REPRESENTATION THEORY, PROBLEM SET 2

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The deadline for submitting the solutions is Oct 19. The solutions are to be submitted electronically (scanned hand-written solutions are fine). E-mail i.loseu@neu.edu.

There are five problems with total number of points equal to 30. The maximal number of points you get for this problem set is 20. Everything above 20 does not count. Partial credit is given.

1) Additivity of $x \mapsto x^p - x^{[p]}$. Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$. Consider the map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ taking x to $x^p - x^{[p]}$.

1) Show that there is a unique homomorphism $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ that maps $x \in \mathfrak{g}$ to $x \otimes 1 + 1 \otimes x \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. It is known as the coproduct (1pt).

2) An element $u \in U(\mathfrak{g})$ is called *primitive* if $\Delta(u) = u \otimes 1 + 1 \otimes u$ (in particular, $\mathfrak{g} \subset U(\mathfrak{g})$ consists of primitive elements). Show that x^p is primitive (1pt).

3) Let us write $U(\mathfrak{g})^{\leqslant m}$ for the span of monomials of degree $\leqslant m$. Show that all primitive elements in $U(\mathfrak{g})^{\leqslant p-1}$ are contained in $\mathfrak{g}$ (3pts).

4) Deduce that ι is additive (2pts).

2) Classification of finite dimensional irreducible $\mathfrak{sl}_2(\mathbb{F})$-modules. Prove Theorem 1.5 from Lecture 4 (6pts – 2pts for each of the three cases).

3) Highest weight vectors are B-semiinvariant. Let V be a rational representation of $\mathrm{SL}_2(\mathbb{F})$ with highest weight n. Prove that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v = v$ for any $v \in V_n, x \in \mathbb{F}$ (4pts).

4) Tensor products and dAHA. Let M be a module over the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ and $V = \mathbb{C}^n$. Our goal in this problem is to produce an action of the degenerate affine Hecke algebra $\mathcal{H}(d)$ on the $\mathfrak{g}$-module $M \otimes V^{\otimes d}$ by $\mathfrak{g}$-linear endomorphisms.

1) Let M' be a $\mathfrak{g}$-module. Define the endomorphism $X_{M'}$ of $M' \otimes V$ by $x_{M'}(m \otimes v) = \sum_{i,j=1}^n E_{ij} m \otimes E_{ji} v$. Show that this endomorphism is $\mathfrak{g}$-linear (2pts).

2) Define the endomorphism $x_i, i = 1, \ldots, d$, of $M \otimes V^{\otimes d}$ as

$$x_{M \otimes V^{\otimes i-1}} \otimes \mathrm{id}^{\otimes d-i}$$

(so that the first factor acts on $M \otimes V^{\otimes i}$). Further, consider the endomorphism t_i of $M \otimes V^{\otimes d}$ that permutes the ith and $i+1$th copies of V, here $i = 1, \ldots, d-1$. Show that $X_i \mapsto x_i, T_i \mapsto t_i$ defines a representation of $\mathcal{H}(d)$ in $M \otimes V^{\otimes d}$ (2pts).

5) Affine Lie algebra $\hat{\mathfrak{sl}}_n$. Here we produce an example of a Kac-Moody algebra $\mathfrak{g}(A)$ with a degenerate matrix A. We take the Dynkin diagram that is a single cycle. The corresponding Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ if $n = 2$ and is given by $(a_{ij})_{i,j=1}^n$, where $a_{ii} = 2, a_{i,i+1} = a_{i+1,i} = a_{1,n} = a_{n,1} = -1$ and $a_{ij} = 0$ else.

1) Consider the space $\mathfrak{g} = \mathfrak{sl}_n[t^{\pm 1}] \oplus \mathbb{C}c$, where the commutator is given by $[x \otimes t^k, y \otimes t^\ell] = [x, y] \otimes t^{k+\ell} + k\delta_{k+\ell,0}\operatorname{tr}(xy)c$, for $x, y \in \mathfrak{sl}_n$, where $\delta_{k+\ell,0}$ is the Kronecker symbol, and $[c, x \otimes t^\ell] = 0$. Show that this space is a Lie algebra, it is denoted by $\hat{\mathfrak{sl}}_n$ (1pt).

2) Let $\mathfrak{h}$ be the subalgebra of diagonal matrices in $\mathfrak{sl}_n$. Set $\hat{\mathfrak{h}} = \mathfrak{h} \oplus c$. Define the elements $e_i, h_i, f_i, i = 0, \ldots, n-1$ as follows: $e_i := E_{i,i+1}$ if $i = 1, \ldots, n-1$, and $e_0 := tE_{n,1}$; $h_i := E_{ii} - E_{i+1,i+1}$ if $i = 1, \ldots, n-1$, and $h_0 := E_{n,n} - E_{1,1} + c$; $f_i := E_{i+1,i}$ if $i = 1, \ldots, n-1$, and $f_0 := t^{-1}E_{1,n}$. Show that the elements e_i, h_i, f_i satisfy the relations of the generators of $\mathfrak{g}(A)$. This yields an epimorphism $\mathfrak{g}(A) \twoheadrightarrow \mathfrak{g}$ (2pts).

3) Show that the Weyl group $W(A)$ of $\mathfrak{g}(A)$ is the semidirect product $S_n \ltimes \{(x_1, \ldots, x_n) | \sum x_i = 0\}$ (2pts).

4) Compute the root system of $\mathfrak{g}(A)$ (2pts).

5*) Show that the epimorphism $\mathfrak{g}(A) \twoheadrightarrow \mathfrak{g}$ is an isomorphism (2pts).