REPRESENTATION THEORY, PROBLEM SET 1

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The deadline for submitting the solutions is Sept 30. The solutions are to be submitted electronically (scanned hand-written solutions are fine). E-mail i.loseu@neu.edu.

There are four problems with total number of points equal to 30. The maximal number of points you get for this problem set is 20. Everything above 20 does not count. Partial credit is given.

1) Infinite dimensional Schur lemma. Let A be a countable dimensional algebra over an uncountable algebraically closed field $\mathbb{F}$. Let V be an infinite dimensional irreducible A-module. We want to prove that $\operatorname{End}_A(V) = \mathbb{F}$.

1) Show that V is countable dimensional (2pts).

2) Show that $\operatorname{End}_A(V)$ is at most countable dimensional (2pts).

3) Now let $\varphi \in \operatorname{End}_A(V)$ be a non-constant element. Show that $\varphi - z$ is invertible for any $z \in \mathbb{F}$. Show that the elements $(\varphi - z)^{-1}, z \in \mathbb{F}$, are linearly independent (2pts).

4) Prove that $\operatorname{End}_A(V) = \mathbb{F}$ (2pts).

2) Basis for $\mathcal{H}(d)$. The algebra $\mathcal{H}(d)$ was introduced in Section 2.4 of Lecture 2.

Note that we have algebra homomorphisms $\iota_1 : \mathbb{C}[X_1, \ldots, X_d] \to \mathcal{H}(d)$ and $\iota_2 : \mathbb{C}S_d \to \mathcal{H}(d)$. The goal of this problem is to show that $\mathbb{C}[X_1, \ldots, X_d] \otimes \mathbb{C}S_d \to \mathcal{H}(d), f \otimes \sigma \mapsto \iota_1(f) \otimes \iota_2(\sigma)$ is a vector space isomorphism.

1) Show that the assignment

$$X_i.f := x_i f, T_i.f := s_i f + \frac{s_i f - f}{x_{i+1} - x_i}$$

extends to a representation of $\mathcal{H}(d)$ in $\mathbb{C}[x_1, \ldots, x_d]$, where, for $f \in \mathbb{C}[x_1, \ldots, x_d]$, we write $s_i f$ for the polynomial obtained from f by swapping x_i and x_{i+1} (3pts).

2) Let $\varphi : \mathcal{H}(d) \to \operatorname{End}(\mathbb{C}[x_1, \ldots, x_d])$ be the representation from part 1). Show that the composition of φ with the map $\mathbb{C}[X_1, \ldots, X_d] \otimes \mathbb{C}S_d \to \mathcal{H}(d)$ is injective (4pts) (2pts if you only consider the case $d = 2$).

3) Deduce the claim in the beginning of the problem (1pt).

3) Center of $\mathcal{H}(d)$. We embed $\mathbb{C}[X_1, \ldots, X_d], \mathbb{C}S_d$ into $\mathcal{H}(d)$ as in Problem 2.

1) Show that, for $F \in \mathbb{C}[X_1, \ldots, X_d] \subset \mathcal{H}(d)$, we have

$$T_i F = (s_i F) T_i + \frac{s_i F - F}{X_{i+1} - X_i}$$

(2pts).

2) Show that $\mathbb{C}[X_1, \ldots, X_d]^{S_d}$ (the algebra of symmetric polynomials) lies in the center of $\mathcal{H}(d)$ (2pts).

3) Use Problem 1 to deduce that all irreducible representations of $\mathcal{H}(d)$ are finite dimensional (2pts).

4) Show that the dimensions of the irreducible representations of $\mathcal{H}(d)$ are bounded by $n!$ (2pts).

5*) Prove that $\mathbb{C}[X_1, \ldots, X_d]^{S_d}$ coincides with the center of $\mathcal{H}(d)$ (2pts).

4) Automorphisms of branching graph. 1) Use the identification $\mathsf{Wt}(n) \cong \mathsf{SYT}(n)$ to give the description of the branching graph in terms of Young diagrams. Namely, show that a diagram μ with $n-1$ boxes is connected to a diagram λ with n boxes if and only if μ is obtained from λ by deleting a box. (2pts)

2) Determine the automorphism group of the branching graph of the symmetric groups (2pts).