# CHAPTER 1: REPRESENTATIONS OF SYMMETRIC GROUPS

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## Contents



## 1. Introduction

1.1. **Representations of finite (almost) simple groups.** Let $G$ be a finite group. We are interested in studying its representations. Let us briefly outline some motivations.

First, the representation theory of finite groups plays a very important part in their structure theory (we will see later that this phenomenon extends beyond finite groups), e.g. in understanding the finite simple groups. Recall that a group $G$ is called *simple* if the only two normal subgroups are $\{1\}$ and $G$. For example, the only commutative simple groups are the cyclic groups of prime order. In what follows, when we talk about simple groups we exclude those.

The representation theory is used to prove many results concerning simple groups: from the Burnside theorem that can be formulated as saying that the order of a simple group must have at least three distinct prime divisors, [E, Section 5.4], to the Brauer-Suzuki theorem on the absense of certain 2-subgroups in finite simple groups, to the celebrated Feit-Thompson theorem that all finite simple groups have even order.

Second, the representation theory of finite groups has various applications outside the core theory of finite groups, see [T, Section 3] for a survey of recent results.

Of course, there are many finite groups, and we should understand which ones we should care about most. Our answer: we care about simple $G$ or "almost simple" $G$ – we will not give a formal definition but explain what this means by an example later. One reason to stick to (almost) simple groups is that many general conjectures in the field can be reduced to that case, see [T, Section 2] for a survey.

The classification of finite simple groups is known. It is as follows.

1) There is an infinite family of the *alternating groups* $\mathfrak{A}_n, n \geqslant 5$. By definition, $\mathfrak{A}_n$ is the subgroup in $S_n$ consisting of all even permutations.

2) There are twenty six exceptional (a.k.a. sporadic) finite simple groups with the Monster (a.k.a. the Friendly giant) being the largest of them.

3) The rest, and, in a sense, the absolute majority of finite simple groups are the (closed relatives of) so called *finite groups of Lie type* (a.k.a. *finite reductive groups*). We will get to them in a later part of this course.

1.2. **The case of symmetric groups.** Now we explain how the symmetric groups $S_n$ enter the picture. By definition, we have $\mathfrak{A}_n \subset S_n$, and $\mathfrak{A}_n$ is a normal subgroup of index 2. Informally, these groups are very close to each other. So one can say $S_n$ is "almost simple" (at least, for $n \geqslant 5$).

The study of irreducible representations of $\mathfrak{A}_n$ reduces to those of $S_n$, Section 6.2. And the representation theory of $S_n$ is nicer. This is why we concentrate on $S_n$.

Of course, the ideology of emphasizing the representation theory of finite simple groups is just one reason to be interested in the representations of symmetric groups. Here are some more:

- A connection to Combinatorics, i.e., partitions – this will be featured very prominently, Section 5, symmetric polynomials (we will only sketch it, Section 6.1.2), etc.
- A connection to the representations of the general linear group via the Schur-Weyl duality, to be mentioned in a later part of the course.
- A connection to the representation theory of affine Lie algebras, which is especially interesting if instead of considering representations over $\mathbb{C}$, we work over a positive characteristic field. This will also be briefly discussed in a later part of this course.

We will concentrate on the representation theory of the symmetric groups $S_n$ over $\mathbb{C}$. For the non-closed fields of characteristic 0 or characteristic $p > n$ the situation turns out to be the same. When the characteristic is $p \leqslant n$, interesting (and complicated!) things happen, we will discuss this briefly in Section 6.4.

Now we summarize a few things that we already know about representations of a finite group $G$ over $\mathbb{C}$ (or a more general algebraically closed field of characteristic 0).

(i) A representation of $G$ is the same thing as a representation of the group algebra $\mathbb{C}G$, [1, Example 1.16].

(ii) The algebra $\mathbb{C}G$ is semisimple, [1, Theorem 3.2]. So, $\mathbb{C}G \cong \bigoplus_V \operatorname{End}_{\mathbb{C}}(V)$, where the summation is taken over isomorphism classes of irreducible $\mathbb{C}G$-modules, see [1, Theorem 2.24]. Every representation of $G$ is completely reducible, so what we need to understand is the irreducible representations.

(iii) The number of irreducible representations of $G$ up to an isomorphism coincides with the number of conjugacy classes in $G$.

The conjugacy classes in $S_n$ are in a natural bijection with partitions of $n$. Namely, we take an element $\sigma \in S_n$ and decompose it into the product of disjoint cycles. The lengthes of cycles form a partition of $n$ that is independent of the choice of $\sigma$ in the conjugacy class. We assign this partition to the conjugacy class of $\sigma$. Conjugacy classes will be denoted like $(***)(**)$. This represents the conjugacy class in $S_{m+5}$ corresponding to the partition $(3,2,1,\ldots,1)$, where $m$ is the number of 1's – we will usually omit the cycles of length 1 in the notation. Our notation for partitions is $(n_1,\ldots,n_k)$, where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k$ are the parts, or $(m_1^{d_1},\ldots,m_\ell^{d_\ell})$, where $m_1 > m_2 > \ldots > m_\ell$ are the distinct parts and $d_1,\ldots,d_\ell$ are their multiplicities. For example, $(2,2,1,1)$ and $(2^2,1^2)$ is two different notations for the same partition of 6.

We would like to emphasize that (iii) does not establish any preferred bijection between the irreducible representations of $S_n$ and the partitions of $n$. To establish such a bijection is our goal in this part. We will follow a "new" approach to the representation theory of the groups $S_n$ due to Okounkov and Vershik, [OV]. Our exposition follows [Kl, Section 2]. For a "traditional" approach based on Young symmetrizers, the reader is welcome to consult [E, Section 5.12-5.15] or [F, Section 7].

**Example 1.1.** In [1, Example 3.4] we have completely classified the irreducible representations of $S_4$. This will be our running example in this chapter. We now describe the resulting bijection with the partitions of 4. The trivial representation $\mathrm{triv}_4$ corresponds to the partition $(4)$. The sign representation $\mathrm{sgn}_4$ corresponds to $(1^4)$. The reflection representation $\mathrm{refl}_4$ corresponds to $(3,1)$, while its twist with the sign, $\mathrm{sgn}_4 \otimes \mathrm{refl}_4$ corresponds to $(2,1^2)$. Finally, the 2-dimensional irreducible representation corresponds to $(2^2)$. Note that 4 has five different partitions and we have listed all of them.

**Remark 1.2.** Let $V$ be a $\mathbb{C}S_n$-module. Note that $V \otimes \mathrm{sgn}_n$ is the same space as $V$ but the action of each permutation is multiplied by its sign.

## 2. Inductive approach

A key observation is that symmetric groups for different $n$ are embedded into one another: $\{1\} = S_1 \subset S_2 \subset S_3 \ldots \subset S_{n-1} \subset S_n \subset \ldots$, where, for $k = 1,\ldots,n-1$, we view $S_k$ as the subgroup of $S_{k+1}$ consisting of all elements that fix $k+1 \in \{1,\ldots,k+1\}$. We could try to use "induction", i.e., to study the irreducible representations of $S_n$ by restricting them to $S_{n-1}$. In fact, this naive idea does not quite work – we will need something more elaborate– but this is our starting point. Our exposition in Sections 2-5 will essentially follow Sections 2.1 and 2.2 in [Kl].

2.1. **Centralizer $Z_B(A)$ and restrictions of representations.** Let $V$ be an irreducible representation of $S_n$. So we will need to "understand" the restriction of $V$ to $S_{n-1}$, where "understand" means: decompose into the direct sum of irreducibles. In fact, for our purposes we will also need to understand the restriction to $S_{n-2} \subset S_n$, we will explain why later. Of course, one can generalize this to the following question known as the "branching problem": let $H \subset G$ be finite groups and $V$ be an irreducible representation of $G$; decompose $V$ into the direct sum of irreducible $\mathbb{C}H$-modules.

This problem can be further generalized. Note that $\mathbb{C}G$ is a semisimple associative algebra and $\mathbb{C}H$ is its subalgebra that is also a semisimple associative algebra. So, given a pair

$B \subset A$ of finite dimensional semisimple associative algebras and an irreducible $A$-module $V$ one could ask to decompose $V$ into the direct sum of irreducible $B$-modules. In fact, one can generalize this further: to a pair of semisimple associative algebras $B$ and $A$ equipped with a homomorphism $\tau : B \to A$. This homomorphism allows to view every $A$-module as a $B$-module.

Below we write $\operatorname{Irr}(A), \operatorname{Irr}(B)$ for the sets of isomorphism classes of irreducible $A$- and $B$-modules so that

$$A \xrightarrow{\sim} \bigoplus_{V \in \operatorname{Irr}(A)} \operatorname{End}(V), \tag{2.1}$$

$$B \xrightarrow{\sim} \bigoplus_{U \in \operatorname{Irr}(B)} \operatorname{End}(U). \tag{2.2}$$

For $U \in \operatorname{Irr}(B)$ consider its multiplicity space in $V \in \operatorname{Irr}(A)$:

$$M_{V,U} := \operatorname{Hom}_B(U, V)$$

so that, thanks to [1, Corollary 2.16], we have a $B$-linear isomorphism

$$\bigoplus_i U_i \otimes M_{V,U_i} \xrightarrow{\sim} V, \tag{2.3}$$

where the $U_i$'s are the elements of $\operatorname{Irr}(B)$. The isomorphism is given by $\sum_i u_i \otimes \varphi_i \mapsto \sum_i \varphi_i(u_i)$.

So our problem is to compute the spaces $M_{V,U}$. It turns out that the nonzero spaces of this form are exactly the irreducible representations of an auxiliary algebra, the *centralizer* of $B$ in $A$ defined as follows.

**Definition 2.1.** By the *centralizer*, $Z_B(A)$, of $B$ in $A$ we mean the subset

$$\{a \in A | a\tau(b) = \tau(b)a, \forall b \in B\}.$$

Note that when $B = A$ and the homomorphism $\tau$ is the identity we recover the definition of the center of $A$, i.e., we have $Z_B(A) = Z(A)$.

**Exercise 2.2.** Show that $Z_B(A) \subset A$ is a subalgebra.

**Lemma 2.3.** *We have an algebra isomorphism* $Z_B(A) \cong \bigoplus_{U,V} \operatorname{End}(M_{V,U})$, *where the sum is taken over all pairs* $U \in \operatorname{Irr}(B), V \in \operatorname{Irr}(A)$ *satisfying* $M_{V,U} \neq \{0\}$.

In other words, the algebra $Z_B(A)$ is semisimple and the irreducible $Z_B(A)$-modules are precisely the nonzero multiplicity spaces $M_{V,U}$.

**Example 2.4.** Let

$$A = \operatorname{Mat}_4(\mathbb{C}) \oplus \operatorname{Mat}_3(\mathbb{C}), B = \operatorname{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 2}, \tag{2.4}$$

and $\tau$ be as follows:

$$\tau(x_1, x_2, x_3) = (\operatorname{diag}(x_1, x_2, x_2), \operatorname{diag}(x_1, x_3)),$$

where $x_1$ is in $\operatorname{Mat}_2(\mathbb{C})$, and $x_2, x_3$ are in the 1st and 2nd copies of $\mathbb{C}$. Let $U_1, U_2, U_3$ be the irreducible $B$-modules (of dimensions $2, 1, 1$), and $V_1, V_2$ be the irreducible $A$-modules (of dimensions $4, 3$), according to decomposition 2.4. We see that $M_{V_1,U_2}$ is 2-dimensional, $M_{V_1,U_1}, M_{V_2,U_1}, M_{V_2,U_3}$ are 1-dimensional, while $M_{V_1,U_3} = M_{V_2,U_2} = \{0\}$. So Lemma 2.3 means that $Z_B(A) \cong \operatorname{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$. Let us verify this directly.

By the definition, $Z_B(A)$ consists of pairs $(y_1, y_2) \in \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ such that $y_1$ commutes with all matrices of the form $\mathrm{diag}(x_1, x_2, x_2)$, while $y_2$ commutes with all matrices of the form $\mathrm{diag}(x_1, x_3)$. A direct check left as an exercise shows that we have

$$y_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & d & e \end{pmatrix}, y_2 = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}, a, b, c, d, e, f, g \in \mathbb{C}.$$

An isomorphism $Z_B(A) \xrightarrow{\sim} \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}$ sends this pair $(y_1, y_2)$ to $(\begin{pmatrix} b & c \\ d & e \end{pmatrix}, a, f, g)$.

*Proof of Lemma 2.3.* Thanks to the decomposition (2.1), we can view $\tau : B \to A$ as a tuple $(\tau_V)_{V \in \mathrm{Irr}(A)}$, where $\tau_V$ is an algebra homomorphism $B \to \mathrm{End}(V)$. For $a = (a_V) \in A = \bigoplus_V \mathrm{End}(V)$, we have

$$a \in Z_B(A) \Leftrightarrow a_V \in Z_B(\mathrm{End}(V)), \forall V.$$

So $Z_B(A) = \bigoplus_V Z_B(\mathrm{End}(V))$. But the subalgebra $Z_B(\mathrm{End}(V)) \subset \mathrm{End}(V)$ is exactly $\mathrm{End}_B(V)$. By [1, Remark 2.15], we have $\mathrm{End}_B(V) \xrightarrow{\sim} \bigoplus_U \mathrm{End}(M_{V,U})$, where the sum is taken over all $U \in \mathrm{Irr}(B)$ such that $M_{V,U} \neq \{0\}$. □

**Remark 2.5.** It is instructive (and useful in what follows) to describe the structure of a $Z_B(A)$-module on $M_{V,U} = \mathrm{Hom}_B(U, V)$ without referring to the decomposition $A = \bigoplus_V \mathrm{End}(V)$. Recall, [1, Remark 2.15], that the $\mathrm{End}_B(V)$ acts on $\mathrm{Hom}_B(U, V)$ via the composition map

$$\mathrm{End}_B(V) \times \mathrm{Hom}_B(U, V) \to \mathrm{Hom}_B(U, V), (\alpha, \varphi) \mapsto \alpha \circ \varphi.$$

So this action is induced from the action of $\mathrm{End}(V)$ on $V$, i.e., $[\alpha\varphi](u) = \alpha[\varphi(u)]$.

Now let $z \in Z_B(A)$ and $\varphi \in \mathrm{Hom}_B(U, V)$. We define $z\varphi \in \mathrm{Hom}_B(U, V)$ by

$$[z\varphi](u) = z\varphi(u), \tag{2.5}$$

for all $u \in U$, where in the right hand side we use the $A$-action on $V$. An easy check shows that $z\varphi$ is well-defined. Thus, under the identification (2.1), the latter action is obtained from the action of $\mathrm{End}(V)$ on $V$ via the projection $A \twoheadrightarrow \mathrm{End}(V)$. Under the identification $Z_B(A) \xrightarrow{\sim} \bigoplus_V \mathrm{End}_B(V)$, (2.5) is obtained from the action of $\mathrm{End}_B(V)$ on $\mathrm{Hom}_B(U, V)$ under the projection $Z_B(A) \to \mathrm{End}_B(V)$. We conclude that (2.5) gives the same action as in the previous paragraph.

Here is a corollary Lemma 2.3 that will be very useful for us in what follows.

**Corollary 2.6.** *The following two conditions are equivalent:*

(1) *For any* $U \in \mathrm{Irr}(B), V \in \mathrm{Irr}(A)$, *we have* $\dim \mathrm{Hom}_B(U, V) \leqslant 1$.
(2) $Z_B(A)$ *is commutative.*

*Proof.* The algebra $Z_B(A) = \bigoplus_{U,V} \mathrm{End}(M_{V,U})$ is commutative if and only if the summands $\mathrm{End}(M_{V,U})$ are. For a nonzero vector space $W$, $\mathrm{End}(W) = \mathrm{Mat}_{\dim W}(\mathbb{C})$ is commutative if and only if $\dim W = 1$. So, (1) $\Leftrightarrow$ (2). □

Under the equivalent conditions of the corollary, we have the decomposition $V = \bigoplus_U U$, where the summation is over all irreducible $B$-modules $U$ that appear in $V$. An important thing to notice is that in this direct sum $U$ is uniquely determined as a subspace of $V$. Indeed, $U$ is embedded into $V$ as the image of any nonzero element in $\mathrm{Hom}_B(U, V)$ – since

$\dim \mathrm{Hom}_B(U,V)$ is one-dimensional all nonzero elements are proportional, so have the same image.

2.2. **The algebra $Z_m(n)$.** We get back to the case of symmetric groups. Set $Z_m(n) := Z_{\mathbb{C}S_m}(\mathbb{C}S_n)$, where the embedding $\mathbb{C}S_m \hookrightarrow \mathbb{C}S_n$ comes from the embedding $S_m \hookrightarrow S_n$ as the subgroup of all permutations fixing each of the elements $m+1,\ldots,n$.

Our first observation is that we can describe a (vector space) basis in $Z_m(n)$. In fact, this can be done in a greater generality.

**Lemma 2.7.** *Let $H \subset G$ be finite groups. Then the subspace $Z_{\mathbb{C}H}(\mathbb{C}G) \subset \mathbb{C}G$ consists of all elements of the form $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ satisfy $a_{hgh^{-1}} = a_g$ for all $h \in H$. In particular, $Z_{\mathbb{C}H}(\mathbb{C}G)$ has basis indexed by the $H$-conjugacy classes in $G$: to a conjugacy class $c$ we assign $b_c := \sum_{g\in c} g \in Z_{\mathbb{C}H}(\mathbb{C}G)$.*

Note that in the case when $H = G$ we recover the description of the center $Z(G)$ of $\mathbb{C}G$, see [1, (3.2)]. The proof in the general case repeats that and is omitted.

Now we get back to the situation of $\mathbb{C}S_m \subset \mathbb{C}S_n$. The conjugation action of $S_m$ on $S_n$ permutes the elements $1,\ldots,m$ in the permutation. So the conjugacy classes are "cycles with marked elements $m+1,\ldots,n$". The notation for these conjugacy classes will be like $(**4)(5*)(6)$ or $(**5)(46)$. Both denote $S_3$-conjugacy classes in $S_6$, the former contains an element $(124)(53)$ (or $(234)(15)$), while the latter contains $(125)(46)$.

The following is a crucial example of the basis element in $Z_{n-1}(n)$ corresponding to an $S_{n-1}$-conjugacy class in $S_n$.

**Example 2.8.** Let $m = n-1$. Take the conjugacy class labelled $(*n)$. It consists of transpositions $(1,n),(2,n),\ldots,(n-1,n)$. The corresponding basis element $b_{(*n)}$ in $Z_{n-1}(n)$ is $\sum_{i=1}^{n-1}(i,n)$. It is called the $n$th *Jucys-Murphy element*. We will denote it by $J_n$.

Our next task will be to determine algebra generators of $Z_m(n)$ (as opposed to a vector space basis). We note that $Z_m(n)$ contains the following elements and subalgebras.

(a) The center of $\mathbb{C}S_m$, i.e., $Z_m(m)$. Note that $Z_m(m)$ lies in the center of $Z_m(n)$: it is contained in $\mathbb{C}S_m$ and every element of $Z_m(n)$ commutes with every element of $\mathbb{C}S_m$, by definition of $Z_m(n)$.
(b) Let $S_{[m+1,n]}$ denote the subgroup of $S_n$ consisting of all permutations fixing each of $1,\ldots,m$. Since each permutation from $S_m$ commutes with each permutation from $S_{[m+1,n]}$, we have $\mathbb{C}S_{[m,n+1]} \subset Z_m(n)$.
(c) We can consider the Jucys-Murphy elements $J_k := \sum_{i=1}^{k-1}(i,k)$. Since $J_k$ commutes with $\mathbb{C}S_{k-1}$ for all $k=1,\ldots,n$, we have $J_{m+1},\ldots,J_n \in Z_m(n)$.

Note that the elements $J_{m+1},\ldots,J_n$ pairwise commute: for $k<\ell$ we have $J_k \in \mathbb{C}S_{\ell-1}$ and $J_\ell \in Z_{\ell-1}(\ell)$.

**Theorem 2.9.** *The algebra $Z_m(n)$ is generated by the subalgebras $Z_m(m), \mathbb{C}S_{[m+1,n]}$ and the elements $J_{m+1},\ldots,J_n$ (as an algebra).*

*Proof.* To an $S_m$-conjugacy class $c$ in $S_n$ we assign its *degree* $\deg c$ that, by definition, is equal to the number of elements in $\{1,\ldots,n\}$ moved by an element in $c$ (this is independent of the choice of the element). For example, $\deg(*n) = \deg(**) = 2$. In particular, either $\deg c = 0$, which is the case precisely for the class of the identity, or $\deg c \geqslant 2$.

Let $A$ be the subalgebra in $Z_m(n)$ generated by $Z_m(m), S_{[m+1,n]}, L_{m+1},\ldots,L_n$. We need to show that $b_c \in A$ for all $c$. Assume the contrary and pick $c$ of minimal degree such that $b_c \notin A$. We will arrive at a contradiction at several steps.

*Step 1.* To start, note that, clearly, $\deg c > 0$. Also $\deg c > 2$. Indeed, we have the following conjugacy classes of degree 2:

- $(*, k)$, where $k > m$. Here $b_{(*,k)} = \sum_{i=1}^{m}(i,k) = J_k - \sum_{i=m+1}^{k-1}(i,k) \in A$, as the sum is in $\mathbb{C}S_{[m+1,n]}$.
- $(k,\ell)$ with $m < k < \ell \leqslant n$. Here $b_{(k,\ell)} = (k,\ell) \in \mathbb{C}S_{[m+1],n} \subset A$.
- $(*,*)$. Here $b_{(*,*)} \in Z_m(m) \subset A$.

*Step 2.* Assume, first, that $c$ has more than one cycle of length at least 2. Break $c$ into the union of two cycle types $c', c''$, e.g., if $c = (6**)(5*)$, then we can take $c' = (6**), c'' = (5*)$. Note that

$$b_{c'}b_{c''} = \alpha b_c + \sum_{c_0, \deg c_0 < \deg c} \alpha_{c_0} b_{c_0},$$

where $\alpha > 0$. Here the first summand incorporates the products of disjoint elements of $c', c''$ (all stars are pairwise distinct), and the sum corresponds to non-disjoint elements, here the degree of the product drops. By the degree minimality assumption on $c$, $b_{c_0} \in A$ and $b_{c'}b_{c''} \in A$. So $b_c \in A$, which contradicts the choice of $c$.

*Step 3.* Now let us pick a cycle $(i_1, i_2, \ldots, i_k) \in S_n$ and consider the product $(i_1, \ldots, i_k)(i_s, j)$. If $j \not\in \{i_1, \ldots, i_k\}$, then we get $(i_1, \ldots, i_s, j, i_{s+1}, \ldots, i_k)$. If $j \in \{i_1, \ldots, i_k\}$, then $(i_1, \ldots, i_k)(i_k j)$ either splits into the product of two cycles of total degree $k$ or is a cycle of degree $k-1$.

*Step 4.* Now suppose that the cycle in $c$ has both an element from $\{1, \ldots, m\}$ (does not matter which, denote it by $*$) and $k \in \{m+1, \ldots, n\}$. We may assume that $k$ is right after $*$ in the cycle. Let $c'$ denote the cycle obtained from $c$ by deleting $k$. Then $b_{c'}b_{(*,k)} = \alpha b_c + \sum_{c_0} \alpha_{c_0} b_{c_0}$, where the summation is over $c_0$ that are products of two disjoint cycles with $\deg c_0 = \deg c$ or have $\deg c_0 < \deg c$. This is a consequence of Step 3, as the left hand side is the sum of products of pairs of cycles that share a common element, $k$. Similarly to Step 2, we arrive at a contradiction with the choice of $c$.

*Step 5.* So either the elements in the only cycle of $c$ are all from $\{1, \ldots, m\}$, in which case $b_c \in Z_m(m)$, or are all from $\{m+1, \ldots, n\}$, in which case $b_c \in S_{[m+1,n]}$. Contradiction. □

Here is an important corollary of Theorem 2.9 concerning the case $m = n-1$.

**Corollary 2.10.** *The following claims are true:*

1. *The algebra $Z_{n-1}(n)$ is commutative.*
2. *For all $U \in \operatorname{Irr}(\mathbb{C}S_{n-1})$ and $V \in \operatorname{Irr}(\mathbb{C}S_n)$, the multiplicity of $U$ in $V$ is 0 or 1.*
3. *The element $J_n$ acts on each irreducible $\mathbb{C}S_{n-1}$-submodule of $V \in \operatorname{Irr}(\mathbb{C}S_n)$ by a scalar (depending on the submodule).*

*Proof.* (1): The algebra $Z_{n-1}(n)$ is generated by its central subalgebra $Z_{n-1}(n-1)$ and $J_n$. In particular, the generators pairwise commute. Since $Z_{n-1}(n)$ is generated by pairwise commuting elements, it is commutative.

(2): this follows from (1) and Corollary 2.6.

(3): since $J_n$ commutes with $\mathbb{C}S_{n-1}$, the operator $J_{n,V}$ of multiplication by $J_n$ is a $\mathbb{C}S_{n-1}$-linear map $V \to V$. Let $U$ be an irreducible $\mathbb{C}S_{n-1}$-module appearing in $V$. By [1, Theorem 2.14(2)], $J_{n,V}$ sends $U$ to $U$. And then by the Schur lemma, [1, Theorem 2.8], this restriction is a scalar. □

**Example 2.11.** Let us see how $J_n$ acts and how $V$ decomposes into irreducible $\mathbb{C}S_{n-1}$-modules in various examples of $V \in \operatorname{Irr}(\mathbb{C}S_n)$.

1) $V = \operatorname{refl}_n$, the $(n-1)$-dimensional reflection representation of $S_n$ realized as $\{(x_1, \ldots, x_n) \in \mathbb{C}^n | x_1 + \ldots + x_n = 0\}$, where $S_n$ acts on $\mathbb{C}^n$ by permuting the coordinates. If $n > 2$, the

representation $\mathrm{refl}_n$ decomposes as the direct sum of two irreducible representations of $S_{n-1}$: $\{(x_1,\ldots,x_{n-1},0)\}$, isomorphic to $\mathrm{refl}_{n-1}$, and the trivial representation $\{(-x,\ldots,-x,(n-1)x)\}$. The element $J_n=\sum_{i=1}^{n-1}(i,n)$ sends $(x_1,\ldots,x_n)$ to $((n-2)x_1+x_n,\ldots,(n-2)x_{n-1}+x_n,x_1+\ldots+x_{n-1})$. The reflection subrepresentation is the eigenspace for $J_n$ with eigenvalue $(n-2)$, while the trivial subresentation is the eigenspace with eigenvalue $-1$.

2) Let $n=4$ and $V$ be the two-dimensional irreducible representation of $S_4$. Recall, [1, Example 3.4], that it is pulled back from the reflection representation of $S_3$ under an epimorphism $S_4\twoheadrightarrow S_3$. The composition $S_3\hookrightarrow S_4\to S_3$ is the identity, while the image of $J_4$ in $S_3$ under $S_4\twoheadrightarrow S_3$ is $(1,2)+(2,3)+(1,3)$. This element acts by 0 on the reflection representation of $S_3$.

To finish this discussion we will make three remarks.

**Remark 2.12.** One can show that the algebra $Z_m(n)$ is generated by $Z_n(n)$ (the center of $\mathbb{C}S_n$), $\mathbb{C}S_{[m+1,n]}$ and the Jucys-Murphy elements $J_{m+1},\ldots,J_n$. The proof is similar to that of Theorem 2.9 and is left as an exercise. This has an interesting corollary, every eigenspace for $J_n$ in an irreducible $\mathbb{C}S_n$, a $\mathbb{C}S_{n-1}$-submodule by (3) of Corollary 2.10, is an irreducible $\mathbb{C}S_{n-1}$-submodule. This is also left as an exercise.

**Remark 2.13.** Once we know the generators of $Z_m(n)$ a natural thing to ask is about the relations. In solving our classification problem below we will need to know "useful" relations between $(n-1,n),J_{n-1},J_n$ in $Z_{n-2}(n)$. This will give rise to the degenerate affine Hecke algebra $\mathcal{H}(2)$.

**Remark 2.14.** Theorem 2.9 remains true if we replace $\mathbb{C}$ with any algebraically closed field of characteristic 0. It may fail in characteristic $p$, as the coefficients $\alpha$ in the proof may be zero modulo $p$.

## 3. Branching graph, paths and weights

3.1. **Branching graph.** Let $V^n$ be an irreducible $\mathbb{C}S_n$-module. Thanks to (2) of Corollary 2.10, we can (uniquely) decompose $V^n$ into the direct sum of pairwise non-isomorphic $\mathbb{C}S_{n-1}$-modules. In its turn, each of the summands can be decomposed into the direct sum of pairwise non-isomorphic $\mathbb{C}S_{n-2}$-modules, etc. So, for each $m<n$, we can decompose $V^n$ into the direct sum of irreducible $\mathbb{C}S_m$-modules. We emphasize that, while every $\mathbb{C}S_m$-module admits such a decomposition, it is not canonical, in general. The reason why we get a canonical decomposition of $V^n$ is the chain of inclusions $S_m\subset S_{m+1}\subset\ldots\subset S_n$ and (2) of Corollary 2.10.

To control the summands we will need the following combinatorial object.

**Definition 3.1.** The branching graph is a directed graph, where the vertices are labeled by the (isomorphism classes of) irreducible $\mathbb{C}S_n$-modules (for all $n$). We have an edge from $U$ to $V$ if and only if $V$ is an irreducible representation of some $\mathbb{C}S_n$ and $U$ is an irreducible representation of $\mathbb{C}S_{n-1}$ that occurs in $V$, i.e., is isomorphic to a summand in the decomposition of $V$ into the direct sum of irreducible $\mathbb{C}S_{n-1}$-module.

So the branching graph shows, for example, how to restrict the irreducible representations of $\mathbb{C}S_n$ to $\mathbb{C}S_{n-1}$ for each $n$.

Here is an example of the piece of the branching graph up to $n=4$. It is based on Example 2.11 and the observation that tensoring with the sign gives a symmetry of the graph (and also that restricting 1-dimensional representations is easy and pleasant).

$$\begin{tikzcd}
\mathrm{triv}_4 & & \mathrm{refl}_4 & & \mathbb{C}^2 & & \mathrm{refl}_4 \otimes \mathrm{sgn}_4 & & \mathrm{sgn}_4 \\
& \mathrm{triv}_3 \arrow[ul] \arrow[ur] & & & \mathrm{refl}_3 \arrow[ull] \arrow[u] \arrow[urr] & & & \mathrm{sgn}_3 \arrow[ul] \arrow[ur] & \\
& & \mathrm{triv}_2 \arrow[ul] \arrow[urr] & & & \mathrm{sgn}_2 \arrow[ul] \arrow[urr] & & & \\
& & & \mathrm{triv}_1 \arrow[ul] \arrow[urr] & & & & &
\end{tikzcd}$$

We can talk about (oriented) paths in the branching graph. For $V^m \in \mathrm{Irr}(\mathbb{C}S_m), V^n \in \mathrm{Irr}(\mathbb{C}S_n)$ with $m < n$, let $\mathsf{Path}(V^m, V^n)$ denote the set of all paths from $V^m$ to $V^n$ If $m = 1$, then we have only one irreducible module $V^1$ and we write $\mathsf{Path}(V^n)$ for $\mathsf{Path}(V^1, V^n)$. We set

$$\mathsf{Path}_n := \bigsqcup_{V^n \in \mathrm{Irr}(\mathbb{C}S_n)} \mathsf{Path}(V^n).$$

Let $\overline{P} = (V^m \to V^{m+1} \to \ldots \to V^n) \in \mathsf{Path}(V^m, V^n)$. We write $V^m(\overline{P})$ for the copy of $V^m$ inside $V^n$ embedded according to $\overline{P}$, i.e., via $V^m \hookrightarrow V^{m+1} \hookrightarrow \ldots \hookrightarrow V^n$ (we emphasize again that any two embeddings of $V^k$ into $V^{k+1}$ are proportionaly). So, we can write the decomposition mentioned in the beginning of the section as

$$V^n = \bigoplus_{V^m \in \mathrm{Irr}(S_m)} \bigoplus_{\overline{P} \in \mathsf{Path}(V^m, V^n)} V^m(\overline{P}). \tag{3.1}$$

Let $\varphi_{\overline{P}}$ denote the inclusion $V^m \hookrightarrow V^n$ corresponding to $\overline{P}$. Note that it is defined uniquely up to rescaling. Define $w_{\overline{P}} = (w_{m+1}, \ldots, w_n) \in \mathbb{C}^{n-m}$ as follows: $w_k$ is the scalar by which $J_k$ acts on $V_{k-1} \subset V_k$, which makes sense by (3) of Corollary 2.10. We call $w_{\overline{P}}$ the *weight* of $\overline{P}$.

Recall that $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ is an irreducible $Z_m(n)$-module, Lemma 2.3, where the action is given by (2.5). In particular, the elements $J_{m+1}, \ldots, J_n \in Z_m(n)$ act on $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$.

Now we can relate the space $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ to the set $\mathsf{Path}(V^m, V^n)$.

**Lemma 3.2.** *The following claims are true.*

(1) *The elements $\varphi_{\overline{P}}$ form a basis in* $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$.
(2) *Each $\varphi_{\overline{P}}$ is an eigenvector for $J_k$ with eigenvalue $w_k$ for all $k = m+1, \ldots, n$, where $(w_{m+1}, \ldots, w_n) = w_{\overline{P}}$.*

*Proof.* (1): Thanks to (3.1), we have

$$\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n) = \bigoplus_{V'^m \in \mathrm{Irr}(S_m)} \bigoplus_{\overline{P} \in \mathsf{Path}(V'^m, V^n)} \mathrm{Hom}(V^m, V'^m(\overline{P})) = \bigoplus_{\overline{P} \in \mathsf{Path}(V^m, V^n)} \mathrm{Hom}(V^m, V^m(\overline{P})).$$

The first equality is thanks to the additivity of Hom, and the second is thanks to the Schur lemma, see [1, Theorem 2.8(1)]. Now we use [1, Theorem 2.8(2)] to see that $\mathrm{Hom}(V^m, V^m(\overline{P})) \cong \mathbb{C}$. By the construction, $\varphi_{\overline{P}}$ is a nonzero element in this 1-dimensional space. (1) follows.

(2): We have $[J_k\varphi_{\overline{P}}](u) = J_k[\varphi_{\overline{P}}(u)]$ for all $u \in V^m$ by (2.5). Note that $\varphi_{\overline{P}}(u) \in V^m(\overline{P})$. By the very construction, $V^m(\overline{P})$ lies in the copy of $V^{k-1}$ inside $V^k$ for all $k = m+1, \ldots, n$. So $J_k$ acts on $V^m(\overline{P})$ by $w_k$. Therefore $J_k\varphi_{\overline{P}} = w_k\varphi_{\overline{P}}$, which proves (2). $\square$

When $m = 1$, we identify $\mathrm{Hom}_{\mathbb{C}S_1}(V^1, V^n) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V^n)$ with $V^n$ in the standard way. For $P \in \mathsf{Path}(V^n)$, we write $v_P$ for $\varphi_P$ viewed as an element of $V^n$. The following is a straightforward corollary of Lemma 3.2.

**Corollary 3.3.** *The following claims are true.*

(1) *The vectors $v_P$ for $P \in \mathsf{Path}_n$ form a basis in $V^n$.*
(2) *Each $v_P$ is an eigenvector for $J_k$ with eigenvalue $w_k$ for all $k = 1, \ldots, n$.*

Note that $w_1 = 0$. This is because $J_1 = \sum_{i=1}^{0}(i,1) = 0$.

**Example 3.4.** 1) Let $V^n = \mathrm{refl}_n$. By Example 2.11, we have a $\mathbb{C}S_{n-1}$-module decomposition $\mathrm{refl}_n \cong \mathrm{refl}_{n-1} \oplus \mathrm{triv}_{n-1}$ for $n > 2$, while for $n = 2$, we have $\mathrm{refl}_2 = \mathrm{triv}_1$. So $\mathsf{Path}(V^n)$ has $n-1$ elements, and they are of the form

$$P = \mathrm{triv}_1 \to \ldots \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \ldots \to \mathrm{refl}_n\,.$$

The weight $w_P$ assigned to this path is $(0, \ldots, i-1, -1, i, \ldots, n-2)$. Indeed, by Example 2.11, $J_k$ acts on $\mathrm{refl}_{k-1} \subset \mathrm{refl}_k$ by $k-2$ and on $\mathrm{triv}_{k-1} \subset \mathrm{refl}_k$ by $-1$. The vector $v_P$ is proportional to $(1, \ldots, 1, -i, 0, \ldots, 0)$, where we have $i$ entries 1.

2) Let $V = \mathbb{C}^2$, the irreducible $\mathbb{C}S_4$-module of dimension 2. There are two elements in $\mathsf{Path}(\mathbb{C}^2)$ and the weights are $(0, 1, -1, 0)$ and $(0, -1, 1, 0)$. This follows from 2) of Example 2.11 combined with 1) of the present example.

3) The sets $\mathsf{Path}(V^n)$ and $\mathsf{Path}(V^n \otimes \mathrm{sgn}_n)$ are identified. Since all $J_k$ are sums of transpositions, passing from $V^n$ to $V^n \otimes \mathrm{sgn}_n$ multiplies these elements by $-1$. In particular all weights get multiplied by $-1$.

We would also like to record the following corollary of the definitions of $\varphi_{\overline{P}}$ and $v_P$.

**Corollary 3.5.** *Let $m < n, V^m \in \mathrm{Irr}(\mathbb{C}S_m), V^n \in \mathrm{Irr}(\mathbb{C}S_n)$. Choose $\underline{P} \in \mathsf{Path}(V^m), \overline{P} \in \mathsf{Path}(V^m, V^n)$ and let $P \in \mathsf{Path}(V^n)$ be the concatenation $\underline{P}\overline{P}$. Then $v_P$ is proportional to $\varphi_{\overline{P}}(v_{\underline{P}})$.*

*Proof.* Both are nonzero vectors in the 1-dimensional subspace $V^1(P) \subset V^n$. $\square$

3.2. **Uniqueness of weights.** The following result will be extremely important in our classification of irreducible representations of $S_n$ and getting information about their bases.

**Theorem 3.6.** *Let $P, P' \in \mathsf{Path}_n$. If $w_P = w_{P'}$, then $P = P'$.*

Before we prove the theorem, let us explain why it is important. We start with a definition/notation.

**Definition 3.7.** Let $\mathsf{Wt}_n := \{w_P | P \in \mathsf{Path}_n\}$. We say that two elements of $\mathsf{Wt}_n$ are r-equivalent ("r" for "representation") if they are weights of two paths into the same irreducible module.

Theorem 3.6 means that the map $\mathsf{Path}_n \to \mathsf{Wt}_n$ (a priori, surjective) is a bijection. This implies several things about the r-equivalence. First, the r-equivalence is indeed an equivalence relation, and the equivalence classes are in bijection with the (isomorphism classes of) irreducible representations. Indeed, the similar claims for the paths are a tautology and then we use $\mathsf{Path}_n \xrightarrow{\sim} \mathsf{Wt}_n$. Second, the theorem implies that the basis vectors $v_P$ for $P \in \mathsf{Path}(V^n)$ are in bijection with the weights in the corresponding r-equivalence class. After the theorem is proved our task is:

**Task 3.8.** To describe the set $\mathsf{Wt}_n$ and the r-equivalence relation.

*Proof of Theorem 3.6.* The proof is by induction. The base, $n = 1$, is vacuous: there is only one irreducible representation and it is 1-dimensional. Now suppose that we have established the claim for $n-1$. Let $\underline{P}, \underline{P}' \in \mathsf{Path}_{n-1}$ be the truncations of $P, P'$. If $w_P = (w_1, \ldots, w_n), w_{P'} = (w_1', \ldots, w_n')$, then $w_{\underline{P}} = (w_1, \ldots, w_{n-1}), w_{\underline{P}'} = (w_1', \ldots, w_{n-1}')$. So, $w_{\underline{P}} = w_{\underline{P}'}$ and, by the inductive assumption, $\underline{P} = \underline{P}'$. Let $V, V' \in \operatorname{Irr}(\mathbb{C}S_n)$ are the end-points of $P, P'$. It remains to show that $V \cong V'$. Let $U \in \operatorname{Irr}(\mathbb{C}S_{n-1})$ be the end-point of $\underline{P} = \underline{P}'$.

The proof of $V \cong V'$ is as follows. We claim that every element $z \in Z_{n-1}(n)$ acts on $U \subset V, U \subset V'$ by scalars, denote them by $\chi(z), \chi'(z)$, and, moreover, $\chi(z) = \chi'(z)$. By Theorem 2.9, the algebra $Z_{n-1}(n)$ is generated by $Z_{n-1}(n-1)$ and $J_n$. So it is enough to check the claim for $z \in Z_{n-1}(n-1)$ and $z = J_n$ separately. Any $z \in Z_{n-1}(n-1)$ acts on $U$ by a scalar, [1, Exercise 2.12], and, tautologically, $\chi(z) = \chi'(z)$ in this case. The element $J_n$ acts on both $U$ embedded to $V, V'$ by $w_n$, by the construction of $w_n$, and so we get $\chi(J_n) = \chi'(J_n) = w_n$.

Note that the center, $Z_n(n)$, of $\mathbb{C}S_n$ is contained in $Z_{n-1}(n)$. Every element $z \in Z_n(n)$ acts on $V$ (resp., $V'$) by a scalar, $\chi_V(z)$ (resp., $\chi_{V'}(z)$) (see [1, Exercise 2.12]). But $\chi_V(z)$ is the same scalar by which $z$ acts on $U$ because $U \subset V$. We deduce $\chi_V(z) = \chi(z) = \chi_{V'}(z)$ for all $z \in Z_n(n)$. From [1, Corollary 2.26] we deduce that $V \cong V'$, which finishes the proof. $\square$

3.3. **Varying the path.** In what follows we will address the following task that is a crucial step in addressing Task 3.8. Suppose we have a path $P = (V^1 \to V^2 \to \ldots V^n) \in \mathsf{Path}(V^n)$. Pick an integer $i$ with $1 \leqslant i < n$. Consider the set of all paths $P' = (V'^1 \to \ldots V'^{n-1} \to V'^n)$ such that $V'^j = V^j$ for $j \neq i$. Denote this set of paths by $\mathsf{Path}(P, i)$.

**Task 3.9.** Describe the possible weights $w_{P'}$ for $P' \in \mathsf{Path}(P, i)$.

Here is our main result related to Task 3.9.

**Theorem 3.10.** *Let $w_P = (w_1, \ldots, w_n)$. The following claims are true.*

(1) $w_i \neq w_{i+1}$.
(2) *If $w_{i+1} = w_i \pm 1$, then $\mathsf{Path}(P, i) = \{P\}$.*
(3) *If $w_{i+1} \neq w_i \pm 1$, then $\mathsf{Path}(P, i)$ consists of two elements $P, P'$ and $w_{P'}$ is obtained from $w_P$ by permuting the entries $i$ and $i+1$.*
(4) *If $i < n-1$ and $w_i = w_{i+1} \pm 1$, then $w_{i+2} \neq w_i$.*

This theorem will be proved in Section 4.3 after some preparation.

The following result is one of the tools to prove Theorem 3.10. Consider the subalgebra $Z_{i-1}(i+1) \subset \mathbb{C}S_n$. Set $V := V^n$ to simplify the notation. Let $V_{P,i} := \operatorname{Span}_{\mathbb{C}}(v_{P'} | P' \in \mathsf{Path}(P, i))$. Note that by Corollary 3.3, the vectors $v_{P'}$ not just span, they form a basis of $V_{P,i}$.

**Proposition 3.11.** *The subspace $V_{P,i} \subset V$ is a $Z_{i-1}(i+1)$-submodule. Moreover, it is irreducible as a $Z_{i-1}(i+1)$-module.*

*Proof of Proposition 3.11.* Let we write $P$ as the concatenation $P_0P_1P_2$ with

$$P_0 \in \mathsf{Path}(V^{i-1}), P_1 \in \mathsf{Path}(V^{i-1}, V^{i+1}), P_2 \in \mathsf{Path}(V^{i+1}, V^n).$$

Then the paths in $\mathsf{Path}(P,i)$ are exactly the paths of the form $P_0P_1'P_2$ with $P_1' \in \mathsf{Path}(V^{i-1}, V^{i+1})$. Moreover, $v_{P_0P_1'P_2} = \varphi_{P_2}(\varphi_{P_1'}(v_{P_0}))$ by Corollary 3.5.

Consider the linear map

$$\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) \to V, \psi \mapsto \varphi_{P_2}(\psi(v_{P_0})). \tag{3.2}$$

The map (3.2) sends $\varphi_{P_1'}$ to $v_{P_0P_1'P_2}$. The former elements form a basis in $\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$ by Lemma 3.2, while the latter elements form a basis in $V_{P,i}$. So (3.2) is injective with image $V_{P,i}$. Since $\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$ is an irreducible $Z_{i-1}(i+1)$-module by Lemma 2.3, it remains to show that (3.2) is $Z_{i-1}(i+1)$-linear.

First of all, $\varphi_{P_2} : V^{i+1} \to V^n$ is $\mathbb{C}S_{i+1}$-linear, and hence $Z_{i-1}(i+1)$-linear since $Z_{i-1}(i+1) \subset \mathbb{C}S_{i+1}$. So, it remains to show that the map

$$\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) \to V^{i+1}, \psi \mapsto \psi(v_{P_0}),$$

is $Z_{i-1}(i+1)$-linear. But this is a direct consequence of (2.5). $\square$

## 4. Degenerate affine Hecke algebra $\mathcal{H}(2)$

Proposition 3.11 is not sufficient to prove Theorem 3.10 because at this point we don't know much about the algebra $Z_{i-1}(i+1)$. What we know is generators. We will determine some relations between them. This will lead us to a new algebra, the degenerate double affine Hecke algebra $\mathcal{H}(2)$. We will determine its irreducible representations and use this to prove Theorem 3.9.

In more detail, thanks to Theorem 2.9, we know that $Z_{i-1}(i+1)$ is generated by $Z_{i-1}(i-1)$, a central subalgebra, and three more elements: $J_i, J_{i+1}, (i, i+1)$. In this section, we will examine some relations between these three generators arriving at the definition of the degenerate affine Hecke algebra $\mathcal{H}(2)$. Using Proposition 3.11 we will see that the space $V_{P,i}$ will be an irreducible representation of this algebra. Then we will study finite dimensional irreducible representations of $\mathcal{H}(2)$. Theorem 3.9 will easily follow from this.

### 4.1. Definition of $\mathcal{H}(2)$.

**Lemma 4.1.** *We have the following identities*

$$J_iJ_{i+1} = J_{i+1}J_i, \quad (i, i+1)^2 = 1, \quad (i, i+1)J_i = J_{i+1}(i, i+1) - 1. \tag{4.1}$$

*Proof.* The element $J_{i+1}$ commutes with $\mathbb{C}S_i$ and hence with $J_i \in \mathbb{C}S_i$. This gives the first relation. The second relation is obvious. The third relation is equivalent to $(i, i+1)J_i(i, i+1) = J_{i+1} - (i, i+1)$ – multiply by $(i, i+1)$ on the right. The left hand side is the conjugation of $J_i$ by $(i, i+1)$ hence equals $\sum_{j=1}^{i-1}(j, i+1)$. This equals $J_{i+1} - (i, i+1)$. $\square$

Define the *degenerate affine Hecke algebra* $\mathcal{H}(2)$ by generators $X_1, X_2, T$ and relations that mirror those found in Lemma 4.1:

$$X_1X_2 = X_2X_1, T^2 = 1, TX_1 = X_2T - 1. \tag{4.2}$$

There is a consequence of these relations:

$$X_1T = TX_2 - 1. \tag{4.3}$$

To see this we multiply the third relation in (4.2) by $T$ both from the left and from the right and use the second relation.

Our conclusion is that we have a unique algebra homomorphism $\mathcal{H}(2) \to Z_{i-1}(i+1)$ given on generators by $X_1 \mapsto J_i, X_2 \mapsto J_{i+1}, T \mapsto (i, i+1)$. In particular, any $Z_{i-1}(i+1)$-module can be viewed as an $\mathcal{H}(2)$-module.

**Corollary 4.2.** *Let $M$ be an irreducible $Z_{i-1}(i+1)$-module. Then it stays irreducible as an $\mathcal{H}(2)$-module.*

*Proof.* Recall, Section 2.2, that $Z_{i-1}(i+1)$ is a central subalgebra of $Z_{i-1}(i+1)$. By [1, Exercise 2.12], every element of the center acts by a scalar on an irreducible $Z_{i-1}(i+1)$-module. In particular, every subspace in $M$ is stable under $Z_{i-1}(i-1)$. Since the subalgebra $Z_{i-1}(i-1)$ and the elements $J_i, J_{i+1}, (i, i+1)$ generate the algebra $Z_{i-1}(i+1)$, a subspace in $M$ is a $Z_{i-1}(i+1)$-submodule if and only if it is stable under $J_i, J_{i+1}, (i, i+1)$. The latter condition is equivalent to being a $\mathcal{H}(2)$-submodule. This finishes the proof. $\square$

**Remark 4.3.** One can ask why we picked these relations. In fact, one can show that (4.2) are the only relations between $J_i, J_{i+1}, (i, i+1)$ that are independent of $i$, in some precise sense.

One can also ask how the algebra $\mathcal{H}(2)$ looks like, e.g., what a vector space basis is. An easy consequence of the relations is that the monomials of the form $X_1^{d_1} X_2^{d_2} \sigma$ for $\sigma \in \{1, T\}$ span $\mathcal{H}(2)$. In fact, one can further show that these monomials form a basis.

**Remark 4.4.** We can generalize this construction and, for $d \geqslant 1$, produce the degenerate affine Hecke algebra $\mathcal{H}(d)$ with a homomorphism to $Z_i(i+d)$. Define $\mathcal{H}(d)$ as the $\mathbb{C}$-algebra generated by $X_1, \ldots, X_d, T_1, \ldots, T_{d-1}$ with the following relations

$$\begin{aligned}
&X_i X_j = X_j X_i, \\
&T_i^2 = 1, \quad T_i T_j = T_j T_i, \text{ for } |i-j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\
&X_i T_j = T_j X_i, \text{ for } i - j \neq 0, 1, \quad T_i X_i = X_{i+1} T_i - 1.
\end{aligned}$$

The motivation for the relations in the second row is that the transpositions $(i, i+1) \in S_d, i = 1, \ldots, d-1$, satisfy these relations. Moreover, the group generated by $T_1, \ldots, T_{d-1}$ with these relations is the symmetric group $S_d$. This can be checked in an elementary way, but there is also a nice topological proof, [Ka, Exercise 3.10].

One can show that we have an algebra homomorphism $\mathcal{H}(d) \to Z_i(i+d)$ with $X_j \mapsto J_{i+j}, T_j \mapsto (i+j, i+j+1)$, left as an exercise.

4.2. **Finite dimensional irreducible representations of $\mathcal{H}(2)$.** Let us classify the finite dimensional irreducible $\mathcal{H}(2)$-modules $M$ (in fact, all irreducible modules over this algebra are finite dimensional, but we will not need this fact).

Since $X_1, X_2$ commute, they have a common eigenvector $m \in M$. Let $X_1 m = am, X_2 m = bm$, where $a, b \in \mathbb{C}$.

Let us consider two cases:

*Case 1.* $Tm$ is proportional to $m$. Since $T^2 = 1$, we have two options:

1.1) $Tm = m$. Let us apply the third relation in (4.2) to $m$. The left hand side gives $TX_1 m = am$, while the right hand side gives $(X_2 T - 1)m = (b-1)m$, so here $b = a+1$.

1.2) $Tm = -m$. Similarly to the previous case, we get $b = a - 1$.

*Case 2.* $m$ and $Tm$ are linearly independent. Let us see how $X_1, X_2$ act on $Tm$:

$$\begin{aligned}
X_1(Tm) &= [X_1 T = TX_2 - 1] = TX_2 m - m = b(Tm) - m, \\
X_2(Tm) &= [X_2 T = TX_1 + 1] = TX_1 m + m = a(Tm) + m.
\end{aligned}$$

In particular, we see that $\operatorname{Span}(m, Tm)$ is stable under $\mathcal{H}(2)$. Since $M$ is irreducible, we see that $m$ and $Tm$ form a basis in $M$. In this basis, we have

$$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, X_2 \mapsto \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}. \tag{4.4}$$

The matrices in (4.4) satisfy the defining relations of $X_1, X_2, T$ for all $a, b \in \mathbb{C}$. So (4.4) defines an $\mathcal{H}(2)$-module structure on $\mathbb{C}^2$, to be denoted by $M(a,b)$. We conclude that $M$ must be isomorphic to $M(a,b)$.

**Lemma 4.5.** *The module $M(a,b)$ is irreducible if and only if $a \neq b \pm 1$. Moreover, if $a \neq b \pm 1$, then $M(a,b) \cong M(a',b')$ if and only if $(a,b) = (a',b')$ or $(b,a) = (a',b')$.*

*Proof.* Assume, first, $a \neq b$ (the only case we will need). In this case, by (4.4) $X_1, X_2$ both have distinct eigenvalues, so act on $M(a,b)$ by diagonalizable operators. But $X_1, X_2$ commute, so they are simultaneously diagonalizable. Therefore there is another eigenvector with eigenvalue $b$ for $X_1$ and $a$ for $X_2$. Since $a \neq b$, every subspace of $M(a,b)$ stable under $X_1$ (or $X_2$) must be the sum of some of the eigenspaces. If this subspace is a proper submodule, then $T$ must preserve it. This implies $a = b \pm 1$ by the analysis of Case 1 above. Conversely, if $a = b \pm 1$, then $m \mp Tm$ is an eigenvector for $X_1, X_2, T$, hence spans a submodule. This implies that $M(a,b)$ is not irreducible. Our conclusion is that $M(a,b)$ is irreducible if and only if $a \neq b \pm 1$.

Since $M(a,b)$ also contains a vector with eigenvalues $(b,a)$, we get $M(a,b) \cong M(b,a)$ by the analysis before the lemma. And if $(a',b')$ is different from both $(a,b)$ and $(b,a)$, then $M(a',b') \not\cong M(a,b)$ as the eigenvalues of $X_1$ on the left hand side are different from $a, b$.

Note that when $a = b$, the operators $X_1, X_2$ on $M(a,b)$ are not diagonalizable. This case is left as an exercise. □

We arrive at the following classification result.

**Proposition 4.6.** *The finite dimensional irreducible $\mathcal{H}(2)$-modules are classified by pairs of complex numbers, $(a,b) \mapsto L(a,b)$, with $L(a,b) \cong L(b,a)$ if $b \neq a, a \pm 1$. The pair $(a,b)$ is a pair of simultaneous eigenvalues of $X_1, X_2$ in $L(a,b)$. Moreover, the following is true.*

1. *If $b = a+1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto 1, X_1 \mapsto a, X_2 \mapsto b$.*
2. *If $b = a-1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto -1, X_1 \mapsto a, X_2 \mapsto b$.*
3. *If $b \neq a \pm 1$, then $L(a,b) \cong M(a,b)$ from (4.4).*
4. *The action of $X_1, X_2$ on $L(a,b)$ is diagonalizable if and only if $a \neq b$.*

**Remark 4.7.** One can also ask how to classify the irreducible $\mathcal{H}(d)$-modules and compute their dimensions. This is no longer elementary, and may be mentioned later in the course when we discuss the category $\mathcal{O}$.

### 4.3. Proof of Theorem 3.10.

*Proof.* Let $w_P = (w_1, \ldots, w_n)$ and $P' \in \mathsf{Path}(V, i)$ with $w_{P'} = (w'_1, \ldots, w'_n)$. By the construction of $w'_j$ in Section 3.1, $w'_j$ depends only on $V_{j-1}, V_j$ (and $w'_1 = 0$). Since $V'_j = V_j$ for all $j \neq i$, we see that $w'_j = w_j$ for $j \neq i, i+1$. Recall that $V_{P,i}$ is an irreducible $Z_{i-1}(i+1)$-module, Proposition 3.11, and hence also an irreducible $\mathcal{H}(2)$-module, Corollary 4.2. The elements $X_1, X_2$ act as $J_i, J_{i+1}$ on $V_{P,i}$ and hence both are diagonalizable, Corollary 3.3. The pairs $(w_i, w_{i+1}), (w'_i, w'_{i+1})$ are pairs of simultaneous eigenvalues for $X_1, X_2$ acting on the irreducible $\mathcal{H}(2)$-module $V_{P,i}$. Now (1)-(3) of the theorem follow from Proposition 4.6.

Let us prove (4): if $w_{i+1} = w_i \pm 1$, then $w_{i+2} \neq w_i$. Since $w_{i+1} = w_i \pm 1$, by part (2), the space $V_{P,i}$ is 1-dimensional. If $w_{i+2} = w_i = w_{i+1} \mp 1$, then $V_{P,i+1}$ is also 1-dimensional. It follows that $\mathbb{C}v_P$ is stable under both transpositions $(i, i+1), (i+1, i+2)$. By Proposition 4.6, $(i, i+1)$ acts on $\mathbb{C}v_P$ by $\pm 1$, while $(i+1, i+2)$ acts by $\mp 1$. But

$$(i, i+1)(i+1, i+2)(i, i+1) = (i, i+2) = (i+1, i+2)(i, i+1)(i+1, i+2)$$

and, so, by looking at how this element acts on $v_P$, we arrive at $\mp 1 = \pm 1$, which gives a contradiction. $\square$

## 5. Completion of classification

In this section we will finish the classification of finite dimensional irreducible $\mathbb{C}S_n$-modules. The two main ingredients are Theorem 3.10 and the basic observation that the number of irreducibles is the same as the number of partitions of $n$. Then we will give a combinatorial parametrization of a basis in a given irreducible module.

5.1. **Combinatorial weights.** Motivated by part (3) of Theorem 3.10 we are going to define an equivalence relation on $\mathbb{C}^n$. By an *admissible transposition* of $\mathbb{C}^n$ we mean a transposition of two adjacent entries if their difference is not $\pm 1$.

**Definition 5.1.** We say that two elements of $\mathbb{C}^n$ are *c-equivalent* ("c" for "combinatorial") if one is obtained from the other by a sequence of admissible transpositions. This is an equivalence relation to be denoted by $\sim_c$. By a *combinatorial weight* we mean an element of $\mathbb{C}^n$ such that every combinatorially equivalent to it element $(w_1, \ldots, w_n)$ satisfies the following three conditions:

(i) $w_1 = 0$.
(ii) For any $i = 1, \ldots, n-1$, we have $w_i \neq w_{i+1}$.
(iii) For any $i = 1, \ldots, n-2$, we have that if $w_{i+1} = w_i \pm 1$, then $w_{i+2} \neq w_i$.

The set of combinatorial weights is denoted by $\mathsf{cWt}_n$.

Here is a consequence of Theorem 3.10.

**Corollary 5.2.** *The following claims hold:*

(1) *We have $\mathsf{Wt}_n \subset \mathsf{cWt}_n$. Moreover, $\mathsf{Wt}_n$ is the union of c-equivalence classes.*
(2) *c-equivalence implies r-equivalence (recall, Definition 3.7, that two weights are r-equivalent if the corresponding paths lead to the same irreducible module).*

We will write $X/\sim$ for the set of equivalence classes in a set $X$ for an equivalence relation $\sim$ and $|\bullet|$ for the cardinality of a set, possibly infinite. Thanks to Corollary 5.2, we have

$$|\mathsf{Wt}_n/\sim_r| \leqslant |\mathsf{Wt}_n/\sim_c| \leqslant |\mathsf{cWt}_n/\sim_c|, \tag{5.1}$$

where the first inequality follows from (2), and the second inequality follows from (1).

Recall, see the discussion after Definition 3.7 that $\mathsf{Wt}_n/\sim_r$ is in bijection with $\mathrm{Irr}(\mathbb{C}S_n)$. In particular, the number of elements coincides with the number of partitions of $n$.

**Lemma 5.3.** *Every c-equivalence class contains an element of the form*

$$(0, 1 \ldots, n_1 - 1, -1, \ldots, n_2 - 2, -2, \ldots, n_3 - 3, \ldots, 1 - k, \ldots n_k - k)$$

*for some positive integers $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k$ with $n_1 \ldots + n_k = n$.*

This lemma combined with (5.1) imply that $|\mathsf{cWt}_n/\sim_c|$ does not exceed the number of partitions of $n$. So, once the lemma is proved, it implies that

- $\mathsf{cWt}_n = \mathsf{Wt}_n$.
- $\sim_c = \sim_r$.
- The numbers $n_1, \ldots, n_k$ in the lemma are uniquely read of the equivalence class.

In particular, to $V \in \mathrm{Irr}(\mathbb{C}S_n)$ we can assign the partition $(n_1, n_2, \ldots, n_k)$ of $n$. Our conclusion is that this gives a bijection between $\mathrm{Irr}(\mathbb{C}S_n)$ and the set of partitions of $n$.

**Example 5.4.** In this example we re-examine partitions corresponding to three of the five irreducible representations of $S_4$: the trivial, the reflection, and the 2-dimensional representations, compare to Example 1.1.

1) Consider the trivial representation $\mathrm{triv}_4$. It has only one weight, $(0,1,2,3)$, so $k=1$ and $n_1=4$. The partition is $(4)$.

2) Consider the reflection representation $\mathrm{refl}_4$. It has three weights, see Example 3.4: $(0,-1,1,2),(0,1,-1,2),(0,1,2,-1)$. The latter is of the form described in Lemma 5.3. So $k=2,n_1=3,n_2=1$ and the partition is $(3,1)$.

3) For the irreducible representation $\mathbb{C}^2$, the weights are $(0,1,-1,0)$ and $(0,-1,1,0)$, Example 3.4. The former is of the form described in Lemma 5.3 and the corresponding partition is $(2,2)$.

*Proof of Lemma 5.3.* Note that all components of a combinatorial weight $(w_1,\ldots,w_n)$ are integers. Indeed, let $i$ be the minimal number with $w_i\not\in\mathbb{Z}$. Then there is an obvious collection of admissible transpositions that moves $w_i$ to the 1st position, giving a contradiction. So, we can consider the lexicographic order on an equivalence class in $\mathsf{cWt}_n$: $(w_1,\ldots,w_n)>(w_1',\ldots,w_n')$ if there is $i$ such that $w_1=w_1',\ldots,w_{i-1}=w_{i-1}'$ and $w_i>w_i'$.

Let $(w_1,\ldots,w_n)$ be a maximal element in its equivalence class. We claim that it has the form described in the lemma.

*Step 1.* Let $n_1$ be such that $n_1-1=\max(w_i)$. Let $k$ be the smallest index such that $w_k=n_1-1$. We claim that $k=n_1$ and $w_i=i-1$ for all $i<n_1$. Indeed, assume the contrary, and pick the largest index $j<k$ with $w_j\neq n_1-1-(k-j)$. We have $w_j<n_1$ by the choice of $k$. We have $w_j\geqslant j-1$. Otherwise, permuting the entries numbered $j$ and $j+1$ gives an admissible transposition that increases $(w_1,\ldots,w_n)$ in the lexicographic order. But if $w_j\geqslant j$, then we can perform a sequence of admissible transpositions to move $w_j$ to the left until we either arrive at one of the fragments $(w_j,w_j)$ or $(w_j,w_j\pm1,w_j)$ or place $w_j\neq 0$ into the 1st position, which is impossible. So $w_j=n_1-1-(k-j)$ for all $j<k$. Since $w_1=0$, we get $k=n_1$.

*Step 2.* If $n_1=n$, then we are done. If not, we claim $w_{n_1+1}=-1$. Assume the contrary. We have $w_{n_1+1}\leqslant n_1-1$ by the choice of $n_1$. Further, we have $w_{n_1+1}\neq n_1-1$ because otherwise $w_{n_1}=w_{n_1+1}$, which is impossible. So we can start moving the entry $w_{n_1+1}$ to the left by a sequence of admissible transpositions. This will end either when we encounter the fragment $(w_{n_1+1},w_{n_1+1}+1,w_{n_1+1})$, which happens for $w_{n_1+1}\geqslant 0$ and lead to a contradiction, or when we place $w_{n_1+1}$ in the first slot, which happens when $w_{n_1+1}<-1$, and also leads to a contradiction.

*Step 3.* Then we repeat the argument of Step 1 and see that $w$ starts with $(0,1,\ldots,n_1-1,-1,0,\ldots,n_2-2)$ for $n_2+2\leqslant n_1+1$. Next, we repeat the argument of Step 2 to see that the next element is $-2$, etc. $\square$

5.2. **Young diagrams and Young tableaux.** The previous section gives a combinatorial classification of $\mathrm{Irr}(\mathbb{C}S_n)$ – by the partitions of $n$ – and also a combinatorial parametrization of basis elements – in terms of combinatorial weights. In this section we will make the latter parametrization more explicit – and more classical – by establishing a bijection between the combinatorial weights and the standard Young tableaux.

Partitions of $n$ that are often depicted as Young diagrams, the following diagram corresponds to the partition $5=3+2$.

$$\begin{array}{|c|c|c}\cline{1-2} \phantom{0} & \phantom{0} & \\ \hline \phantom{0} & \phantom{0} & \multicolumn{1}{c|}{\phantom{0}} \\ \hline \end{array}$$

The advantage of this description is that we can fill the boxes with numbers. Our goal is to relate $\mathsf{Wt}_n = \mathsf{cWt}_n$ to the set of *standard Young tableaux* (SYT) $\mathsf{SYT}(n)$. Recall that a standard Young tableau on a Young diagram with $n$ boxes is a filling of this diagram with numbers from 1 to $n$ that strictly increase bottom to top and left to right (in particular, each number occurs exactly once). The underlying Young diagram of an SYT $T$ will be called the *shape* of $T$. For example, these two fillings are examples of SYT's of shape $(3, 2)$.

$$\begin{array}{|c|c|c}\cline{1-2} 4 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c}\cline{1-2} 3 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \tag{5.2}$$

**Definition 5.5.** To a Young tableau $T$ we assign its *content* as follows. Let $(x_i, y_i)$ be the coordinate of the box numbered by $i$. By its content we mean the difference $x_i - y_i$. The content $c(T)$ of $T$ is, by definition, the collection of contents of the individual boxes, i.e., $(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$.

The following two collections are contents of the tableaux in the previous example: $(0, 1, 2, -1, 0)$ and $(0, 1, -1, 2, 0)$. In other words, we record the diagonals where the boxes of $T$ lie.

**Exercise 5.6.** Show that the map $T \mapsto c(T)$ is injective and explain how to recover $T$ from $c(T)$.

**Proposition 5.7.** *The map $T \mapsto c(T)$ is a biijection $\mathsf{SYT}(n) \to \mathsf{cWt}(n)$, moreover, the shape of $T$ coincides with the partition assigned to $c(T)$ in Lemma 5.3.*

*Proof.* We can define an *admissible transposition* of entries $i$ and $i + 1$ in a tableau $T$: we permute $i$ and $i + 1$ if the result is still a SYT. For example, the two tableaux above are obtained from one another by permuting 3 and 4.

The admissible permutations give rise to an equivalence relation $\sim_c$ on $\mathsf{SYT}(n)$. It is easy to see that an admissible permutation of $k, k + 1$ in $T$ corresponds exactly to the admissible permutation of entries numbered $k$ and $k + 1$ in $c(T)$. The collection $c(T) = (c_1, \dots, c_n)$ satisfies conditions (i)-(iii) of Definition 5.1. For example, let us check (iii). The condition $c_i = c_{i+2}$ means that the numbers $i, i + 2$ appear on the same diagonal in $T$. Consider the square in $T$ with the bottom left corner containing $i$ and the top right corner containing $i+2$ (it is in this order by the definition of a SYT). The remaining elements inside the square are strictly between $i$ and $i + 2$, and there are, at least, two of them. Since all elements are distinct, we arrive at a contradiction.

So $c(T)$ is indeed an element of $\mathsf{CWt}(n)$ and the image of $c$ is the union of equivalence classes for $\sim_c$.

We can define *normal* SYT's, where we fill the first row by numbers from 1 to some $n_1$, then the second row by the numbers from $n_1+1$ to $n_1+n_2$, etc., for example, the first tableau in (5.2) is normal. Clearly, if $T$ is normal, then $c(T)$ is of the form described in Lemma 5.3. From this lemma it follows that $c$ is surjective. On the other hand, $c$ is injective, Exercise 5.6.

The argument in the beginning of the previous paragraph also implies that the shape of $T$ is the Young diagram corresponding to the partition assigned to the equivalence class of $c(T)$. □

Now we can restate Corollary 3.3 in the language of SYT.

**Corollary 5.8.** *Let $\lambda$ be a Young diagram with $n$ boxes and $V_\lambda$ be the corresponding irreducible $\mathbb{C}S_n$-module. There is a basis, $v_T$, in $V_\lambda$ labelled by SYT's $T$ of shape $\lambda$. Moreover, each $v_T$ is an eigenvector for the Jucys-Murphy elements $J_i, i = 1, \ldots, n$. The eigenvalue of $J_i$ on $v_T$ is the content of the box labelled $i$ in $T$.*

We finish this section by giving a combinatorial description of the branching graph from Section 3.1. Let $(w_1, \ldots, w_n) \in \mathsf{Wt}_n$ and let $T$ be the corresponding SYT. Let $T'$ denote the SYT corresponding to $(w_1, \ldots, w_{n-1})$. Then $T'$ is obtained from $T$ by removing the box labelled $n$. Note that $n$ is located in a corner box, i.e., a box with nothing above or the right. And for any Young diagram and any corner box, we can find a SYT with $n$ in that box.

Here is a corollary of this discussion.

**Corollary 5.9.** *Let $\lambda$ be a partition of $n$ and $V_\lambda$ be the corresponding irreducible module over $\mathbb{C}S_n$. As a $\mathbb{C}S_{n-1}$-module, $V_\lambda$ decomposes as $\bigoplus_\mu V_\mu$, where $\mu$ runs over all Young diagrams obtained from $\lambda$ by removing a box. Moreover, $J_n$ acts on $V_\mu$ by the scalar equal to the content of the removed box.*

For example, if $\lambda = (3, 2)$, then there are two summands, corresponding to the partitions $(3, 1)$ and $(2, 2)$. The corresponding eigenvalues of $J_5$ are 0 and 2, respectively.

**Definition 5.10.** By the Young graph we mean a directed graph whose vertices are Young diagrams and we have an edge $\mu \to \lambda$ if and only if $\mu$ is obtained from $\lambda$ by removing one box.

**Corollary 5.11.** *Under our identification of $\bigsqcup_{n\geqslant 1} \operatorname{Irr}(\mathbb{C}S_n)$ with the set of Young diagrams, the branching graph becomes the Young graph.*

Finally, we give a combinatorial description of tensoring with $\operatorname{sgn}_n$.

**Exercise 5.12.** Show that tensoring with the sign representation corresponds to transposing the Young diagram (about the main diagonal; for example for the partition $(3, 1)$ its transpose is $(2, 1^2)$).

## 6. Complements

6.1. **Further results.** Once the irreducible representations are classified one can ask to compute their characters. And once we have a basis one can ask to describe how the elements of $S_n$, or, at least, generators, act on basis elements. The latter is actually quite easy with the approach we have discussed so we start with that.

6.1.1. *Action of generators.* Let $\lambda$ be a partition of $n$ and $V_\lambda$ be the corresponding irreducible representation of $S_n$. We have a basis in $V_\lambda$ labelled by the standard Young tableaux of shape $\lambda$, Corollary 5.8. Let $v_T$ denote a basis element labeled by a SYT $T$. We emphasize that $v_T$ is defined uniquely up to a scalar factor. Let $\underline{w} = (w_1, \ldots, w_n)$ be the weight of $v_T$, i.e., the content of $T$.

The symmetric group $S_n$ is generated by the transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$. We want to understand how $(i, i+1)$ acts on $v_T$. For this we recall that $(i, i+1) \in Z_{i-1}(i+1)$ and

(a) either $\mathbb{C}v_T$ is a $Z_{i-1}(i+1)$-submodule
(b) or there is another SYT, say $T'$, such that $\mathrm{Span}_{\mathbb{C}}(v_T, v_{T'})$ is a $Z_{i-1}(i+1)$-submodule, Theorem 3.10.

(a) happens when we cannot permute $i, i+1$ in $T$. In terms of weights, this means $w_{i+1} = w_i \pm 1$, or, equivalently, in $T$ the number $i+1$ is next to the right of $i$, the case when $w_{i+1} = w_i + 1$, or right above $i$, the case when $w_{i+1} = w_i - 1$. (b) happens when we can permute $i, i+1$ inside of $T$, and $T'$ is the SYT obtained by this permutation. Using Proposition 4.6 we conclude

- $(i, i+1)v_T = v_T$ if $w_{i+1} - w_i = 1$,
- $(i, i+1)v_T = -v_T$ if $w_{i+1} - w_i = -1$,
- $(i, i+1)v_T = av_T + bv_{T'}$ with $b \neq 0$ if $w_{i+1} - w_i \neq \pm 1$.

Note that $b$ is not defined uniquely as both $v_T$ and $v_{T'}$ can be independently rescaled. But $a$ is determined uniquely.

**Exercise 6.1.** Prove that $a = (w_{i+1} - w_i)^{-1}$.

In fact, we can normalize the vectors $v_T$ such that in the above formulas we can set $b = \sqrt{1-a^2}$, see [Kl, Proposition 2.3.5]. One can also normalize so that all coefficients $b$ are rational numbers, [Kl, Theorem 2.3.1]. From here one can deduce that all irreducible representations of $S_n$ are defined over $\mathbb{Q}$, equivalently, that $\mathbb{Q}S_n$ is isomorphic to the direct sum of matrix algebras over $\mathbb{Q}$ (and not some skew-fields).

6.1.2. *Frobenius character formula.* Now we discuss the character of $V_\lambda$. The most famous result here is the Frobenius character formula relating the characters of representations of $S_n$ to symmetric polynomials, see, e.g., [E, Section 5.15] or [F, Section 7.3]. It is not directly related to the inductive approach that we took to study the irreducible representations but the connection between irreducible representations of $S_n$ and the symmetric polynomials is so important that we cannot bypass it.

We start by explaining the notion of the Frobenius character. Fix $N \geqslant n$. For $d > 0$, we can consider the power symmetric polynomial in $N$ variables: $p_m = \sum_{i=1}^N x_i^d$. Now take a permutation $\sigma \in S_n$ and let $(n_1, \ldots, n_k)$ be the corresponding permutation, where the entries are the lengths of cycles in $\sigma$. Let $p_\sigma = p_{n_1} p_{n_2} \ldots p_{n_k}$. For example, $p_1 = (x_1 + \ldots + x_N)^n$. Note that $p_\sigma = p_{\sigma'}$ if and only if $\sigma$ and $\sigma'$ correspond to the same permutation, i.e., are conjugate. So, for a conjugacy class $c$ in $S_n$ we write $p_c$ for $p_\sigma$ with $\sigma \in c$.

**Definition 6.2.** Let $V$ be a finite dimensional $\mathbb{F}S_n$-module and $\chi_V : S_n \to \mathbb{C}$ be its character. By the *Frobenius character* $F_V$ we mean the symmetric polynomial

$$F_V(x_1, \ldots, x_N) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) p_\sigma.$$

It is easy to show that, since $N \geqslant n$, the elements $p_c$ form a basis in the space of symmetric polynomials of degree $n$. So one can recover the usual character from the Frobenius character.

Before we proceed we should explain the dependence on $N$. We note that $F_V(x_1, \ldots, x_N)$ is obtained from $F_V(x_1, \ldots, x_{N+1})$ by setting $x_{N+1} = 0$. So, to make things more convenient, people talk about symmetric polynomials in infinitely many variables, they are infinite sums. We are not going to do this.

Here is the crucial property of $F_\bullet$. Let $V_n, V_m$ be representations of $S_n, S_m$ respectively. We can view their tensor product as a representation of $S_n \times S_m$: $(\sigma, \tau)(u \otimes v) = (\sigma u) \otimes (\tau v)$.

Then we can induce to $S_{n+m}$. Let

$$V_{n+m} = \operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(V_n\otimes V_m).$$

**Proposition 6.3.** *Suppose $N\geqslant n+m$. We have $F_{V_{n+m}}=F_{V_n}F_{V_m}$.*

*Proof.* Let $H\subset G$ be finite groups, and $U$ be a finite dimensional $\mathbb{C}H$-module, let $\chi_U$ denote its character. Set $V:=\operatorname{Ind}_H^G U$. The Frobenius formula for the character of an induced module, [E, Section 5.9], says that

$$\chi_V(g)=|H|^{-1}\sum_{x\in G|xgx^{-1}\in H}\chi_U(xgx^{-1}),$$

the proof is not hard, the readers can do this as an exercise. So, in our situation, the coefficient of $p_c$ in $F_{V_{n+m}}$ is

$$\frac{1}{(n+m)!n!m!}\sum_{\sigma\in c}\sum_{x\in S_{n+m}|x\sigma x^{-1}\in S_n\times S_m}\chi_{V_n\otimes V_m}(x\sigma x^{-1}) \tag{6.1}$$

Let $c':=c\cap(S_n\times S_m)$, this is the union of conjugacy classes in $S_n\times S_m$. Note that if $\sigma'=(\sigma_1,\sigma_2)$ for $\sigma_1\in S_n,\sigma_2\in S_m$, then $\chi_{V_n}(\sigma_1)\chi_{V_m}(\sigma_2)=\chi_{V_n\otimes V_m}\sigma'$. Therefore, the coefficient of $p_c$ in

$$F_{V_n}F_{V_m}=\frac{1}{n!m!}\sum_{\sigma_1\in S_n,\sigma_2\in S_m}\chi_{V_n}(\sigma_1)\chi_{V_m}(\sigma_2)p_{\sigma_1}p_{\sigma_2}$$

equals

$$\frac{1}{n!m!}\sum_{\sigma'\in c'}\chi_{V_n\otimes V_m}(\sigma'). \tag{6.2}$$

Let $c'=c'_1\sqcup c'_2\sqcup\ldots\sqcup c'_\ell$ be the decomposition into $S_n\times S_m$-conjugacy classes. For $i=1,\ldots,\ell$, let $\chi_i$ be the common value of $\chi_{V_n\otimes V_m}$ on the elements of $c'_i$. Then we can rewrite (6.1) as

$$\frac{1}{(n+m)!n!m!}\sum_{i=1}^{\ell}|\{(\sigma,x)\in c\times S_{n+m}|x\sigma x^{-1}\in c'_i\}|\chi_i, \tag{6.3}$$

while (6.2) can be rewritten as

$$\frac{1}{n!m!}\sum_{i=1}^{\ell}|c'_i|\chi_i. \tag{6.4}$$

We reduce to proving that

$$|\{(\sigma,x)\in c\times S_{n+m}|x\sigma x^{-1}\in c'_i\}|=(n+m)!|c'_i|$$

The right hand side is the cardinality of the set $c'_i\times S_{n+m}$. We have a bijection

$$\{(\sigma,x)\in c\times S_{n+m}|x\sigma x^{-1}\in c'_i\}\xrightarrow{\sim}c'_i\times S_{n+m},(\sigma,x)\mapsto(x\sigma x^{-1},x).$$

This implies that (6.3) and (6.4) coincide and finishes the proof. $\square$

It turns out that $F_{V_\lambda}$ is the so called *Schur polynomial* $s_\lambda$. There are several ways to define it. For example, let $\lambda=(n_1\geqslant n_2\geqslant\ldots\geqslant n_k)$ be a partition of $n$. Fix $N\geqslant n$. We can assume $k=N$ by adjoining zero parts to $(n_1,\ldots,n_k)$. Then the polynomial $\det(x_j^{n_i+N-i})_{i,j=1}^N$ is sign-symmetric (if we permute $x_1,\ldots,x_N$ according to a permutation

$\pi \in S_N$, then the polynomial gets multiplied by $\mathrm{sgn}(\pi)$). Therefore it is divisible by the Vandermonde determinant

$$\det(x_j^{N-i})_{i,j=1}^N = \prod_{i<j}(x_i - x_j).$$

The ratio is a symmetric polynomial.

**Definition 6.4.** This ratio is called the *Schur polynomial* and is denoted by $s_\lambda$.

For an alternative description, see [F, Section 4.3]. The following result is in (2) of the theorem in [F, Section 7.3]. Note that Fulton defines $F_V$ differently but his definition agrees with ours, see equation (12) in [F, Section 7.3].

**Theorem 6.5.** *We have $F_{V_\lambda} = s_\lambda$.*

This theorem is also equivalent to [E, Theorem 5.15.1].

6.1.3. *Murnaghan-Nakayama rule.* Theorem 6.5 is not directly deduced using the inductive approach to the representations of symmetric groups that we were pursuing. But there is a character formula that follows relatively easily from our approach: the Murnaghan-Nakayama rule. The reader is referred to [Kl, Section 2.3], in particular, Theorem 2.3.6 and Corollary 2.3.7 there.

6.2. **Irreducible representations of $\mathfrak{A}_n$.** Here we will explain how to reduce the study of the irreducible representations of $\mathfrak{A}_n$ to those of $S_n$. Here is the result.

**Theorem 6.6.** *Let $V_\lambda$ denote the irreducible $\mathbb{C}S_n$-module corresponding to a partition $\lambda$. Then the following claims are true:*

(1) *If $\lambda$ is different from its transpose, $\lambda^t$, then $V_\lambda$ is irreducible as a module over $\mathbb{C}\mathfrak{A}_n$.*
(2) *If $\lambda$ coincides with its transpose, then $V_\lambda$ decomposes into the sum of two non-isomorphic $\mathbb{C}\mathfrak{A}_n$-modules permuted by any odd permutation.*
(3) *Moreover, every irreducible $\mathbb{C}\mathfrak{A}_n$-module appears in $V_\lambda$ and $\lambda$ is defined uniquely up to taking the transpose.*

The moral reason why the transpose should play a role is that $V_\lambda \otimes \mathrm{sgn}_n \cong V_{\lambda^t}$, Exercise 5.12, and the sign representation restricts to the trivial representation of $\mathfrak{A}_n$.

The proof is based on understanding the induction from $\mathfrak{A}_n$ to $S_n$. Recall that $\mathfrak{A}_n$ is a normal subgroup of index 2 in $S_n$. Also recall, [1, Section 3.5], that the induction for representations of groups can be understood as follows: for a pair of finite groups $H \subset G$ and a representation $U$ of $H$, the induced representation $\mathrm{Ind}_H^G(U)$ is realized as

$$\mathsf{Fun}_H(G, U) := \{f : G \to U | f(hg) = hf(g), \forall h \in H, g \in G, \},$$

where the action of $G$ is given by $[g.f](g') = f(g'g^{-1})$.

Here is a key lemma that will be used to prove Theorem 6.6.

**Lemma 6.7.** *Suppose $H$ is normal in $G$. Let $V$ be a representation of $G$ viewed as a representation of $H$. Then we have*

$$\mathrm{Ind}_H^G(V) \xrightarrow{\sim} \mathrm{Ind}_H^G(\mathrm{triv}) \otimes V. \tag{6.5}$$

*Proof.* We have $\mathrm{Ind}_H^G(\mathrm{triv}) \otimes V \cong \{f : G \to V | f(hg) = f(g)\}$ with action of $G$ on the right hand side given by $[g.f](g') = gf(g'g^{-1})$. An isomorphism (6.5) is given by $f \mapsto \tilde{f}$ with $\tilde{f}(g') = g'^{-1}f(g')$. To check that this map is an isomorphism onto $\{f : G \to V | f(hg) = f(g)\}$ and is $G$-equivariant is left as an exercise. $\square$

*Proof of Theorem 6.6.* The proof is in several steps.

*Step 1.* We claim that $\operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(\mathrm{triv}) \cong \mathrm{triv}_n \oplus \mathrm{sgn}_n$. Indeed, $\operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(\mathrm{triv})$ is the module of functions on $S_n/\mathfrak{A}_n$. It has the basis $e_+, e_-$ of characteristic functions of $\mathfrak{A}_n$ and $S_n \setminus \mathfrak{A}_n$. For a permutation $\sigma$ we have $\sigma e_\pm = e_{\mathrm{sgn}(\sigma)\pm}$. So $e_+ + e_-$ spans the trivial representation, while $e_+ - e_-$ spans the sign representation.

*Step 2.* By Frobenius reciprocity, [1, Corollary 3.13], for two partitions $\lambda, \mu$ of $n$ we have

$$\operatorname{Hom}_{\mathfrak{A}_n}(V_\lambda, V_\mu) = \operatorname{Hom}_{S_n}(V_\lambda, \operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(V_\mu)). \tag{6.6}$$

Further, we get

$$\operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(V_\mu) \cong \operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(\mathrm{triv}_n) \otimes V_\mu \cong (\mathrm{triv}_n \oplus \mathrm{sgn}_n) \otimes V_\mu = V_\mu \oplus V_{\mu^t}.$$

Here the first isomorphism follows from Lemma 6.7, the second follows from Step 1, and the third, where we write $\mu^t$ for the transpose of $\mu$, follows from Exercise 5.12. We conclude that the left hand side of (6.6) is

- zero, if $\lambda \neq \mu, \mu^t$.
- 1-dimensional, if $\lambda = \mu$ or $\mu^t$, and $\mu \neq \mu^t$,
- 2-dimensional, if $\lambda = \mu = \mu^t$.

*Step 3.* Apply the conclusion of Step 2 to $\lambda = \mu$. If $\lambda \neq \lambda^t$, then $\operatorname{End}_{\mathbb{C}\mathfrak{A}_n}(V_\lambda)$ is 1-dimensional. For a completely reducible module, to have the 1-dimensional endomorphism space is equivalent to being irreducible. This shows (1). Similarly, if $\lambda = \lambda^t$, then $V_\lambda$ decomposes into the direct sum of two pairwise non-isomorphic $\mathbb{C}\mathfrak{A}_n$-modules. Any odd permutation sends an irreducible $\mathbb{C}\mathfrak{A}_n$-submodule of $V_\lambda$, say $U$, to an irreducible submodule, that must be different from $U$ because $U$ is not $S_n$-stable. This shows (2).

*Step 4.* Now we show (3). Similarly to Step 3, we see that for $\mu \neq \lambda, \lambda^t$, the $\mathbb{C}\mathfrak{A}_n$-modules $V_\lambda, V_\mu$ do not contain common irreducible summands. This establishes the uniqueness part of (3). It remains to show that every irreducible $\mathbb{C}\mathfrak{A}_n$-module $U$ arises in the construction of Step 3, this is the existence part. Consider the $\mathbb{C}S_n$-module $V := \operatorname{Ind}_{\mathfrak{A}_n}^{S_n}(U)$. By Frobenius reciprocity,

$$\operatorname{Hom}_{\mathfrak{A}_n}(V, U) \xrightarrow{\sim} \operatorname{Hom}_{S_n}(V, V).$$

The target is nonzero, hence so is the source. So $U$ must appear in some irreducible $\mathbb{C}S_n$-submodule of $V$, which finishes the proof. $\square$

6.3. **Induction.** In this section we will discuss the induction from $S_{n-1}$ to $S_n$, we write $\operatorname{Ind}_{n-1}^n$ for $\operatorname{Ind}_{S_{n-1}}^{S_n}$. Let $\mu$ be a partition of $n-1$ and $\lambda$ be a partition of $n$. Let $V_\mu, V_\lambda$ be the corresponding irreducible modules over $\mathbb{C}S_{n-1}$ and $\mathbb{C}S_n$. By the Frobenius reciprocity, [1, Corollary 3.13], we have

$$\operatorname{Hom}_{\mathbb{C}S_n}(\operatorname{Ind}_{n-1}^n V_\mu, V_\lambda) \cong \operatorname{Hom}_{\mathbb{C}S_{n-1}}(V_\lambda, V_\mu).$$

Here we use the realization of $\operatorname{Ind}_{n-1}^n V_\mu$ as $\mathbb{C}S_n \otimes_{\mathbb{C}S_{n-1}} V_\mu$ and the isomorphism from the left hand side to the right hand side is given by restricting to $V_\mu = \mathbb{C}S_{n-1} \otimes_{\mathbb{C}S_{n-1}} V_\mu \subset \mathbb{C}S_n \otimes_{\mathbb{C}S_{n-1}} V_\mu$. The modifications for the other realization of the induction (as the right adjoint of the restriction) are left as an exercise.

By Corollary 5.9, the right hand side is 1-dimensional if $\mu$ is obtained from $\lambda$ by removing a box, and is zero else. We arrive at the following result.

**Lemma 6.8.** *We have $\operatorname{Ind}_{n-1}^n V_\mu = \bigoplus_\lambda V_\lambda$, where the summation is over all Young diagrams obtained from $\mu$ by adding a box.*

For example, if $\mu = (2,2)$, then we have two summands in the right hand side, they correspond to the partitions $(3,2)$ and $(2,2,1)$.

It turns out that the element $J_n$ gives rise to an endomorphism of the $\mathbb{C}S_n$-module $\operatorname{Ind}_{n-1}^n U$ for any $\mathbb{C}S_{n-1}$-module $U$ so that, for $U = V_\mu$, the irreducible summands of $\operatorname{Ind}_{n-1}^n V_\mu$ are its eigenspaces.

Consider a more general situation. Let $H \subset G$ be finite groups and $Z := Z_{\mathbb{C}H}(\mathbb{C}G)$. Let $U$ be a $\mathbb{C}H$-module. Recall that, by the construction, $\operatorname{Ind}_H^G U = \mathbb{C}G \otimes_{\mathbb{C}H} U$. For $z \in Z$, define the operator $z^*$ on this space by

$$z^*(a \otimes u) = (az) \otimes u.$$

Note that since $z$ commutes with $\mathbb{C}H$, this operator is well-defined. Moreover, it commutes with the $G$-action because the action of $G$ because it is by left multiplications on the first tensor factor. We also remark that $(z_1 z_2)^* = z_2^* z_1^*$ for $z_1, z_2 \in Z$, so $\operatorname{Ind}_H^G U$ becomes a right $Z$-module.

We note that for any $\mathbb{C}H$-module $U$ and $\mathbb{C}G$-module $V$, the algebra $Z$ acts on $\operatorname{Hom}_H(U,V)$ as explained in Remark 2.5. It also acts on $\operatorname{Hom}_G(\operatorname{Ind}_H^G U, V)$: $z\varphi := \varphi \circ z^*$. Both $\operatorname{Hom}_H(U,V)$ and $\operatorname{Hom}_G(\operatorname{Ind}_H^G U, V)$ become left $Z$-modules.

**Lemma 6.9.** *The isomorphism* $\operatorname{Hom}_G(\operatorname{Ind}_H^G U, V) \to \operatorname{Hom}_H(U,V)$ *is* $Z$*-linear.*

*Proof.* The isomorphism is given by restricting $\varphi : \operatorname{Ind}_H^G U \to V$ to $U \hookrightarrow \mathbb{C}G \otimes_{\mathbb{C}H} U, u \mapsto 1 \otimes u$. We write $\varphi'$ for the restriction. Then

$$[z\varphi]'(u) = [z\varphi](1 \otimes u) = \varphi(z \otimes u) = z\varphi(1 \otimes u) = z(\varphi'(u)) = [z\varphi'](u),$$

where the third equality holds because $\varphi$ is $\mathbb{C}G$-linear. So $[z\varphi]' = z[\varphi']$, and we are done. □

Now take $G = S_n, H = S_{n-1}$. Take $z = J_n$.

**Corollary 6.10.** *Let $\lambda, \mu$ be two diagrams with $n$ and $n-1$ boxes respectively. Suppose $\mu$ is obtained from $\lambda$ by removing a box (so that, by Lemma 6.8, $V_\lambda$ is a direct summand of the $\mathbb{C}S_n$-module $\operatorname{Ind}_{n-1}^n V_\mu$). Then $J_n^*$ acts on $V_\lambda$ by a scalar and that scalar is the content of the box removed from $\lambda$ to get $\mu$.*

*Proof.* By Lemma 6.9, the isomorphism $\operatorname{Hom}_{S_{n-1}}(V_\mu, V_\lambda) \xrightarrow{\sim} \operatorname{Hom}_{S_n}(\operatorname{Ind}_{n-1}^n V_\mu, V_\lambda)$ intertwines the operators on these (1-dimensional) spaces induced by $J_n$. The action of $J_n$ on $\operatorname{Hom}_{S_{n-1}}(V_\mu, V_\lambda)$ is by the same scalar as the action of $J_n$ on the copy of $V_\mu$ inside $V_\lambda$. Since $J_n^*$ is an endomorphism of the $\mathbb{C}S_n$-module $\operatorname{Ind}_{n-1}^n V_\mu$, and every irreducible there occurs with multiplicity 1, the operator $J_n^*$ acts on every irreducible summand, including $V_\lambda$, by a scalar – compare to the proof of (3) of Corollary 2.10. This scalar is the same as the scalar by which $J_n$ acts on $\operatorname{Hom}_{S_n}(\operatorname{Ind}_{n-1}^n V_\mu, V_\lambda)$. This finishes the proof. □

**Remark 6.11.** It is very useful to revisit the construction of the present section from a more categorical perspective. We have used a natural isomorphism

$$\operatorname{Hom}_{\mathbb{C}S_n}(\operatorname{Ind}_{n-1}^n U, V) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}S_{n-1}}(U,V), \tag{6.7}$$

i.e., that the induction functor $\operatorname{Ind}_{n-1}^n : \mathbb{C}S_{n-1}\text{-mod} \to \mathbb{C}S_n\text{-mod}$ is left adjoint to the restriction functor $\mathbb{C}S_n\text{-mod} \to \mathbb{C}S_{n-1}\text{-mod}$, to be denoted by $\operatorname{Res}_n^{n-1}$. Both functors are $\mathbb{C}$-linear (meaning that they respect the $\mathbb{C}$-vector space structure on the Hom sets) and the adjunction is $\mathbb{C}$-linear as well meaning that (6.7) is an isomorphism of vector space rather than just a bijection of sets. The endomorphisms of a $\mathbb{C}$-linear functor naturally form an (associative, unital) $\mathbb{C}$-algebra. And the adjunction induces an isomorphism between the endomorphism algebras, where we need to take one of them with the opposite multiplication, e.g.,

$$\operatorname{End}(\operatorname{Ind}_{n-1}^n)^{opp} \xrightarrow{\sim} \operatorname{End}(\operatorname{Res}_n^{n-1}). \tag{6.8}$$

We have algebra homomorphisms

$$Z_{n-1}(n) \to \operatorname{End}(\operatorname{Ind}_{n-1}^n)^{opp}, \operatorname{End}(\operatorname{Res}_n^{n-1})$$

that intertwine (6.8) – for the readers who understand how (6.8) is constructed, this is a very useful exercise.

In particular, the element $J_n \in Z_{n-1}(n)$ gives functor endomorphisms of both $\operatorname{Ind}_{n-1}^n, \operatorname{Res}_n^{n-1}$. Both these functors map to categories, where objects are finite dimensional vector spaces over $\mathbb{C}$. Once we equip such a vector space with a linear operator, we can decompose it into the direct sum of generalized eigenspaces. So, say, for a $\mathbb{C}S_{n-1}$-module, we can decompose $\operatorname{Ind}_{n-1}^n M$ as $\bigoplus_{a\in\mathbb{C}}(\operatorname{Ind}_{n-1}^n M)_a$, where the summands are the generalized eigenspaces for the operator $J_n^*$. All eigenvalues are integers by Corollary 6.10.

Here are two claims that the readers are strongly encouraged to check, both follow from the observation that $J_n^*$ gives a functor endomorphism:

- all $(\operatorname{Ind}_{n-1}^n M)_a$ are $\mathbb{C}S_n$-submodules (zero if $a$ is not an integer);
- for all $a \in \mathbb{C}$, the assignment $M \mapsto (\operatorname{Ind}_{n-1}^n M)_a$ is a part of a functor $\mathbb{C}S_{n-1}\text{-mod} \to \mathbb{C}S_n\text{-mod}$.

So we can say that the functor $\operatorname{Ind}_{n-1}^n$ decomposes as the direct sum of functors $(\operatorname{Ind}_{n-1}^n \bullet)_a$. The same is true for the restriction functor: $\operatorname{Res}_n^{n-1} = \bigoplus_{a\in\mathbb{Z}}(\operatorname{Res}_n^{n-1})_a$. Lemma 6.9 implies that $(\operatorname{Ind}_{n-1}^n \bullet)_a$ is still the left adjoint functor of $(\operatorname{Res}_n^{n-1} \bullet)_a$.

**Exercise 6.12.** Each of the functors $(\operatorname{Res}_n^{n-1} \bullet)_a$ and $(\operatorname{Ind}_{n-1}^n \bullet)_a$ sends an irreducible module to an irreducible module or 0.

6.4. **Characteristic $p$.** Now we proceed to a brief discussion of the representation theory of $\mathbb{F}S_n$, where $\mathbb{F}$ is an algebraically closed field of characteristic $p$ (in fact, one doesn't need to impose the condition of being algebraically closed). This is usually referred to as the modular (from "modulo $p$") Representation theory.

We can still consider $J_n \in \mathbb{F}S_n$.

**Lemma 6.13.** *The element $J_n$ acts on any $\mathbb{F}S_n$-module with eigenvalues in $\mathbb{F}_p$.*

*Proof.* Consider $J_n \in \mathbb{Z}S_n$. We claim that there is $F \in \mathbb{Z}[x]$ that is the product of factors of the form $(x-m)$ for $m \in \mathbb{Z}$ such that $F(J_n) = 0$. Indeed, $J_n$ acts on every $\mathbb{C}S_n$-module with integral eigenvalues, this follows, for example, from Corollary 5.9. Apply this to the regular module $\mathbb{C}S_n$ and take $F$ to be the minimal polynomial of the operator of left multiplication by $J_n$. We get $F(J_n) = 0$ in $\mathbb{C}S_n$, and hence in $\mathbb{Z}S_n$.

Of course, $F(J_n) = 0$ in $\mathbb{F}S_n$ as well. The possible eigenvalues of $J_n \in \mathbb{F}S_n$ are the reductions mod $p$ of the roots of $F$. Those are integers. This finishes the proof. $\square$

**Example 6.14.** Unlike in characteristic 0, $J_n$ may fail to act by a diagonalizable operator even on an irreducible module. For example, let $p = 2$ and $V$ be the reflection representation of $\mathbb{F}S_3$, still defined as the subspace $\{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\} \subset \mathbb{F}^3$. It is irreducible. The operator $J_3 = (1,3) + (2,3)$ sends $(x_1, x_2, x_3)$ to $(x_3 + x_1, x_2 + x_3, x_2 + x_1) = (x_2, x_1, x_3)$. It has a unique (up to rescaling) eigenvector $(1,1,0)$ (with eigenvalue 1).

For the same reason as in Remark 6.11, we have the functor decompositions

$$\mathrm{Res}_n^{n-1} = \bigoplus_{a\in\mathbb{F}_p}(\mathrm{Res}_n^{n-1}\bullet)_a, \mathrm{Ind}_{n-1}^n = \bigoplus_{a\in\mathbb{F}_p}(\mathrm{Ind}_{n-1}^n\bullet)_a.$$

For example, for an $\mathbb{F}S_n$-module $M$, the $\mathbb{F}S_{n-1}$-module $\mathrm{Res}_n^{n-1} M)_a$ is the generalized eigenspace for $J_n$ with eigenvalue $a$ in $M$.

Unlike in Exercise 6.12, even if $M$ is irreducible, $(\mathrm{Res}_n^{n-1} M)_a$ may be reducible. We see this already in Example 6.14: there is a unique irreducible $\mathbb{F}S_2$-module, and it is 1-dimensional, while $(\mathrm{Res}_n^{n-1} M)_1 = \mathrm{Res}_n^{n-1} M$ is 2-dimensional. Nevertheless, there is the following theorem whose proof uses the representation theory of degenerate affine Hecke algebras. It is a special case of [Kl, Corollary 5.1.7, 5.3.2].

**Theorem 6.15.** *Recall that $\mathbb{F}$ is an algebraically closed field of characteristic $p$. Let $V$ be an irreducible $\mathbb{F}S_n$-module. Pick $a \in \mathbb{F}_p$ such that $(\mathrm{Res}_n^{n-1} M)_a$ is nonzero. Then this module has the unique irreducible submodule, the unique irreducible quotient module, and these two irreducible modules are isomorphic.*

We will return to the discussion of the modular representation theory of $S_n$, which is an active area of research, later in the course.

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