LECTURE 6: KAC-MOODY ALGEBRAS, REDUCTIVE GROUPS, AND REPRESENTATIONS

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Introduction

We start by introducing Kac-Moody algebras and completing the classification of finite dimensional semisimple Lie algebras.

We then discuss the classification of finite dimensional representations of semisimple Lie algebras (and, more generally, integrable highest weight representations of Kac-Moody algebras).

We finish by discussing the structure and representation theory of reductive algebraic groups.

1. Generators and relations. Kac-Moody algebras

1.1. **Relations in simple Lie algebras.** Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}\subset\mathfrak{g}$ a Cartan subalgebra, $\alpha_1,\ldots,\alpha_n$ a simple root system. We write e_i,h_i,f_i for $e_{\alpha_i},h_{\alpha_i},f_{\alpha_i}$. Let $A=(a_{ij})_{i,j=1}^n$ be the Cartan matrix, where, recall, $a_{ij}=\alpha_j(h_i)$. Set $\mathfrak{n}:=\bigoplus_{\alpha>0}\mathfrak{g}_\alpha, \mathfrak{n}^-:=\bigoplus_{\alpha<0}\mathfrak{g}_\alpha$, these are Lie subalgebras of $\mathfrak{g}$ because $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta}$. We have $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ (as vector spaces) because every root is either positive or negative.

Lemma 1.1. *The elements e_i (resp., f_i) generate $\mathfrak{n}$ (resp., $\mathfrak{n}^-$). In particular, $e_i,h_i,f_i,i=1,\ldots,n$, generate $\mathfrak{g}$. Further, we have the following relations (a.k.a. Serre relations):*

1. $[h_i,h_j]=0$.
2. $h_i=[e_i,f_i],[e_i,f_j]=0$ *for* $i\neq j$.
3. $[h_i,e_j]=a_{ij}e_j,[h_i,f_j]=-a_{ij}f_j$.
4. $\operatorname{ad}(e_i)^{1-a_{ij}}e_j=0,\operatorname{ad}(f_i)^{1-a_{ij}}f_j=0,i\neq j$.

Example 1.2. Let $\mathfrak{g}=\mathfrak{sl}_n$. Then $\mathfrak{n}$ (resp., $\mathfrak{n}^-$) is the subalgebra of all strictly upper (resp., lower) triangular matrices. We have $h_i:=E_{i,i}-E_{i+1,i+1},e_i=E_{i,i+1},f_i=E_{i+1,i}$. We have $[E_{ij},E_{jk}]=E_{ik}$ if $i\neq k$. Lemma 1.1 basically follows from this identity.

Proof. Let us check that the elements e_i generate $\mathfrak{n}$ (the claim about f_i's and $\mathfrak{n}^-$ is analogous). Assume the contrary: there is $\alpha>0$ such that $\mathfrak{g}_\alpha$ does not lie in the subalgebra $\mathfrak{n}_0$ generated by the e_i's. We have $\alpha=\sum_{i=1}^n m_i\alpha_i$ with $m_i\geqslant 0$. We may assume that $\sum_{i=1}^n m_i$ is minimal possible such that $\mathfrak{g}_\alpha\not\subset\mathfrak{n}_0$. Since all $m_j\geqslant 0$ and $0<(\alpha,\alpha)=\sum_j m_j(\alpha,\alpha_j)$, we see that there is j with $\alpha(h_j)=\frac{2(\alpha,\alpha_j)}{(\alpha_j,\alpha_j)}>0$. The elements α and $s_j(\alpha)=\alpha-\alpha(h_j)\alpha_j$ are roots. (6) of Proposition 1.4 of Lecture 5 implies that $\alpha-\alpha_j$ is a root. By the inductive assumption, $\mathfrak{g}_{\alpha-\alpha_j}\subset\mathfrak{n}_0$. Consider the $\mathfrak{sl}_2$-module $\sum_{z\in\mathbb{Z}}\mathfrak{g}_{\alpha+z\alpha_j}$. By the representation theory of $\mathfrak{sl}_2$, we see that $[e_j,\mathfrak{g}_{\alpha-\alpha_j}]=\mathfrak{g}_\alpha$, and we are done.

Let us check the relations. (1) is obvious. The first equality in (2) is the definition of h_i. The second one follows from $[e_i,f_j]\in\mathfrak{g}_{\alpha_i-\alpha_j}=0$ because $\alpha_i-\alpha_j$ is neither positive nor negative root. (3) follows from $a_{ij}=\alpha_j(h_i)$.

Let us prove that $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$. We know that

$$[e_i, f_j] = 0, [h_i, f_j] = -a_{ij} f_j. \tag{1.1}$$

Consider the $\mathfrak{sl}_2$-subalgebra spanned by e_i, h_i, f_i and its module $\bigoplus_{z\in\mathbb{Z}} \mathfrak{g}_{-\alpha_j+z\alpha_i}$. Considering f_j as an element of this module and using (1.1) and the representation theory of $\mathfrak{sl}_2$, we see that $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$. The other equality in (4) is proved similarly. □

1.2. **Kac-Moody algebras.** Now let A be an arbitrary symmetrizable irreducible Cartan matrix. We can define the Lie algebra $\mathfrak{g}(A)$ by generators h_i, e_i, f_i and relations (1)-(4). This is the Kac-Moody algebra (with Cartan matrix A).

Let $\mathfrak{n}$ (resp., $\mathfrak{n}^-$, $\mathfrak{h}$) be the subalgebra in $\mathfrak{g}(A)$ generated by the elements e_i (resp., f_i, resp., h_i). Then we have the following important result

Proposition 1.3. *We have $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ (as vector spaces). Moreover, $h_1, \ldots, h_n$ form a basis in $\mathfrak{h}$.*

To simplify the notation let us write $\mathfrak{g}$ for $\mathfrak{g}(A)$. Consider the vector space $\mathfrak{h}^\vee$ with basis $\alpha_1, \ldots, \alpha_n$. This space comes with a pairing $\mathfrak{h}^\vee \times \mathfrak{h} \to \mathbb{C}$ given by $(\alpha_i, h_j) \to a_{ij}$ and with a symmetric form defined by S (note that the pairing and the form are degenerate if and only if $\det A = 0$ – which is an interesting case, but it requires some extra care because the elements α_i cannot be viewed as functions on $\mathfrak{h}$).

Let Q^+ (resp., Q) denote the sub-semigroup (resp., subgroup) in $\mathfrak{h}^\vee$ spanned by $\alpha_1, \ldots, \alpha_n$. We assign degrees $\alpha_i, 0, -\alpha_i$ to e_i, h_i, f_i. Since the relations in $\mathfrak{g}$ are Q-homogeneous, we see that $\mathfrak{n}$ is Q^+-graded and $\mathfrak{n}^-$ is $-Q^+$-graded. For $\beta \in Q^+ \sqcup -Q^+$, let $\mathfrak{g}_\beta$ denote the corresponding graded component. We say that β is a *root* if $\mathfrak{g}_\beta \neq \{0\}$. The notions of positive and negative roots are introduced in an obvious way.

We still can define the bijections $s_i : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$ as before (the simple reflections). The subgroup of $\mathrm{GL}(\mathfrak{h}^\vee)$ generated by the elements s_i is called the Weyl group of $\mathfrak{g}$ or of A. The Weyl group elements map roots to roots preserving the dimensions of root spaces, this again follows from the representation theory of $\mathfrak{sl}_2$. Roots obtained from $\alpha_1, \ldots, \alpha_n$ by applying Weyl group elements are called *real*, for a real root α we have $\dim \mathfrak{g}_\alpha = 1$. The other roots are called imaginary.

Lemma 1.4. *If β is an imaginary root, then $(\beta, \beta) \leqslant 0$.*

1.3. **Case of positive definite Cartan matrix.** Now suppose that A is positive definite (and irreducible).

Theorem 1.5. *The algebra $\mathfrak{g}(A)$ is finite dimensional and simple.*

Sketch of proof. The Weyl group W is a subgroup in $\mathrm{O}(\mathfrak{h}^\vee)$ that preserves the lattice generated by $\alpha_1, \ldots, \alpha_n$. Such a group is finite. By Lemma 1.4, every root is real. So the root system of $\mathfrak{g}$ is finite and all $\mathfrak{g}_\alpha$'s have dimension 1. So $\dim \mathfrak{g}(A) < \infty$. Any ideal in $\mathfrak{g}(A)$ can be shown to intersect $\mathfrak{h}$ which contradicts the irreducibility of A. □

When A is not positive definite, the algebra $\mathfrak{g}(A)$ is not finite dimensional. A family of examples will be described in the homework.

2. Finite dimensional representations

2.1. **The case of finite dimensional algebras.** Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and V be its finite dimensional module. Since $\mathfrak{h}$ is commutative and consists of semisimple elements, we have the *weight* decomposition $V = \bigoplus_{\nu\in\mathfrak{h}^*} V_\nu$, where V_ν is the eigenspace for $\mathfrak{h}$ with eigenvalue ν. If $V_\nu \neq \{0\}$, we say that ν is a *weight* of V. The number of $\nu(h_i)$ is a weight for an $\mathfrak{sl}_2$-module and so is integral. Set $P := \{\mu \in \mathfrak{h}^* | \mu(h_i) \in \mathbb{Z}, \forall i\}$. We conclude that any weight ν is in P (and hence P is called the weight lattice).

On $\mathfrak{h}^*$ we introduce a partial order $\leqslant$: $\nu' \leqslant \nu$ if $\nu - \nu'$ is the sum of positive roots. By a *highest weight* of an irreducible module V, we mean a maximal weight λ (existing because the set of weights is finite). Note that any $v \in V_\lambda$ is annihilated by $\mathfrak{n}$.

Lemma 2.1. *Let λ be a highest weight of V. Then*

(1) $\lambda(h_i) \geqslant 0$.
(2) *Further,* $f_i^{\lambda(h_i)+1} v_\lambda = 0$.

Proof. (1) follows from $e_i v = 0$. (2) follows from the representation theory of $\mathfrak{sl}_2$, compare to the proof of Lemma 1.1. ☐

The elements $\lambda \in \mathfrak{h}^*$ satisfying (1) are called *dominant*. The set of dominant weights is denoted by P^+.

The following theorem generalizes the corresponding result for $\mathfrak{sl}_2$.

Theorem 2.2. *There is a bijection between* $\mathrm{Irr}_{fin}(\mathfrak{g})$ *and* P^+ *that sends a module to its (unique) highest weight.*

The proof is similar to the $\mathfrak{sl}_2$-case, see Lecture 3. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Now for an arbitrary $\lambda \in \mathfrak{h}^*$ we can form the one-dimensional $\mathfrak{b}$-module $\mathbb{C}_\lambda$ ($\mathfrak{h}$ acts by λ and $\mathfrak{n}$ acts by 0, compare to $\mathbb{C}_z$ in the case $\mathfrak{g} = \mathfrak{sl}_2$). So we can form the Verma module $\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ that has the universal property:

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), V) = \{v \in V_\lambda | \mathfrak{n} v = 0\}.$$

Let v_λ denote the image of $1 \in U(\mathfrak{g})$ in $\Delta(\lambda)$.

Lemma 2.3. *The following is true:*

(1) $\Delta(\lambda) = \bigoplus_{\nu\leqslant\lambda} \Delta(\lambda)_\nu$ *and* $\dim \Delta(\lambda)_\nu < \infty$.
(2) *There is a unique simple quotient $L(\lambda)$ of $\Delta(\lambda)$.*
(3) *Moreover, if $v \in L(\lambda)$ is annihilated by $\mathfrak{n}$, then it is proportional to v_λ.*

Proof. By the PBW theorem, $\Delta(\lambda)$ has basis $\prod_{\alpha>0} f_\alpha^{-n_\alpha} v_\lambda$ (for some fixed ordering of positive roots). The weight of this basis vector is $\lambda - \sum_{\alpha>0} n_\alpha \alpha$. This implies (1).

Note that any proper submodule of $\Delta(\lambda)$ is contained in $\bigoplus_{\nu<\lambda} \Delta(\lambda)_\nu$. This implies (2).

Let us prove (3). Assuming it is false, we can find $v \in L(\lambda)_\nu$ with $\mathfrak{n} v = 0$. We have $\nu < \lambda$ and hence the image of $\Delta(\nu)$ in $L(\lambda)$ is proper. Contradiction. ☐

Corollary 2.4. *Any irreducible finite dimensional $\mathfrak{g}$-module has a single highest weight. Two finite dimensional irreducible modules with the same highest weight are isomorphic.*

To prove Theorem 2.2 it remains to prove the following proposition.

Proposition 2.5. *If λ is dominant, then* $\dim L(\lambda) < \infty$.

Sketch of the proof. The proof is in several steps.

Step 1. Deduce that f_i, e_i act on $L(\lambda)$ locally nilpotently (for e_i this follows from the weight considerations, while for f_i one needs to use the second part of Lemma 2.1).

Step 2. Deduce that $L(\lambda)$ is the sum of its finite dimensional $\mathfrak{sl}_2$-modules for any $\mathfrak{sl}_2 = \mathrm{Span}(e_i, h_i, f_i)$. So the action of this $\mathfrak{sl}_2$ integrates to $\mathrm{SL}_2(\mathbb{C})$.

Step 3. By using the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, show that the set of weights of $L(\lambda)$ is closed under s_i.

Step 4. By Step 3, the set of weights is closed under the action of the Weyl group W. On the other hand, any weight μ has the property that $\lambda - \mu$ is the sum of positive roots. From here one can deduce that the set of weights is finite.

Step 5. Since the dimensions of all weight spaces are finite (this is so even in $\Delta(\lambda)$ by (1) of Lemma 2.3), we see that $\dim L(\lambda) < \infty$. □

2.2. **Fundamental weights.** For $i = 1, \ldots, n$, define $\omega_i \in \mathfrak{h}^*$ by $\omega_i(\alpha_j^\vee) = \delta_{ij}$. This elements form bases in the group P and in the monoid P^+.

The irreducible representations corresponding to fundamental weights are important because of the following lemma.

Lemma 2.6. *Let λ, μ be dominant weights. Then $L(\lambda + \mu)$ coincides with the submodule in $L(\lambda) \otimes L(\mu)$ generated by $v_\lambda \otimes v_\mu$.*

Proof. First of all, note that a $\mathfrak{g}$-submodule in any $\mathfrak{g}$-module generated by $v \in V_\nu$ with $\mathfrak{n}v = 0$ is isomorphic to $L(\nu)$. Indeed, it is a finite dimensional image of a homomorphism $\Delta(\nu) \to V$. Any finite dimensional image of $\Delta(\nu)$ is $L(\nu)$ by (2) of Lemma 2.3 and the complete reducibility of finite dimensional modules.

The vector $v_\lambda \otimes v_\mu$ has weight $\lambda + \mu$ and is annihilated by $\mathfrak{n}$. Our claim follows. □

Example 2.7. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We have $\omega_i = \sum_{j=1}^i \epsilon_j$ and $L(\omega_i) = \Lambda^i \mathbb{C}^{n+1}$. Indeed, let $v_1, \ldots, v_{n+1}$ be a natural basis in $\mathbb{C}^{n+1}$. The vector $v_1 \wedge v_2 \wedge \ldots \wedge v_i$ has weight ω_i and is annihilated by $\mathfrak{n}$. It is not difficult to see that ω_i is the only dominant weight in $\Lambda^i \mathbb{C}^{n+1}$ so the latter is irreducible.

Example 2.8. Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. We have $\omega_i = \sum_{j=1}^i \epsilon_j$ for $i < n$, and $\omega_n = \frac{1}{2}\sum_{j=1}^n \epsilon_j$. One can show that $\bigwedge^i \mathbb{C}^{2n+1} = L(\omega_i)$ for $i < n$. The irreducible representation $L(\omega_n)$ (the spinor representation) is not realized in this way.

Example 2.9. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. We have $\omega_i = \sum_{j=1}^i \epsilon_i$, and $L(\omega_i)$ is a direct summand in $\Lambda^i \mathbb{C}^{2n}$ generated by $v_1 \wedge v_2 \wedge \ldots \wedge v_i$ for any $i = 1, \ldots, n$.

Example 2.10. Let $\mathfrak{g} = \mathfrak{so}_{2n}$. We have $\omega_i = \sum_{j=1}^i \epsilon_i$ for $i = 1, \ldots, n-2$, $\omega_{n-1} = \omega_{n-2} + \frac{1}{2}(\epsilon_{n-1} - \epsilon_n), \omega_n = \omega_{n-2} + \frac{1}{2}(\epsilon_{n-1} + \epsilon_n)$. We have $L(\omega_i) = \bigwedge^i \mathbb{C}^{2n}$ for $i \leqslant n-2$. The representations $L(\omega_{n-1}), L(\omega_n)$ are the so called half-spinor representations.

2.3. **Integrable highest weight representations of Kac-Moody algebras.** Now let A be a symmetrizable Cartan matrix and $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding Kac-Moody algebra. Recall the space $\mathfrak{h}^\vee$ spanned by the simple roots $\alpha_1, \ldots, \alpha_n$. We are interested in studying highest weight representations of $\mathfrak{g}$, i.e., representations V equipped with a grading $V = \bigoplus_{\mu \in -Q^+} V(\mu)$, where $V(\mu)$ is a finite dimensional space, where $\mathfrak{h}$ acts diagonalizably and $e_i V(\mu) \subset V(\mu + \alpha_i), f_i V(\mu) = V(\mu - \alpha_i)$. Note that V decomposes into the sum of weight spaces for $\mathfrak{h}$ but this decomposition does not need to agree with $V = \bigoplus_{\mu \in -Q^+} V(\mu)$ when A is degenerate. If $V(0)$ is a single weight space, then the weight of $\mathfrak{h}$ in it, an element of $\mathfrak{h}^*$, is called the highest weight of V.

We say that a weight $\mathfrak{g}$-module V is *integrable* if e_i, f_i act on V locally nilpotently (so that V integrates to the corresponding infinite dimensional group).

Theorem 2.11. *Any integrable highest weight $\mathfrak{g}$-module is completely reducible. The irreducibles are classified by dominant weights (via taking the highest weight).*

3. Reductive algebraic groups

3.1. **Classification.** An algebraic group is called *unipotent* if it is represented by unipotent operators in any rational representation (equivalently, in some faithful rational representation). A typical example is the group of uni-triangular matrices.

We say that an algebraic group is reductive if it does not have normal unipotent subgroups. For example, an *algebraic tori* (or just tori) $(\mathbb{C}^\times)^n$ are reductive. Those are the only connected commutative reductive groups.

We say that an algebraic group is semisimple if its Lie algebra is semisimple. The following theorems classifies (connected) semisimple and reductive algebraic groups.

Theorem 3.1. *The following is true.*

1. *For any semisimple Lie algebra $\mathfrak{g}$, there is a unique connected and simply connected algebraic group with this Lie algebra.*
2. *Let G be a connected semisimple algebraic group. Then there is a simply connected semisimple group $\tilde{G}$ and a finite central subgroup $Z \subset \tilde{G}$ such that $G = \tilde{G}/Z$.*
3. *A connected algebraic group G is reductive if and only if there is a semisimple algebraic group G', a torus T and a finite central subgroup $Z \subset G' \times T$ such that $G = (T \times G')/Z$. For T we can take the connected component of 1 in the center $Z(G)$. The Lie algebra $\mathfrak{g}'$ coincides with $[\mathfrak{g}, \mathfrak{g}]$.*

So basically, to understand connected reductive algebraic groups, we need to compute the centers of simply connected semisimple groups (that are necessarily finite).

Example 3.2. The groups $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are simply connected. The centers consist of scalar matrices and so have order n for $\mathrm{SL}_n(\mathbb{C})$ and 2 for $\mathrm{Sp}_{2n}(\mathbb{C})$. The group $\mathrm{SO}_n(\mathbb{C})$ is not simply connected, it has a $2:1$ cover, called the *spinor group* $\mathrm{Spin}_n(\mathbb{C})$.

3.2. **Structure theory.** Let us discuss important structural features of a (connected) reductive algebraic group G. By a *Borel subgroup*, one means a maximal (with respect to inclusion) connected solvable subgroup B. An important result is that all such subgroups are G-conjugate. Moreover, B coincides with its normalizer. The Lie algebra of B, a subalgebra in $\mathfrak{g}$, is called a Borel subalgebra. For example, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$ is a Borel subalgebra. The homogeneous space G/B is a smooth projective variety known as the flag variety for G. This terminology is motivated by the following example.

Example 3.3. Let $G = \mathrm{GL}_n(\mathbb{C})$. One can show that every connected solvable subgroup fixes a complete flag of subspaces in $\mathbb{C}^n$. So the Borel subgroups are precisely the subgroups of upper-triangular matrices with respect to some basis. And G/B is the flag variety.

Another important class of subgroups in G are maximal tori (with respect to inclusion). Note that any torus in G is contained in a Borel subgroup.

Theorem 3.4. *The following is true:*

1. *Any two maximal tori in a Borel subgroup B are conjugate. Hence any two maximal tori in G are conjugate.*
2. *If G is semisimple, then the Lie algebra of a maximal torus is a Cartan subalgebra.*

(3) *Let T be a maximal torus and $\mathfrak{h} \subset \mathfrak{g}$ be its Lie subalgebra. Note that since T is abelian its adjoint action on $\mathfrak{h}$ is trivial. So $N_G(T)/T$ acts on $\mathfrak{h}$. This action is faithful and the image coincides with W.*

(4) *T contains the center of G.*

Example 3.5. Consider $G = \mathrm{SL}_n(\mathbb{C})$. Any torus acts by diagonal matrices in some basis. So any maximal torus consists of all diagonal matrices in some basis. The normalizer $N_G(T)$ is the subgroup of all monomial matrices, i.e., non-degenerate matrices that have exactly one nonzero entry in every row. We see that $N_G(T)/T \cong S_n$. The action of $N_G(T)/T$ on $\mathfrak{h}$ is by permuting the entries.