LECTURE 8: REPRESENTATIONS OF $\mathfrak{g}_{\mathbb{F}}$ AND OF $\mathrm{GL}_n(\mathbb{F}_q)$

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Introduction

We start by briefly explaining key results on the representation theory of semisimple Lie algebras in large enough positive characteristic.

After that, we start a new topic: the complex representation theory of finite groups of Lie type and Hecke algebras. Today we consider the most basic group: $G := \mathrm{GL}_n(\mathbb{F}_q)$. We introduce the Hecke algebra as the endomorphism algebra of the G-module $\mathbb{C}[B \setminus G]$. We describe the basis in this algebra and some of the multiplication rules that allow to present the Hecke algebra by generators and relations. Then we use the Tits deformation principle to show that the Hecke algebra is isomorphic to $\mathbb{C}S_n$.

1. Representations of semisimple Lie algebras in positive characteristic

Let $G_{\mathbb{F}}$ be a simple algebraic group over an algebraically closed field $\mathbb{F}$ of positive characteristic and $\mathfrak{g}_{\mathbb{F}}$ be its Lie algebra. In this section we assume that the characteristic of $\mathbb{F}$ is large enough. This will guarantee that $(\cdot,\cdot)$ is non-degenerate, that $\mathfrak{g}_{\mathbb{F}}$ is simple (by root considerations), and several more subtle things. In a sentence, the structure theory of $\mathfrak{g}_{\mathbb{F}}$ will be the same as of $\mathfrak{g}$, while the representation theory will be crucially different.

1.1. **Case of nilpotent p-character.** Recall that any irreducible $\mathfrak{g}_{\mathbb{F}}$-module has the so called p-character, an element of $\mathfrak{g}_{\mathbb{F}}$ to be denoted by α. In this section, we assume that α is nilpotent. As we will see below, the general case can be reduced to this one. Since $p \gg 0$, the nilpotent $G_{\mathbb{F}}$-orbits in $\mathfrak{g}_{\mathbb{F}}$ are in a natural bijection with the nilpotent orbits of $G(= G_{\mathbb{C}})$ in $\mathfrak{g}$ (this should be clear when $\mathfrak{g} = \mathfrak{sl}_n$, and is easy to show when $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$). So we can view α also as an element of $\mathfrak{g}$ (defined up to G-conjugacy).

Consider an irreducible $\mathfrak{g}$-module M with p-character α. We still have the so called *Harish-Chandra* center $U(\mathfrak{g}_{\mathbb{F}})^{G_{\mathbb{F}}}$. As in characteristic 0, $U(\mathfrak{g}_{\mathbb{F}})^{G_{\mathbb{F}}} = \mathbb{F}[\mathfrak{h}_{\mathbb{F}}^*]^W$. For $\lambda \in \mathfrak{h}_{\mathbb{F}}^*$ let $U_{\alpha,\lambda}(\mathfrak{g}_{\mathbb{F}})$ denote the corresponding quotient of $U_\alpha(\mathfrak{g}_{\mathbb{F}})$. One can show that if $U_{\alpha,\lambda}(\mathfrak{g}_{\mathbb{F}}) \neq \{0\}$, then $\lambda \in \mathfrak{h}_{\mathbb{F}_p}^*$.

Below we will consider the case when the stabilizer of $\lambda + \rho$ in W is trivial (the regular case). One reduces the general case to this one using translation functors. Consider the variety $\mathcal{B}$ of all Borel subalgebras in $\mathfrak{g}$. This is nothing else but the flag variety G/B. Inside, we have the closed (generally, singular) subvariety $\mathcal{B}_\alpha$ of all subalgebras containing α.

Example 1.1. Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $\mathcal{B}$ is the variety Fl of full flags, and $\mathcal{B}_\alpha$ consists of all flags $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n$ such that $\alpha V_i \subset V_{i-1}$ for all i (recall that α is nilpotent and so if α preserves V_i, V_{i-1}, then it maps V_i to V_{i-1}).

Theorem 1.2. *Let α be nilpotent, and λ be regular. Then* $|\operatorname{Irr}(U_{\alpha,\lambda}(\mathfrak{g}))| = \dim H_*(\mathcal{B}_e)$.

Example 1.3. Let $\mathfrak{g} = \mathfrak{sl}_2$. If $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\mathcal{B}_e$ consists of one point, $\{0\} \subsetneq \operatorname{im}\alpha \subsetneq \mathbb{C}^2$. We have $(p+1)/2$ points in $\mathfrak{h}_{\mathbb{F}_p}^*/\{\pm 1\}$, and $(p+1)/2$ irreducible $U_\alpha(\mathfrak{g})$-modules. Taking the scalar of action of C, we get a bijection between these two sets, as (partially) predicted by Theorem 1.2.

If $\alpha = \{0\}$, then $\mathcal{B}_e = \mathcal{B}$ and the homology has dimension 2. We have p irreducible $U_0(\mathfrak{g})$-modules, $L_0(z), z = 0, \ldots, p-1$. The eigenvalue of C on $L_0(z)$ is $\frac{1}{2}((z+1)^2-1)$. Regular λ corresponds to $z \neq -1$. We see that $\mathrm{Irr}(U_{0,\lambda}(\mathfrak{g})) = \{L(\lambda), L(-2-\lambda)\}$.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $n_1, \ldots, n_k$ be the sizes of Jordan blocks in the Jordan decomposition of α. One can show that

$$\dim H_*(\mathcal{B}_e) = \frac{n!}{n_1! \ldots n_k!}.$$

In this case, the number of finite dimensional irreducible modules can be computed by more elementary techniques than those of [BMR].

Let us proceed to explaining what is known about the dimensions of the simple $U_\alpha(\mathfrak{g})$-modules. They are known in principle, [BM], but the answer is involved and quite unexplicit. There is a nice general fact proved by Premet, [P].

Theorem 1.5. *We have $U_\alpha(\mathfrak{g}_{\mathbb{F}}) \cong \mathrm{Mat}_{p^d}(\mathcal{W}_{\alpha,\mathbb{F}})$, where $\mathcal{W}_{\alpha,\mathbb{F}}$ is some associative algebra and $d = \frac{1}{2}\dim \mathfrak{g} \cdot \alpha$. In particular, the dimension of any $U_\alpha(\mathfrak{g}_{\mathbb{F}})$-module is divisible by p^d.*

1.2. **Reduction to a nilpotent p-character.** Let $\alpha \in \mathfrak{g}_{\mathbb{F}}$. We can decompose α into the sum $\alpha_s + \alpha_n$ of commuting diagonalizable and nilpotent elements (Jordan decomposition). Let $\mathfrak{g}_{0,\mathbb{F}}$ stand for the centralizer of α_s in $\mathfrak{g}_{\mathbb{F}}$ (a so called Levi subalgebra), when $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{g}_0$ is conjugate a subalgebra of block-diagonal matrices.

Then we have the following result.

Proposition 1.6. *$U_\alpha(\mathfrak{g}_{\mathbb{F}}) \cong \mathrm{Mat}_{p^k}(U_\alpha(\mathfrak{g}_{0\mathbb{F}}))$, where $k = \frac{1}{2}\dim \mathfrak{g} \cdot \alpha_s$. In particular, there is a natural bijection $\mathrm{Irr}(U_\alpha(\mathfrak{g}_{\mathbb{F}})) \cong \mathrm{Irr}(U_\alpha(\mathfrak{g}_{0\mathbb{F}}))$.*

Since α_s is central in $\mathfrak{g}_0$, we have an isomorphism $U_\alpha(\mathfrak{g}_{0\mathbb{F}}) \cong U_{\alpha_n}(\mathfrak{g}_{0\mathbb{F}})$.

2. Representations of $\mathrm{GL}_n(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field with q elements (so that $q = p^\ell$ for some prime p and positive integer ℓ). We are interested in representations of the finite group $G := \mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$. In particular, such representations are completely reducible and we only need to classify the irreducible representations. The number of those is the same as the number of conjugacy classes in G. We will explain the classification of conjugacy classes later. In this lecture we will produce the irreducible representations that correspond to unipotent conjugacy classes. Recall that the classification of unipotent matrices up to conjugacy does not depend on the field: the Jordan normal form theorem holds for all operators with eigenvalues in the base field. In particular, we see that the unipotent conjugacy classes are in one-to-one correspondence with the partitions of n.

The idea of construction of the corresponding representations comes from the representation theory of reductive groups. Namely, let B be the subgroup of all upper-triangular matrices in G. We are looking at the irreducible representations of G that have a B-fixed vector. We will see that these irreducible representations are classified by the partitions of n. A crucial tool here is the so called Hecke algebra, a deformation of $\mathbb{C}S_n$.

2.1. $\mathbb{C}[B\setminus G]$ **and its endomorphisms.** We are interested in the irreducible G-modules V such that $V^B \neq \{0\}$, equivalently, such that $(V^*)^B = (V^B)^* \neq 0$. Of course, $(V^*)^B = \mathrm{Hom}_B(V,\mathbb{C})$, where we write $\mathbb{C}$ for the trivial B-module. Recall the coinduced module $\mathrm{Hom}_B(\mathbb{C}G,\mathbb{C}) = \mathbb{C}[B\setminus G]$, where we write $B\setminus G$ for the set of left B-cosets in G and G acts on $\mathbb{C}[B\setminus G]$ by $g.f(g') = f(g'g)$. By its universal property, $\mathrm{Hom}_B(V,\mathbb{C}) = \mathrm{Hom}_G(V,\mathbb{C}[B\setminus G])$. So $V^B \neq \{0\}$ if and only if V is a summand of $\mathbb{C}[B\setminus G]$. Recall that the assignment $V \mapsto \mathrm{Hom}_B(V,\mathbb{C}) = \mathrm{Hom}_G(V,\mathbb{C}[B\setminus G]))$ gives rise to a bijection between the set of $V \in \mathrm{Irr}(G)$ that are summands in $\mathbb{C}[B\setminus G]$ and the $\mathrm{Irr}(\mathrm{End}_G(\mathbb{C}[B\setminus G]))$. So we need to understand the structure of the algebra $\mathrm{End}_G(\mathbb{C}[B\setminus G])$.

First of all, we will give an alternative description of this algebra. Consider the space $\mathbb{C}[B\setminus G]^B$ of B-invariant functions on $B\setminus G$ that is naturally identified with the set of $B\times B$-invariant functions on G. One can define the convolution product $C[B\setminus G]^B \otimes \mathbb{C}[B\setminus G] \to \mathbb{C}[B\setminus G]$ as follows:

$$F * f(g) = |B|^{-1}\sum_{h\in G} F(h)f(h^{-1}g).$$

We have $F*f \in \mathbb{C}[B\setminus G]$ because

$$F*f(bg) = |B|^{-1}\sum_{h\in G} F(h)f(h^{-1}bg) = |B|^{-1}\sum_{h\in G} F(b^{-1}h)f(h^{-1}g) = F*f(g).$$

Also note that $F*? : \mathbb{C}[B\setminus G] \to \mathbb{C}[B\setminus G]$ is a G-equivariant homomorphism. It follows that it restricts to a bilinear map $\mathbb{C}[B\setminus G]^B \otimes \mathbb{C}[B\setminus G]^B \to \mathbb{C}[B\setminus G]^B$. It is straightforward to check that $(F'*F)*f = F'*(F*g)$. So we see that $\mathbb{C}[B\setminus G]^B$ is an associative algebra with respect to convolution that acts on $\mathbb{C}[B\setminus G]$ by G-equivariant endomorphisms. In particular, we have an algebra homomorphism $\mathbb{C}[B\setminus G]^B \to \mathrm{End}_G(\mathbb{C}[B\setminus G])$.

Lemma 2.1. *The homomorphism $\mathbb{C}[B\setminus G]^B \to \mathrm{End}_G(\mathbb{C}[B\setminus G])$ is an isomorphism.*

Proof. We have $\mathrm{Hom}_G(\mathbb{C}[B\setminus G],\mathbb{C}[B\setminus G]) = \mathrm{Hom}_B(\mathbb{C}[B\setminus G],\mathbb{C}) = \mathbb{C}[B\setminus G]^B$. So the two algebras have the same dimension. It remains to check that the homomorphism is injective. Applying $\sum_{h\in G} F(h)f(h^{-1}g) = 0$ to the characteristic functions f of B-orbits in G, we see that $\sum_{h\in B} F(hg) = 0$ for any $g\in G$. We conclude that $F(g) = 0$. □

The realization $\mathrm{End}_G(\mathbb{C}[B\setminus G]) = \mathbb{C}[B\setminus G]^B$ is beneficial for several reasons. First of all, we can find a basis in the right hand side. Embed $W := S_n$ into $G = \mathrm{GL}_n(\mathbb{F}_q)$ as the group of monomial matrices with unit nonzero coefficients. The Gauss elimination algorithm proves the following fact known as the *Bruhat* decomposition.

Lemma 2.2. *We have $G = \bigsqcup_{w\in W} BwB$. In particular, we have the basis $T_w, w\in W$, in $\mathbb{C}[B\setminus G]^B$, where T_w is the characteristic function of BwB.*

Now let us study the product $T_u * T_w$. Let $\mu_{u,w} : BuB\times BwB \to G$ be the multiplication map. Note that B acts freely on $BuB\times BwB, b.(x,y) = (xb^{-1}, by)$ and $\mu_{u,w}$ is B-equivariant so that the fibers are unions of B-orbits. Then

$$T_u * T_w(g) = \frac{1}{|B|}|\mu_{u,w}^{-1}(g)|. \tag{2.1}$$

In particular, we see that T_1 is the unit in $\mathbb{C}[B\setminus G]^B$. Now consider the case when $u = s_i$, the simple transposition $(i,i+1)$. Consider the length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$ that to $w\in W$ assigns the minimal number ℓ such that $w = s_{i_1}\dots s_{i_\ell}$ for some $i_1,\dots,i_\ell$ (such decompositions are called *reduced*). It equals to the number of inversions in w. Note that $\ell(s_iw) = \ell(w)\pm 1$.

Proposition 2.3. *We have* $T_sT_w = T_{sw}$ *if* $\ell(sw) = \ell(w) + 1$ *and* $T_sT_w = qT_{sw} + (q-1)T_w$ *if* $\ell(sw) = \ell(w) - 1$ *(where we write* s *for* s_i*).*

Proof. We have $|BwB|/|B| = q^{\ell(w)}$. This can be deduced from the Gauss elimination algorithm or from the equality $|BwB|/|B| = |B|/|B \cap wBw^{-1}|$ (the intersection can be described explicitly).

Consider the case $\ell(sw) = \ell(w)+1$ so that $|BswB|/|B| = (|BsB|/|B|)(|BwB|/|B|)$. Note that $BswB$ lies in the image of $\mu_{s,w}$ and so we get $|\mu_{s,w}^{-1}(g)| = |B|$ if $g \in BswB$ and $\mu_{s,w}^{-1}(g)$ is empty else. We deduce that $T_sT_w = T_{sw}$.

Now let us consider the case when $\ell(sw) = \ell(w) - 1$. Let $u = sw$. By the previous case, $T_w = T_s * T_u$. So we just need to prove that $T_s^2 = q + (q-1)T_s$. We have the inclusion $BsB \subset P_i$, where P_i consists of all matrices (a_{jk}) such that $a_{jk} \neq 0$ implies $j \leqslant k$ or $j = i+1, k = i$. Therefore $BsBBsB \subset BsB \sqcup B$ so the only basis elements that can occur with nonzero multiplicities in T_s^2 are $T_s, 1$. The preimage of 1 under $\mu_{s,s}$ is isomorphic to BsB and so the coefficient of 1 equals $|BsB|/|B| = q$. Since $|BsB|^2 = |\mu^{-1}(1)||B| + |\mu^{-1}(s)||BsB| = q|B| + |\mu^{-1}(s)|q|B|$, we deduce that $|\mu^{-1}(s)|/|B| = q-1$. This proves $T_s^2 = q + (q-1)T_s$. ☐

Below we will write T_i instead of T_{s_i}.

Corollary 2.4. *We have* $T_i^2 = (q-1)T_i + q, T_iT_j = T_iT_j$ *if* $|i-j| > 1$ *and* $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$.

An advantage of looking at $\operatorname{End}_G(\mathbb{C}[B \setminus G])$ is that this algebra is manifestly semisimple.

2.2. **Hecke algebra over** $\mathbb{Z}[v^{\pm 1}]$**.** Let v be an independent variable. We define the $\mathbb{Z}[v^{\pm 1}]$-algebra $H_v(n)$ by the generators $T_i, i = 1, \ldots, n-1$ and the relations as in Corollary 2.4, where q is replaced with v. For $w \in W$, we define an element T_w as follows. Choose a reduced expression $w = s_{i_1} \ldots s_{i_\ell}$, where $\ell = \ell(w)$. It is a classical fact that any two reduced expressions of w are obtained from one another by a sequence of *braid moves*: replacing s_is_j with s_js_i when $|i-j| > 1$, and replacing $s_is_{i+1}s_i$ with $s_{i+1}s_is_{i+1}$ and vice versa. Set $T_w = T_{i_1} \ldots T_{i_\ell}$, this is well-defined.

Theorem 2.5. *The algebra* $H_v(n)$ *is a free* $\mathbb{Z}[v^{\pm 1}]$*-module with basis* $T_w, w \in W$.

Proof. We note that $T_iT_w = T_{s_iw}$ if $\ell(s_iw) = \ell(w) + 1$, and $T_iT_w = (v-1)T_w + vT_{s_iw}$ if $\ell(s_iw) = \ell(w) - 1$. So the span of T_w's is closed under the multiplication by the generators and hence T_w's span $H_v(n)$. In order to show that the elements T_w are linearly independent over $\mathbb{Z}[v^{\pm 1}]$, consider the free $\mathbb{Z}[v^{\pm 1}]$-module U with basis $u_w, w \in W$. Define an action of the generators T_i on U by

$$T_iu_w = \begin{cases} u_{s_iw}, & \ell(s_iw) = \ell(w) + 1, \\ (v-1)u_w + vu_{s_iw}, & \ell(s_iw) = \ell(w) - 1. \end{cases} \tag{2.2}$$

It is straightforward (but tedious) to check that this extends to an $H_v(n)$-action. Since $T_wu_1 = u_w$, we see that the elements T_w are linearly independent. ☐

For $z \in \mathbb{C}^\times$, set $H_{\mathbb{C},z}(n) := \mathbb{C}_z \otimes_{\mathbb{Z}[v^{\pm 1}]} H_v(n)$, where the homomorphism $\mathbb{Z}[v^{\pm 1}] \to \mathbb{C}_z$ is given by $v \mapsto z$. We note that $\mathbb{C}[B \setminus G]^B = H_{\mathbb{C},q}(n)$, while $\mathbb{C}S_n = H_{\mathbb{C},1}(n)$.

2.3. **Structure of Hecke algebras.** The following result is known as the Tits deformation principle.

Theorem 2.6. *Let X be a principal open subset in $\mathbb{C}^n$ and A is a free $\mathbb{C}[X]$-algebra of finite rank. For any two points $x, y \in X$, if the specializations A_x, A_y are semisimple, then they are isomorphic.*

Corollary 2.7. *We have an isomorphism $\mathcal{H}_{\mathbb{C},q}(n) \cong \mathcal{H}_{\mathbb{C},1}(n)$.*

Proof. We apply Theorem 2.6 to $X = \mathbb{C}^\times, A = \mathbb{C}[v^{\pm 1}] \otimes_{\mathbb{Z}[v^{\pm 1}]} H_v(n), x = 1, y = q$. □

Proof of Theorem 2.6. The proof is in several steps.

Step 1. Let r be the rank of A. Pick some basis $v_1, \ldots, v_r$ of the $\mathbb{C}[X]$-module A. The coefficient of v_ℓ in $v_i v_j$ is an element of $\mathbb{C}[X]$. So we get a morphism $\varphi : X \to \mathbb{C}^{r*} \otimes \mathbb{C}^{r*} \otimes \mathbb{C}^r$ of algebraic varieties that sends $x \in X$ to the multiplication of A_x (in basis $v_1, \ldots, v_r$).

Step 2. Recall the form $(\cdot, \cdot)$ on the associative algebra A given by $(a, b) = \mathrm{tr}_A(ab)$. Its entries are again functions on $\mathbb{C}[X]$. The locus where this form is non-degenerate is the locus of $x \in X$ such that A_x is semisimple. So we can replace X with a principal open subset and assume that A_x is semisimple for any $x \in X$.

Step 3. The group $\mathrm{GL}_r(\mathbb{C})$ acts on the space of products $\mathbb{C}^{r*} \otimes \mathbb{C}^{r*} \otimes \mathbb{C}^r$ by base changes. There are finitely many orbits of this group corresponding to semisimple associative algebras. We see that the image of φ lies in the union of these orbits.

Step 4. Pick a point x and let $y_1, \ldots, y_n$ be affine coordinates on X centered at x. Set $R := \mathbb{C}[[y_1, \ldots, y_n]]$. Consider the algebra $\hat{A} = R \otimes_{\mathbb{C}[X]} A$. This is an R-algebra that is a free finite rank module over R such that $\hat{A}/\mathfrak{m}\hat{A} = A_x$, where $\mathfrak{m} \subset R$ is the maximal ideal. We want to prove that $\hat{A} \cong R \otimes A_x$ (in other words, A is a trivial bundle of algebras over the formal neighborhood of x in X).

Step 5. We will use the result about lifting of idempotents: if e is an element in A_x such that $e^2 = e$, then there is an element $\hat{e} \in \hat{A}$ that maps to e under the projection $\hat{A} \twoheadrightarrow A_x$ and satisfies $\hat{e}^2 = \hat{e}$.

Pick primitive idempotents (=diagonal matrix unit) $e_1, \ldots, e_k$, one per each direct summand. Lift them to idempotents $\hat{e}_1, \ldots, \hat{e}_k \in \hat{A}$. So $\hat{V}_i := \hat{A}\hat{e}_i$ is free over R and $\hat{V}_i/\mathfrak{m}\hat{V}_i = V_i (= A_x e_i)$. We have an algebra homomorphism $\hat{A} \to \bigoplus_{i=1}^k \mathrm{End}_R(\hat{V}_i)$ given by the action of $\hat{A}$ on $\hat{V}_1 \oplus \ldots \oplus \hat{V}_k$. It specializes to the homomorphism $A_x \to \bigoplus_{i=1}^k \mathrm{End}(V_i)$ given by the action of A_x on $V_1 \oplus \ldots \oplus V_k$. But the latter homomorphism is an isomorphism. Now we are done by a standard fact: let $\psi : M \to N$ be a homomorphism of free finite rank R-modules that is an isomorphism after specializing to the residue field. Then ψ is an isomorphism.

Step 6. The preimage of a locally closed subvariety under a morphism is a locally closed subvariety. The claim that A is a trivial bundle of algebras over a formal neighborhood of x in X shows that the preimage of any orbit of a semisimple associative algebra under φ is open. Since X is irreducible, we see that only one of these preimages is nonzero. □

One can ask, for which q the algebra $\mathcal{H}_{\mathbb{C},q}(n)$ is semisimple. The answer is: if and only if q is not a root of 1 of order $\leqslant n$. In fact, one can develop the representation theory of $\mathcal{H}_{\mathbb{C},q}(n)$ for q as above in the same fashion as for $\mathbb{C}S_n$ by using the multiplicative versions of the Jucys-Murphi elements to be introduced in Homework 3. Using this construction one can produce a natural bijection between $\mathrm{Irr}(\mathcal{H}_{\mathbb{C},q}(n))$ and $\mathrm{Irr}(S_n)$ (that cannot be deduced from Theorem 2.6).

References

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