LECTURE 9: FINITE GROUPS OF LIE TYPE AND HECKE ALGEBRAS

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Introduction

We continue to study the representation theory of finite groups of Lie type and its connection to Hecke algebras, now in a more general setting. We start by defining Hecke algebras of arbitrary Weyl groups. Then we introduce finite groups G of Lie type and relate the co-induced representations $\mathbb{C}[B\setminus G]$ to Hecke algebras, similarly to what was done for $\mathrm{GL}_n(\mathbb{F}_q)$.

In the second part of this lecture we discuss a way to produce representations of finite groups of Lie type that is of crucial importance for the classification of irreducibles and computation of their characters: the Deligne-Lusztig induction. We start by explaining the induction (or, in our conventions, co-induction) of a character of B lifted from a character of T. Then we describe maximal tori of G, since our field is not algebraically closed two maximal tori do not need to be conjugate. Finally, we discuss the Deligne-Lusztig induction, it is constructed using étale cohomology and produces a virtual representation of G starting with a character of a torus.

1. Hecke algebras and finite groups of Lie type

1.1. **Generic Hecke algebras.** Let $G_{\mathbb{C}}$ be a semisimple algebraic group over $\mathbb{C}$ and W be its Weyl group. Recall that each reflection in W is conjugate to a simple reflection. Let S denote the set of simple reflections in W. We define independent variables v_s such that $v_s = v_t$ and $s,t \in S$ are such that s and t are conjugate in W. For example, in types A, D, E (simply laced types) all reflections are conjugate (any two adjacent simple roots are conjugate in W and hence the corresponding reflections are conjugate). So here we have one variable. In types B, C, F, G we have two possible lengths of roots that lead to two conjugacy classes and so will have two variables v_s. For $s,t \in S$ let m_{st} be the number of edges between s,t in the Dynkin diagram plus 2 so that $m = m_{st}$ is the minimal positive integer such that $(st)^m = 1$, equivalently, $sts\ldots = tst\ldots$, where in each side we have m factors.

We define the generic Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ as the algebra over $Z[v_s^{\pm 1}]_{s\in S}$ generated by the elements $T_s, s \in S$, with relations

$$T_sT_tT_s\ldots = T_tT_sT_t\ldots \ (m_{st} \text{ factors}), \quad (T_s - v_s)(T_s + 1) = 0.$$

If $W = S_n$, we get the algebra $\mathcal{H}_v(n)$ introduced in the previous lecture. As another example, consider the Weyl group of type B_n (or C_n, they are the same). We number the simple reflections as follows $s_0 = s_{\epsilon_n}, s_i = s_{\epsilon_{n+1-i}-\epsilon_{n-i}}, i = 1,\ldots,n-1$. We have $m_{ij} = 2$ if $|i-j| > 1, m_{01} = 4, m_{i,i+1} = 3$ if $i > 0$. We write v for $v_i, i > 0$ and V for v_0. The relations become as follows: $T_1,\ldots,T_{n-1}$ have relations as in $\mathcal{H}_v(n)$, while $T_0T_i = T_iT_0$ for $i > 1$, $T_0T_1T_0T_1 = T_1T_0T_1T_0, (T_0 - V)(T_0 + 1) = 0$.

For $w \in W$, we have an element $T_w = T_{i_1}\ldots T_{i_k}$ for a reduced expression $w = s_{i_1}\ldots s_{i_k}$, again, T_w is well-defined. Similarly to the S_n-case we have the following theorem.

Theorem 1.1. *The algebra $\mathcal{H}(W)$ is a free module over $\mathbb{Z}[v_s^{\pm 1}]$ with basis $T_w, w \in W$.*

For numerical values $q_s \in \mathbb{C}^\times$ of the variables v_s, we can consider the specialization $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$ of $\mathcal{H}(W)$, a $\mathbb{C}$-algebra. The Tits deformation principle implies the following.

Proposition 1.2. *Suppose the algebra $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$ is semisimple. Then it is isomorphic to $\mathbb{C}W$.*

Remark 1.3. All constructions here generalize to real reflection groups (finite groups of isometries of a Euclidian space generated by reflections), e.g. to dihedral groups.

We can also define Hecke algebras for arbitrary symmetrizable Cartan matrices, they will deform the group algebras of the Weyl groups. These Hecke algebras are going to be important for us when we discuss the character formulas for irreducible rational representations of reductive algebraic groups. Note that Proposition 1.2 no longer holds because the Hecke algebra is no longer finite dimensional.

1.2. **Finite groups of Lie type.** Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Consider a connected reductive algebraic group $G_{\mathbb{F}}$. This group is known to be defined over $\mathbb{F}_p$. Further, we can find a maximal torus and Borel subgroup $T_{\mathbb{F}} \subset B_{\mathbb{F}} \subset G_{\mathbb{F}}$ defined over $\mathbb{F}_p$. For the classical groups $\mathrm{SL}_n(\mathbb{F}), \mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$, this can be checked directly (we take all diagonal matrices for $T_{\mathbb{F}}$ and all upper triangular matrices for $B_{\mathbb{F}}$, recall that we take forms given by anti-diagonal matrices to define $\mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$).

Now pick $\ell > 0$ and set $q = p^\ell$. Let Fr denote the automorphism $x \mapsto x^q$ of $\mathbb{F}$ so that $\mathbb{F}_q$ is the fixed point locus of Fr (we will write Fr_q when we want to indicate the dependence on q). Since $G_{\mathbb{F}}$ is defined over $\mathbb{F}_p$, we get the Frobenius homomorphism $\mathsf{Fr} : G_{\mathbb{F}} \to G_{\mathbb{F}}$. We can define the group $G = G_{\mathbb{F}_q}$ as the fixed point set of Fr in $G_{\mathbb{F}}$. The group G is a special case of a finite group of Lie type. Examples include $\mathrm{SL}_n(\mathbb{F}_q)$ (or $\mathrm{GL}_n(\mathbb{F}_q)$), $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and $\mathrm{SO}_n(\mathbb{F}_q)$. Let us write $N_{\mathbb{F}}$ for the normalizer of $T_{\mathbb{F}}$ in $G_{\mathbb{F}}$. Clearly, $N_{\mathbb{F}}$ is Fr-stable and we set $N := N_{\mathbb{F}}^{\mathsf{Fr}}$. So we have subgroups $T := T_{\mathbb{F}_q} \subset N, B := B_{\mathbb{F}_q} \subset G$ (a maximal torus and a Borel subgroup).

More generally, let Φ be an automorphism of $G_{\mathbb{F}}$ such that some power of Φ is Fr_{q^k}. Suppose that $T_{\mathbb{F}}, B_{\mathbb{F}}$ are Φ-stable (so that $N_{\mathbb{F}}$ is also Φ-stable). We get the fixed point subgroup $G := G_{\mathbb{F}}^{\Phi} \subset G_{\mathbb{F}_{q^k}}$. This is a general case of a *split* finite group of Lie type. We also get subgroups $T := T_{\mathbb{F}}^{\Phi} \subset N := N_{\mathbb{F}}^{\Phi}, B := B_{\mathbb{F}}^{\Phi}$.

We want to give a classical example of $G = G_{\mathbb{F}}^{\Phi}, \Phi \neq \mathsf{Fr}_q$: a finite unitary group. Recall that if we have a hermitian form h on a complex vector space V, we can define the unitary group $U(h)$ of all linear transformations of V that are unitary with respect to h (when h is positive definite we get the usual unitary group). If J is the matrix of h, then $U(h) = \{A \in \mathrm{GL}_n(\mathbb{C}) | \bar{A}^t J A = J\}$ (the superscript "t" means the transposed matrix).

Now consider the group $\mathrm{GL}_n(\mathbb{F})$. Let J denote the matrix with 1's on the main anti-diagonal and zeroes elsewhere. We have an automorphism α of $\mathrm{GL}_n(\mathbb{F})$ given by $A \mapsto JA^tJ$. Note that $\alpha^2 = 1$ and $\alpha \circ \mathsf{Fr}_q = \mathsf{Fr}_q \circ \alpha$. Set $\Phi := \alpha \circ \mathsf{Fr}_q$ so that $\Phi = \alpha \circ \Phi, \Phi^2 = \mathsf{Fr}_{q^2}$. Consider the group $\mathrm{GL}_n(\mathbb{F})^{\Phi} = \{A \in \mathrm{GL}_n(\mathbb{F}_{q^2}) | \bar{A}^t J A = J\}$, where $\bar{A}$ is obtained from A by applying $\mathsf{Fr}_q : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$. The group $\mathrm{GL}_n(\mathbb{F})^{\Phi}$ is called the finite unitary group and is denoted by $\mathrm{GU}_n(\mathbb{F}_q)$.

One reason to care about finite groups of Lie type comes from the theory of finite simple groups. Each finite simple group is either

- the alternating group A_n,
- or a (generally, non-split) finite group of Lie type,
- or one of finitely many sporadic finite simple groups.

1.3. **Hecke algebras and representations of G.** Let $G := G_{\mathbb{F}}^{\Phi}$, where Φ is as above. Then we have subgroups $T \subset N, B \subset G$. Clearly $T \subset N$ is a normal subgroup and we can form the quotient $W := N/T$. We set $BwB := BnB$, where n is any representative of w in N. The following theorem gives the Bruhat decomposition.

Theorem 1.4. *We have* $G = \bigsqcup_{w \in W} BwB$.

Inside W we can find a subset S of involutions such that (W, S) becomes a Coxeter system (so that W is a real reflection group and S is the set of simple reflections). Furthermore, we have $BswB = BsBBwB$ if $\ell(sw) = \ell(w)+1$ and $BswB \sqcup BwB = BswB$ if $\ell(sw) = \ell(w)-1$. Set $q_s := |BsB|/|B|$, one can show $q_s = q_t$ if s, t are conjugate in W. Moreover, if $\Phi = \mathsf{Fr}_q$, we have $q_s = q$ for all $s \in S$.

Example 1.5. Consider the case $G = \mathrm{GU}_n(\mathbb{F}_q)$. In this case, W is the Weyl group of type $B_{\lfloor n/2 \rfloor}$. We have $q_i = q^2$ for $i > 0$. If n is even, we get $q_0 = q$, and if n is odd, then $q_0 = q^3$. This is a part of Homework 3.

Consider the specialization $\mathcal{H}_{\mathbb{C}, q_\bullet}(W)$, where the variable v_s goes to q_s.

Theorem 1.6. *The endomorphism algebra* $\mathrm{End}_G(\mathbb{C}[B \setminus G])$ *is isomorphic to* $\mathcal{H}_{\mathbb{C}, q_\bullet}(W)$.

This allows to produce some irreducible representations of G (the number is equal to the number of W-irreps). Of course, this is just a tiny portion of all representations.

2. Deligne-Lusztig induction

2.1. **Induction from Borel.** Let us produce more irreducible representations. Before we co-induced from the trivial B-module. Now let us consider one-dimensional B-modules with trivial action of $U := U_{\mathbb{F}}^{\Phi}$, the unipotent subgroup (in our examples, this is the group of all strictly upper-triangular matrices). Such a module is given by a character of T, say χ. The coinduced module $\mathrm{Hom}_B(\mathbb{C}G, \mathbb{C}_\chi)$ coincides with $\mathbb{C}[B \setminus_\chi G] = \{f \in \mathbb{C}[G] | f(bg) = \chi(b) f(g)\}$. The endomorphism algebra $\mathrm{End}_G(\mathbb{C}[B \setminus_\chi G])$ is $\mathrm{Hom}_G(\mathbb{C}[B \setminus_\chi G], \mathbb{C}_\chi) = \mathbb{C}[G]^{B \times B, \chi \times \chi^{-1}}$, where, by definition, the right hand side is $\{f \in \mathbb{C}[G] | f(b_1 g b_2) = \chi(b_1) f(g) \chi(b_2)\}$. The latter coincides with $\bigoplus_{w \in W} \mathbb{C}[BwB]^{B \times B, \chi \times \chi^{-1}}$. The space $\mathbb{C}[BwB]^{B \times B, \chi \times \chi^{-1}}$ is one dimensional if $\chi \in \mathrm{Hom}(T, \mathbb{C}^\times)$ is fixed by w^{-1} and is zero else. Indeed, take $b_1, b_2 \in T, b_1 = n b_2 n^{-1}$, where n is a representative of w. Then we get $f(n)\chi(b_2) = f(nb_2) = f(b_1 n) = \chi(n b_2 n^{-1}) f(n)$ meaning $\chi(b_2) = \chi(n b_2 n^{-1}) = (w^{-1}.\chi)(b_2)$. So if $f(n) \neq 0$, then χ is fixed by w^{-1}. A converse is an exercise.

As we have seen, if $\chi = 1$, then each space $\mathbb{C}[BwB]^{B \times B, \chi \times \chi^{-1}}$ is one-dimensional. The other extreme is when χ is generic, i.e., it is not fixed by any non-trivial Weyl group element. In this case $\dim \mathrm{End}_G(\mathbb{C}[B \setminus_\chi G]) = 1$ and so $\mathbb{C}[B \setminus_\chi G]$ is irreducible. The general case interpolates between these two: the endomorphism algebra will be isomorphic to the (suitably understood) Hecke algebra of W_χ, the stabilizer of χ. Let us point out that W_χ does not need to be a Coxeter group. This does not happen for $G = \mathrm{GL}_n(\mathbb{F}_q)$: here χ can be thought as an element of $(\mathbb{F}_q^\times)^n$, and the group W_χ is the product of symmetric groups.

Lemma 2.1. *The modules* $\mathbb{C}[B \setminus_\chi G]$ *and* $\mathbb{C}[B \setminus_{\chi'} G]$ *have common direct summands if and only if* χ, χ' *are* W*-conjugate. In the latter case, they are isomorphic.*

Proof. Consider the space $\{f \in \mathbb{C}[BwB] | f(b_1 g b_2) = \chi(b_1) f(g) \chi'(b_2)\}$. It is one-dimensional if and only if $\chi = w\chi'$ (this follows from an argument above) and is zero else. This implies the first claim.

The second claim for $G = \mathrm{GL}_n(\mathbb{F}_q)$ will be a part of Homework 3. ☐

2.2. **Conjugacy classes and tori.** Now let us discuss the structure of conjugacy classes in G. We start with $G = \mathrm{GL}_n(\mathbb{F}_q)$. Let α be a primitive element in $\mathbb{F}_{q^m}$. Consider the endomorphism $A_\alpha : \mathbb{F}_q^m \to \mathbb{F}_q^m$ (given by the multiplication by α in $\mathbb{F}_q^m = \mathbb{F}_{q^m}$). After a base change to $\mathbb{F}$, the operator A_α becomes diagonalizable with eigenvalues $\alpha, \mathsf{Fr}_q(\alpha), \ldots, \mathsf{Fr}_q^{m-1}\alpha$. So the operators A_α, A_β are conjugate if and only if $\beta = \mathsf{Fr}_q^k\alpha$ in this case we say that α, β are equivalent.

We say that an element $A \in \mathrm{GL}_n(\mathbb{F}_q)$ is semisimple if it becomes diagonalizable over $\mathbb{F}$. Any semisimple element is conjugate to an operator of the form $\mathrm{diag}(A_{\alpha_1}, \ldots, A_{\alpha_k})$, where $\alpha_1, \ldots, \alpha_k$ are defined uniquely up to a permutation and equivalences. More generally, we have an obvious analog of the Jordan normal form theorem.

Now fix a partition $\nu = (\nu_1, \ldots, \nu_k)$ of n and consider the set T_ν of all elements of the form $\mathrm{diag}(A_{\alpha_1}, \ldots, A_{\alpha_k})$, where $\alpha_i \in \mathbb{F}_{q^{\nu_i}}$ (not necessarily primitive). This is a subgroup. For $\nu = (1, 1, \ldots, 1)$, we get $T_\nu = T$. The subgroups T_ν are the maximal tori in $G = \mathrm{GL}_n(\mathbb{F}_q)$ (up to conjugacy). All of them but T are not included into a Borel subgroup (=do not preserve a complete flag).

Let us extend this construction to a general G (with $\Phi = \mathsf{Fr}_q$) and make it more conceptual. If $S_\mathbb{F} \subset G_\mathbb{F}$ a Fr-stable maximal torus, then $(S_\mathbb{F})^{\mathsf{Fr}}$ is an abelian subgroup in G. But it does not need to be conjugate to T. We have $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$ for some $g \in G_\mathbb{F}$. The equality $\mathsf{Fr}(S_\mathbb{F}) = S_\mathbb{F}$ is equivalent to $g^{-1}\mathsf{Fr}(g) \in N_\mathbb{F}$. Now we have the following theorem of Lang.

Theorem 2.2. *The map $L : G_\mathbb{F} \to G_\mathbb{F}$ given by $g \mapsto g^{-1}\mathsf{Fr}(g)$ is surjective.*

Proof. The proof is based on the observation that the tangent map of Fr is zero at all points. Consider the action of $G_\mathbb{F}$ on itself given by $g.h := gh\mathsf{Fr}(g)^{-1}$. For any fixed h, the map $g \mapsto g.h$ is etale (all tangent maps are iso). So any $G_\mathbb{F}$-orbit on $G_\mathbb{F}$ has dimension $\dim G_\mathbb{F}$ and is open. Since $G_\mathbb{F}$ is connected, we get a single orbit. This shows that the map $g \mapsto g\mathsf{Fr}(g)^{-1}$ is surjective. It follows that L is surjective. □

So pick $n \in N_\mathbb{F}$ and set $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$, where $g \in G_\mathbb{F}$ is such that $g^{-1}\mathsf{Fr}(g) = n$. Up to G-conjugacy, the subgroup $(S_\mathbb{F})^{\mathsf{Fr}}$ depends only on the image w of n in W. So we denote it by T_w. Moreover, it only depends on the conjugacy class of w.

Example 2.3. When $G = \mathrm{GL}_n(\mathbb{F})$, then $T_\nu = T_w$, where ν is the cycle type of w. It is enough to check this when w is a single cycle. Under the identification of $T_\mathbb{F}$ with $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$ given by $t \mapsto gtg^{-1}$, the morphism $\mathsf{Fr} : S_\mathbb{F} \to S_\mathbb{F}$ becomes $t \mapsto w(\mathsf{Fr}(t))$. We can identify $T_\mathbb{F}$ with $(\mathbb{F}^\times)^n$ such that w permutes the coordinates on $T_\mathbb{F}$ (in a cycle). So the fixed points in $T_\mathbb{F}$ become $(z, \mathsf{Fr}(z), \ldots, \mathsf{Fr}^{n-1}(z))$, where $\mathsf{Fr}^n(z) = z$, i.e., $z \in \mathbb{F}_{q^n}$. From here it is easy to see that $(S_\mathbb{F})^{\mathsf{Fr}}$ is conjugate to $T_{(n)}$.

In fact, any semisimple element in G is conjugate to an element in T_w, note that the conjugacy class of w is not uniquely determined by the element, for example, the constant matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ lie in all tori T_w.

2.3. **Deligne-Lusztig induction.** Recall that we can produce a representation $\mathbb{C}[B \setminus_\chi G]$ from a character χ of T. This involves the choice of B but as different B's containing T are conjugate by W, by Lemma 2.1, this choice does not affect $\mathbb{C}[B \setminus_\chi G]$. We denote this representation by $R_T^G(\chi)$.

We want to have a similar construction for an arbitrary maximal torus T_w of G. The problem is that this torus is not included into a Borel subgroup. Deligne and Lusztig solved this problem in [DL] defining what is now called the Deligne-Lusztig induction.

Let $S_{\mathbb{F}} \subset G_{\mathbb{F}}$ be a Fr-stable maximal torus and let $B_{\mathbb{F}} = S_{\mathbb{F}} \ltimes U_{\mathbb{F}}$ be a Borel subgroup, it is not Fr-stable, in general. Consider the subvariety $Y := L^{-1}(U_{\mathbb{F}}) \subset G_{\mathbb{F}}$, i.e., $\{g \in G | g^{-1}\mathsf{Fr}(g) \in U_{\mathbb{F}}\}$. Note that G acts on Y by left multiplications: for $h \in G$, we have $L(hg) = (hg)^{-1}\mathsf{Fr}(hg) = g^{-1}h^{-1}\mathsf{Fr}(h)\mathsf{Fr}(g) = g^{-1}\mathsf{Fr}(g) = L(g)$. The group $T_w = (S_{\mathbb{F}})^{\mathsf{Fr}}$ acts by right multiplications because $S_{\mathbb{F}}$ normalizes $U_{\mathbb{F}}$. Clearly, the actions of G and T_w commute.

We want to get a virtual representation of $G \times T_w$ from its action on Y ("virtual" means that it is a formal linear combination of irreducibles with integral coefficients). This representation will be the Euler characteristic of Y. In order to define the Euler characteristic, we need some cohomology theory. The variety Y is over $\mathbb{F}$ so it is not a topological space in any reasonable sense.

Yet there is a suitable cohomology theory, it is called étale cohomology. More precisely, we pick a prime ℓ that does not divide q. Consider the ℓ-adic field $\mathbb{Q}_\ell$. Then, for an algebraic variety X over $\mathbb{F}$, we can define the ith cohomology group with compact support $H^i_c(X, \mathbb{Q}_\ell)$. This cohomology group is a finite dimensional vector space over $\mathbb{Q}_\ell$, it is zero when $i < 0$ or i is large enough so it makes sense to speak about the Euler characteristic $\chi(X, \mathbb{Q}_\ell)$. When X has an action of a group H, all cohomology groups $H^i_c(X, \mathbb{Q}_\ell)$ carry a representation of H. In particular, $\chi(Y, \mathbb{Q}_\ell)$ is a virtual representation of $G \times T_w$ (over $\mathbb{Q}_\ell$). The algebraic closure $\overline{\mathbb{Q}}_\ell$ is known to be isomorphic to $\mathbb{C}$. So $\chi(Y, \mathbb{C}) := \mathbb{C} \otimes_{\mathbb{Q}_\ell} \chi(Y, \mathbb{Q}_\ell)$ is a complex virtual representation of $G \times T_w$ that can be shown to be independent of the choice of $B_{\mathbb{F}}$.

Example 2.4. Suppose that $U_{\mathbb{F}}$ is Fr-stable. Then the Lang map $L_U : U_{\mathbb{F}} \to U_{\mathbb{F}}$ is surjective. The fiber over 1 is U. It follows that $Y = G \times_U U_{\mathbb{F}}$. The group $U_{\mathbb{F}}$ is an affine space and so $H^i_c(U_{\mathbb{F}}, \mathbb{Q}_\ell)$ vanishes unless $i = 2\dim U_{\mathbb{F}}$, in the latter case, $\dim H^i_c(U_{\mathbb{F}}, \mathbb{Q}_\ell) = 1$. So, $\chi(Y, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Q}_\ell} H^{2\dim U_{\mathbb{F}}}_c(Y, \mathbb{Q}_\ell) = \mathbb{C}[U \setminus G]$ (an equality of $G \times T$-modules).

Now let us explain the construction of $R^G_{T_w}(\chi)$, where χ is a character of T_w. We simply take $R^G_{T_w}(\chi) := [\chi(Y, \mathbb{C}) \otimes \chi]^{T_w}$. In the case when $T_w = T$, we recover the co-induced module $\mathbb{C}[B \setminus_\chi G]$.

The virtual representations $R^G_{T_w}(\chi)$ were extensively studied by Deligne and Lusztig, and then by Lusztig, see, e.g. [L], there is also an exposition of these results in [C]. For example, one can show that every irreducible representation of G appears in one of $R^G_{T_w}(\chi)$.

One can define the notion of conjugate characters χ of T_w, and χ' of $T_{w'}$. It is easy to define this notion when w is conjugate to w', while in general one should view χ, χ' as elements of the so called Langlands dual group. For example, the trivial characters of different tori are all conjugate. It was shown in [DL] that if χ, χ' are not conjugate, then $R^G_{T_w}(\chi)$ and $R^G_{T_{w'}}(\chi')$ do not have common summands.

The most interesting and complicated case is when $\chi = 1$. The representations that appear in $R^G_{T_w}(1)$ are called *unipotent*. In the case when $G = \mathrm{GL}_n(\mathbb{F}_q)$ all unipotent representations are already realized for $w = 1$, while outside of type A, this is not so. Morally, the unipotent representations should correspond to unipotent conjugacy classes, but this correspondence is subtle (for example, there are three unipotent conjugacy classes for $\mathrm{SL}_2(\mathbb{F}_q)$ for odd q but only two unipotent representations).

References

[C] R. Carter. *Finite groups of Lie type. Conjugacy classes and complex characters.* 2nd ed. John Wiley and sons, 1993.

[DL] P. Deligne, G. Lusztig. *Representations of reductive groups over finite fields.* Ann. of Math. (2), 103 (1976), 103-161.

[L] G. Lusztig. *Characters of reductive groups over a finite field*, Ann. Math. Studies 107, Princeton University Press (1984).