REPRESENTATION THEORY, PROBLEM SET 3

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The deadline for submitting the solutions is Nov 6. The solutions are to be submitted electronically (scanned hand-written solutions are fine). E-mail i.loseu@neu.edu.

There are five problems with total number of points equal to 30. The maximal number of points you get for this problem set is 20. Everything above 20 does not count. Partial credit is given.

Problem 1. Consider the Hecke algebra $\mathcal{H}_v(n)$ (for S_n, over $\mathbb{Z}[v^{\pm 1}]$). Set

$$L_m := v^{-m} T_{m-1} T_{m-2} \ldots T_2 T_1^2 T_2 \ldots T_{m-1}$$

(we assume $L_1 := 1$).

1) Prove that L_n commutes with $\mathcal{H}_v(n-1)$ (embedded into $\mathcal{H}_v(n)$ as the subalgebra generated by $T_1, \ldots, T_{n-2}$) (2pts).

2) Define the *affine Hecke algebra* $\mathcal{H}_v^{aff}(n)$ (for S_n) by generators $X_1, \ldots, X_n, T_1, \ldots, T_{n-1}$ such that the X_i's are invertible and the following relations hold:

a) the elements T_i satisfy the relations in $\mathcal{H}_v(n)$.

b) $X_i X_j = X_j X_i$, $T_i X_j = X_j T_i$ if $i \neq j, j+1$ and $T_i X_i T_i = v X_{i+1}$.

Show that $\mathcal{H}_v^{aff}(n)/(X_1 - 1) \cong \mathcal{H}_v(n)$ (1pt).

3) Note that $L_n - 1$ lies in the ideal generated by $v - 1$ (this is not a problem yet...). Determine the image of $(L_n - 1)/(v - 1)$ in $\mathcal{H}_v(n)/(v-1) = \mathbb{Z}S_n$ (2pts).

Problem 2. This was an example in Lecture 9. We consider the group $G = \mathrm{GU}_n(\mathbb{F}_q) := \{A \in \mathrm{GL}_n(\mathbb{F}_{q^2}) | \bar{A}^t J A = J\}$, where $J = (\delta_{i+j,n+1})_{i,j=1}^n$ and $\bar{A} := \mathsf{Fr}_q(A)$. We assume that q is sufficiently large and is not a power of 2.

1) Show that $|G| := q^{n(n-1)/2} \prod_{i=1}^n (q^i - (-1)^i)$, (2pts).[1]

2) Let T be the subgroup of G consisting of all diagonal matrices. Check that $W := N_G(T)/T$ is the Weyl group of type $B_{\lfloor n/2 \rfloor}$ (2pts).

3) Check that the parameters for the Hecke algebra of G are computed as follows: $q_i = q^2$ for $i > 0$, and $q_0 = q$ if n is even, while $q_0 = q^3$ if n is odd (we use notation from Lecture 9), (2pts).

Problem 3. Let $G := \mathrm{GL}_n(\mathbb{F}_q)$ and let T, B be the subgroups of all upper-triangular and diagonal matrices. Let $\chi \in \mathrm{Hom}(T, \mathbb{C}^\times)$. Show that the co-induced modules $\mathbb{C}[B \setminus_\chi G]$ and $\mathbb{C}[B \setminus_{w\chi} G]$ are isomorphic (6pts).

Problem 4. This problem concerns computation of Kazhdan-Lusztig basis elements $C_w \in \mathcal{H}_q(W)$.

1) Check equalities $C_{st} = T_{st} - q(T_s + T_t) + q^2, C_{ts} = T_{ts} - q(T_s + T_t) + q^2$ in $\mathcal{H}_q(S_3)$ (2pts).

2) Show that $C_s C_w = -(q + q^{-1}) C_w$ if $\ell(sw) = \ell(w) - 1$ (3pts).

3) Show that $C_{w_0} = \sum_{w \in W} (-q)^{\ell(w_0) - \ell(w)} T_w$ (3pts).

[1] An interesting observation: up to a sign this coincides with the number of elements of "$\mathrm{GL}_n(\mathbb{F}_{-q})$"

Problem 5. This problem concerns the structure of the category $\mathcal{O}_0$ for $\mathfrak{sl}_2$.

1) Prove that the projective cover $P(-2)$ of $L(-2)$ is a non-split extension $0 \to \Delta(0) \to P(-2) \to \Delta(-2) \to 0$ (1pt).

2) Prove that $\operatorname{End}_{\mathcal{O}_0}(P(-2)) = \mathbb{C}[x]/(x^2)$ (2pts).

3) Prove that the functor $\mathbb{V} := \operatorname{Hom}_{\mathcal{O}_0}(P(-2), \bullet) : \mathcal{O}_0 \to \mathbb{C}[x]/(x^2)$-mod is fully faithful on the projective objects (2pts).