n(\mathbb{F})$.

Proposition 1.1. *The element $x^p - x^{[p]}$ is central in $U(\mathfrak{gl}_n(\mathbb{F}))$.*

In the proof we will use the following lemma.

Lemma 1.2. *Let A be an associative $\mathbb{F}$-algebra and $x, y \in A$. Recall that we write $\mathrm{ad}(x)$ for the map $A \to A, \mathrm{ad}(x)(a) := (xa - ax)$. We have $\mathrm{ad}(x)^p y = [x^p, y]$.*

Proof. We distribute $\mathrm{ad}(x)^p y$. Let l_x, r_x denote the operators $y \mapsto xy$ and $y \mapsto yx$ so that $\mathrm{ad}(x) = l_x - r_x$. Since l_x, r_x commute, we get

$$\mathrm{ad}(x)^p y = \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^i y x^{p-i}.$$

Since $\binom{p}{i} = 0$ for $0 < i < p$ (we are in characteristic p), the right hand side is $x^p y - y x^p$. □

Proof of Proposition 1.1. Pick $x, y \in \mathfrak{gl}_n(\mathbb{F})$. Applying Lemma 1.2 to $A = U(\mathfrak{gl}_n(\mathbb{F}))$, we have $[x^p, y] = \mathrm{ad}(x)^p y$ in $U(\mathfrak{gl}_n(\mathbb{F}))$. Note that the right hand side is in $\mathfrak{g}$. Now apply Lemma 1.2 to $A = \mathrm{Mat}_n(\mathbb{F})$. We get $[x^{[p]}, y] = \mathrm{ad}(x)^p y$. So $[x^p - x^{[p]}, y] = 0$. Since $\mathfrak{gl}_n(\mathbb{F})$ generates $U(\mathfrak{gl}_n(\mathbb{F}))$, we see that $x^p - x^{[p]}$ is central. □

Now let $G \subset \mathrm{GL}_n(\mathbb{F})$ be an algebraic group defined over $\mathbb{F}_p$ (i.e., by polynomials with coefficients in $\mathbb{F}_p$). One can show that $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ is closed with respect to the map $x \mapsto x^{[p]}$, see [J, Section 7] for details. In any case, for the subalgebras $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}), \mathfrak{sp}_n(\mathbb{F})$, the claim that $\mathfrak{g}$ is closed with respect to $x \mapsto x^{[p]}$ can be checked by hand. For example, in the case of $\mathfrak{sl}_n(\mathbb{F})$, the condition that $x \in \mathfrak{sl}_n(\mathbb{F})$ is equivalent to $\lambda_1 + \ldots + \lambda_n = 0$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x counted with multiplicities. The eigenvalues of $x^{[p]}$ are $\lambda_1^p, \ldots, \lambda_n^p$, we get $\lambda_1^p + \ldots + \lambda_n^p = (\lambda_1 + \ldots + \lambda_n)^p = 0$ because the map $\lambda \mapsto \lambda^p$ is a ring automorphism of $\mathbb{F}$.

So we get a map $\iota : \mathfrak{g} \to Z(U(\mathfrak{g})), x \mapsto x^p - x^{[p]}$. Recall that the group G acts on $\mathfrak{g}$ (adjoint representation). This action is by Lie algebra automorphisms so it extends to an action of G by algebra automorphisms on $U(\mathfrak{g})$. Therefore, it preserves the center $Z(U(\mathfrak{g}))$.

Proposition 1.3. *The map $\iota : \mathfrak{g} \to Z(U(\mathfrak{g}))$ is additive, semi-linear in the sense that $\iota(\lambda x) = \lambda^p x$ for $\lambda \in \mathbb{F}$, and G-equivariant. Moreover, if $x_1, \ldots, x_n$ form a basis in $\mathfrak{g}$, then the elements $x_i^p - x_i^{[p]} \in U(\mathfrak{g})$ are algebraically independent.*

The claims that ι is semilinear and G-equivariant are straightforward. The claim that ι is additive is more subtle and is left for the homework. The claim about the algebraic independence follows from the PBW theorem.

The subalgebra generated by $\iota(\mathfrak{g})$ inside $U(\mathfrak{g})$ is called the *p-center*. For $\mathfrak{g} = \mathfrak{sl}_2$, it is generated by $e^p, f^p, h^p - h$. It does not coincide with the whole center of $U(\mathfrak{sl}_2)$. For example, the Casimir element $C = ef + fe + h^2/2 \in U(\mathfrak{sl}_2)$ lies in the center but not in the p-center.

1.2. **p-reductions.** Pick an element $\alpha \in \mathfrak{g}^*$. Consider the quotient $U_\alpha(\mathfrak{g})$ of $U(\mathfrak{g})$ by the relations $\iota(x) - \langle \alpha, x \rangle, x \in \mathfrak{g}$. The algebras $U_\alpha(\mathfrak{g})$ below will be called *p-reductions* (short from "p-central reductions"). The element α is called the *p-character*.

Since any element of the p-center acts on $M \in \operatorname{Irr}(\mathfrak{g})$ by a scalar, we see that the $U(\mathfrak{g})$-action on M factors through $U_\alpha(\mathfrak{g})$ for α uniquely determined by M. So it is a natural question to describe the structure of $U_\alpha(\mathfrak{g})$.

Proposition 1.4. *The dimension of $U_\alpha(\mathfrak{g})$ equals $p^{\dim \mathfrak{g}}$. Moreover, if $x_1, \ldots, x_n$ form a basis of $\mathfrak{g}$, then the ordered monomials $x_1^{m_1} \ldots x_n^{m_n}$ with $0 \leqslant m_i \leqslant p-1$ form a basis in $U_\alpha(\mathfrak{g})$.*

Proof. Recall that the monomials

$$x_1^{pd_1+m_1} \ldots x_n^{pd_n+m_n} \tag{1.1}$$

$d_i \in \mathbb{Z}_{\geqslant 0}, m_i \in \{0, \ldots, p-1\}$ form a basis in $U(\mathfrak{g})$ (the PBW theorem). Now consider a new set of element indexed by the same d_i, m_i:

$$x_1^{m_1} \ldots x_n^{m_n} (x_1^p - x_1^{[p]} - \langle x_1, \alpha \rangle)^{d_1} \ldots (x_n^p - x_n^{[p]} - \langle x_n, \alpha \rangle)^{d_n}. \tag{1.2}$$

We claim that the monomials (1.2) form a basis in $U(\mathfrak{g})$. Indeed, the difference between (1.1) and (1.2) has degree that is strictly less than the total degree of any of these monomials, this implies our claim about basis.

Now note that the elements $x_i^p - x_i^{[p]} - \langle \alpha, x_i \rangle$ are central. So the two-sided ideal generated by these elements coincides with the linear span of basis elements (1.2), where at least one of the degrees d_i is positive. This easily implies the proposition. $\square$

So $\operatorname{Irr}_{fin}(U(\mathfrak{g})) = \bigsqcup_{\alpha \in \mathfrak{g}^*} \operatorname{Irr}_{fin}(U_\alpha(\mathfrak{g}))$ (in the left hand side we have the set of finite dimensional irreducible representations, note that there are no others by Problem 1 in Homework 1 because $U(\mathfrak{g})$ is a finitely generated module over its center). For every α, $\operatorname{Irr}(U_\alpha(\mathfrak{g})) = \operatorname{Irr}_{fin}(U_\alpha(\mathfrak{g}))$ is non-empty because $\dim U_\alpha(\mathfrak{g}) < \infty$.

In particular, in the case of $\mathfrak{sl}_2(\mathbb{F})$ the elements e, f no longer need to act nilpotently on a finite dimensional representations.

Note that, if $\alpha, \beta \in \mathfrak{g}^*$ are such that $\alpha = g \cdot \beta$, then the action of $g \in G$ on $U(\mathfrak{g})$ induces an algebra isomorphism $U_\beta(\mathfrak{g}) \xrightarrow{\sim} U_\alpha(\mathfrak{g})$. So it is enough to compute $\mathrm{Irr}(U_\alpha(\mathfrak{g}))$ for one α per every G-orbit.

1.3. **The case of $\mathfrak{sl}_2(\mathbb{F})$.** Since we assume that $p > 2$, the form $(x, y) = \mathrm{tr}(xy)$ on $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$ is non-degenerate. This form is $\mathrm{SL}_2(\mathbb{F})$-invariant. So the G-module $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified. We know how to classify elements of $\mathfrak{sl}_2(\mathbb{F})$ (and, more generally, $\mathfrak{sl}_n(\mathbb{F})$) up to G-conjugacy, the classification is given by the Jordan normal form theorem. So up to G-conjugacy, we have three possibilities for $M \in \mathrm{Irr}_{fin}(\mathfrak{sl}_2)$:

(1) $e^p, f^p, h^p - h$ act by 0. Here $\alpha = 0$.
(2) $e^p, f^p - 1, h^p - h$ act by 0. Here $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
(3) $e^p, f^p, h^p - h - \lambda$ act by 0 for $\lambda \neq 0$. Here $\alpha = \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix}$.

In the first case, M is called *restricted* – we will see below that it is obtained from an irreducible representation of $\mathrm{SL}_2(\mathbb{F})$. In all three cases e acts nilpotently, and as in the characteristic 0 case, we can find an h-eigenvector $v \in M$ annihilated by e. Let z be an eigenvalue. Note that $z^p - z = 0$ in cases (1),(2) (equivalently $z \in \mathbb{F}_p$), while, in case (3), z is one of the p solutions of $z^p - z = \lambda$.

Now we are going to proceed as in the characteristic 0 case: we introduce analogs of Verma modules for $U_\alpha(\mathfrak{g})$. Namely, $h, e \in U_\alpha(\mathfrak{g})$ generate a subalgebra in $U_\alpha(\mathfrak{g})$ with basis $h^\ell e^k, 0 \leqslant \ell, k \leqslant p-1$. We denote this subalgebra by $U_\alpha(\mathfrak{b})$. Consider the one-dimensional $U(\mathfrak{b})$-module $\mathbb{F}_z$, where e acts by 0 and h acts by z, where z is above. It factors through $U_\alpha(\mathfrak{b}) = U(\mathfrak{b})/(h^p - h - \alpha(h), e^p)$.

Form the induced module $\Delta_\alpha(z) := U_\alpha(\mathfrak{g}) \otimes_{U_\alpha(\mathfrak{b})} \mathbb{F}_z$ (the baby Verma modules). This module has basis $f^i v_z, i = 0, \ldots, p-1$, where $hv_z = zv_z, ev_z = 0$.

The following is the classification irreducible $U_\alpha(\mathfrak{g})$-modules (as well as the description of the structure of the modules $\Delta_\alpha(z)$).

Theorem 1.5. *The following is true.*

(i) *In case (1), there are p pairwise non-isomorphic irreducible $U_\alpha(\mathfrak{g})$-modules $L(z), z = 0, \ldots, p-1$ of dimension $z+1$, where $L_\alpha(z)$ is a simple quotient of $\Delta_\alpha(z)$. The module $\Delta_\alpha(p-1)$ is irreducible, while for $i \neq p-1$, there is an exact sequence $0 \to L_\alpha(-2-i) \to \Delta_\alpha(i) \to L_\alpha(i) \to 0$.*
(ii) *In case (2), all $\Delta_\alpha(z)$ are irreducible. We have $\Delta_\alpha(z) \cong \Delta_\alpha(z')$ if and only if $z + z' = -2$. We have $(p+1)/2$ irreducible $U_\alpha(\mathfrak{g})$-modules in this case.*
(iii) *In case (3), the $\mathfrak{g}$-modules $\Delta_\alpha(z)$ are non-isomorphic and form a complete collection of the irreducible $U_\alpha(\mathfrak{g})$-modules.*

The proof of this theorem will be in the homework.

2. Representations of $\mathrm{SL}_2(\mathbb{F})$

2.1. **Correspondence between group and Lie algebra.** A connection between the representation theories of an algebraic group G and the corresponding Lie algebra $\mathfrak{g}$ is much more loose than in characteristic 0. In a sentence, the representation theory of G in characteristic p is much closer to characteristic 0, than the representation theory of the corresponding Lie algebra.

Let V be a representation of G. It is also a representation of $\mathfrak{g}$.

Lemma 2.1. *Suppose V is irreducible over G and $\mathfrak{g}^{*G} = \{0\}$. Then the only eigenvalue of $\iota(\mathfrak{g})$ in V is zero.*

In fact, a stronger statement is true: for a rational representation V of G, the $U(\mathfrak{g})$-action on V factors through $U_0(\mathfrak{g})$.

Proof. For a $\mathfrak{g}$-module M and $g \in G$, define a new $\mathfrak{g}$-module M^g that coincides with M as a vector space but the action of $\mathfrak{g}$ is modified: if φ is a representation of $\mathfrak{g}$ in M, then $\varphi \circ \mathrm{Ad}(g)$ is a representation of $\mathfrak{g}$ in M^g. Similarly, we can define the G-module V^g. But $V \cong V^g$ as a G-module, and so also as a $\mathfrak{g}$-module. On the other hand, if α is a p-character of M, then $g^{-1} \cdot \alpha$ is a p-character of M^g. Since $\mathfrak{g}^{*G} = \{0\}$, we see that the only eigenvalue of $\iota(\mathfrak{g})$ in V is zero. □

On the other hand, many nontrivial G-modules give rise to trivial $\mathfrak{g}$-modules, we have seen this already in Example 1.2 of Lecture 3. Consider the Frobenius automorphism, $\mathsf{Fr} : \mathbb{F} \to \mathbb{F}, x \mapsto x^p$. It lifts to an automorphism of $\mathrm{GL}_n(\mathbb{F})$ (entry-wise) also denoted by Fr. For any algebraic subgroup of $\mathrm{GL}_n(\mathbb{F})$ defined over $\mathbb{F}_p$, the automorphism Fr of $\mathrm{GL}_n(\mathbb{F})$ restricts to an automorphism $G \to G$. So, for a rational representation V of G, we can consider its pullback Fr^*V, which is also a rational representation of G (if ρ is a representation of G in V, then $\rho \circ \mathsf{Fr}$ is the representation of G in Fr^*V). Note that the tangent map of Fr is zero at any point (derivatives of the pth powers are equal to 0). So $\mathfrak{g}$ acts on Fr^*V by 0. From here it is easy to that even if a representation of $\mathfrak{g}$ integrates to a representation of G, then it does so in infinitely many non-isomorphic ways.

2.2. **Weight decomposition.** The first thing to notice is that a rational representation of $\mathrm{SL}_2(\mathbb{F})$ still has a weight decomposition, which in this case is a decomposition with respect to the action of the subgroup T of diagonal matrices in $\mathrm{SL}_2(\mathbb{F})$ that is isomorphic to the multiplicative group $\mathbb{F}^\times$.

Lemma 2.2. *Any rational representation V of $\mathbb{F}^\times$ splits into the direct sum of one dimensional rational representations of $\mathbb{F}^\times$.*

Proof. From Linear algebra, we know that any two commuting diagonalizable operators are simultaneously diagonalizable. In fact, this is true for any collection (even infinite) of diagonalizable operators. The collection we take consists of all elements of $\mathbb{F}^\times$ that are of finite order coprime to p hence diagonalizable. Denote this collection by Λ. So the elements from Λ are simultaneously diagonalizable. The non-diagonal matrix coefficients of V vanish on Λ and so are zero. □

Recall (Example 1.2 of Lecture 3) that the one-dimensional rational representations of $\mathbb{F}^\times$ are classified by integers: to $n \in \mathbb{Z}$ we assign the representation given by $z \mapsto z^n$.

Now let V be a rational representation of $\mathrm{SL}_2(\mathbb{F})$. We can decompose it into the direct sum $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where $V_n = \{v \in V | \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} v = z^n v\}$. By a highest (resp., lowest) weight of V we mean the maximal (resp., minimal) n such that $V_n \neq \{0\}$. Note that $V_n \cong V_{-n}$. An isomorphism between these spaces is provided by the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ because conjugating $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we get $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$. So the lowest weight is negative the highest weight.

Theorem 2.3. *For each $n \in \mathbb{Z}_{\geqslant 0}$, there is a unique irreducible representation $L(n)$ of $\mathrm{SL}_2(\mathbb{F})$ with highest weight n.*

This theorem will be proved below. An idea of the proof is similar to the case of $\mathfrak{sl}_2(\mathbb{C})$: we will produce an analog of a Verma module called a *Weyl module.*

2.3. **(Dual) Weyl modules.** Consider the representation $W^\vee(n) := S^n(\mathbb{F}^2)$ (a *dual Weyl module*) of $\mathrm{SL}_2(\mathbb{F})$. Let $x, y \in \mathbb{F}^2$ be a basis with weights $1, -1$. Then $x^n, x^{n-1}y, \ldots, y^n$ form a basis in $W^\vee(n)$ with weights $n, n-2, \ldots, -n$. The reason why this representation is useful is that it has the following universality property. Let B denote the group of upper triangular matrices. Consider the one-dimensional representation $\mathbb{F}_{-n}$ of B, where $b := \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}$ acts by $\chi(b)^{-n}$, $\chi(b) := z$.

Proposition 2.4. *For a rational representation V of $G = \mathrm{SL}_2(\mathbb{F})$, we have a natural isomorphism* $\mathrm{Hom}_G(V, W^\vee(n)) \cong \mathrm{Hom}_B(V, \mathbb{F}_{-n})$.

Proof. In the proof we will need a geometric realization of $W^\vee(n)$: as the global sections $\Gamma(\mathcal{O}(n))$ of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The group G acts on $\mathbb{P}^1 = \mathbb{P}(\mathbb{F}^2)$ and moreover, $\mathbb{P}^1$ is the homogeneous space G/B. The bundle $\mathcal{O}(n)$ carries an action of G that is compatible with the action of G on $\mathbb{P}^1$. In other words, this is a homogeneous vector bundle. To give such a bundle one only needs to specify a fiber at one point that is a representation of the stabilizer. Pick a point $[1:0] \in \mathbb{P}^1$ whose stabilizer is B. The fiber $\mathcal{O}(-1)_{[x:y]}$ of $\mathcal{O}(-1)$ at a point $[x:y] \in \mathbb{P}^1$ is the line passing through this point. It follows that $\mathcal{O}(-1)_{[1:0]} \cong \mathbb{F}_1$ (an isomorphism of B-modules). So $\mathcal{O}(n)_{[1:0]} = (\mathcal{O}(-1)^*)^{\otimes n}_{[1:0]}$ is the B-module $\mathbb{F}_{-n}$.

Now we are ready to produce an isomorphism $\mathrm{Hom}_G(V, W^\vee(n)) \to \mathrm{Hom}_B(V, \mathbb{F}_{-n})$. Note that $W^\vee(n) = \Gamma(\mathcal{O}(n))$ coincides with $\{f \in \mathbb{F}[G] | f(gb) = \chi(b)^n f(g)\}$. It follows that $\mathrm{Hom}(V, W^\vee(n))$ coincides with the space of polynomial maps $\varphi : G \to V^*$ such that $\varphi(gb) = \chi(b)^n \varphi(g)$. So $\mathrm{Hom}_G(V, W^\vee(n)) = (V^* \otimes W^\vee(n))^G$ coincides with $H := \{\varphi : G \to V^* | \varphi(gb) = g\chi^n(b)\varphi(1)\}$ (such a map is automatically polynomial). The map $\varphi \mapsto \varphi(1)$ establishes an isomorphism of H with $(V^* \otimes \mathbb{F}_{-n})^B$. $\square$

We define the *Weyl module* $W(n)$ as $W^\vee(n)^*$. It follows from Proposition 2.4 that

$$\mathrm{Hom}_G(W(n), V) = \mathrm{Hom}_B(\mathbb{F}_n, V) = \{v \in V | bv = \chi(b)^n v, \forall b \in B\}. \tag{2.1}$$

Proof of Theorem 2.3. Let V be an irreducible $\mathrm{SL}_2(\mathbb{F})$-module with highest weight n. Any vector in V_n is invariant under the subgroup $\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\}$ (this is a homework problem). From (2.1) we conclude that V is an irreducible quotient of $W(n)$. Now the proof repeats that of Proposition 2.6, Lecture 3. In particular, $L(n)$ is the unique irreducible quotient of $W(n)$. $\square$

Remark 2.5. Note that $L(n) \cong L(n)^*$ (both sides are irreducibles with highest weight n). So $L(n) \hookrightarrow W^\vee(n)$. The kernel of $W(n) \twoheadrightarrow L(n)$ only has simple composition factors $L(m)$ with $m < n$ because all the weights in the kernel are less than n. The same is true for the cokernel of $L(n) \hookrightarrow W^\vee(n)$.

2.4. **Steinberg decomposition.** Here we are going to explain the structure of $L(n)$. The description we are going to give is "inductive". The base is $n < p$.

Lemma 2.6. *Let $n < p$. Then $L(n) \cong W^\vee(n) \cong W(n)$. As a $\mathfrak{g}$-module, $L(n) \cong L_0(n)$.*

Proof. Since $n < p$, we have $S^n(V)^* \cong S^n(V^*)$ for any finite dimensional $\mathbb{F}$-space V. Further, the $\mathrm{SL}_2(\mathbb{F})$-module $\mathbb{F}^2$ is self-dual. So the module $W^\vee(n) = S^n(\mathbb{F}^2)$ is self-dual. From Remark 2.5, it follows that $W^\vee(n) = L(n)$. The claim about the isomorphism of $\mathfrak{g}$-modules is straightforward. □

Here is an "induction step".

Lemma 2.7. *For $n < p$ and $d \geqslant 0$, we have an isomorphism $L(dp+n) \cong L(n) \otimes \mathsf{Fr}^* L(d)$ of $\mathrm{SL}_2(\mathbb{F})$-modules.*

Proof. The highest weight of the right hand side is $dp + n$ so it is enough to show that the right hand side is irreducible. As a $\mathfrak{g}$-module, $\mathrm{Fr}^* L(d)$ is trivial. If V is a G-submodule of $L(n) \otimes \mathsf{Fr}^* L(d)$, then it is also a $\mathfrak{g}$-submodule. By the proof of the Burnside theorem in Lecture 1, $V = L(n) \otimes V_0$, where $V_0 \subset \mathsf{Fr}^* L(d)$. We have $V_0 = \mathrm{Hom}_{\mathfrak{g}}(L(n), V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(n), L(n) \otimes \mathsf{Fr}^* L(d)) = \mathsf{Fr}^* L(d)$. Note that $\mathrm{Hom}_{\mathfrak{g}}(U_1, U_2)$, where U_1, U_2 are G-modules, is a G-module (unlike in the characteristic 0, this module can be nontrivial). The equality $\mathrm{Hom}_{\mathfrak{g}}(L(n), L(n) \otimes \mathsf{Fr}^* L(d)) = \mathsf{Fr}^* L(d)$ is that of G-modules. But the pull-back of an irreducible module under a group homomorphism is itself irreducible. So the G-module $\mathrm{Hom}_{\mathfrak{g}}(L(n), V)$ is included into an irreducible G-module. It follows that $V_0 = \mathsf{Fr}^* L(d)$ that finishes the proof. □

Our conclusion is that any $L(n)$ decomposes (in a unique way) into the tensor product of iterated pullbacks under Fr of $L(m)$'s with $m < p$ (Steinberg decomposition). This allows to determine the weight decomposition of $L(n)$. It also allows to determine the multiplicities of $L(n)$'s in the composition series of $W(n')$.

References

[J] J.C. Jantzen. *Representations of algebraic groups.* Academic Press, 1987.