REPRESENTATIONS OF ALGEBRAIC GROUPS AND LIE ALGEBRAS

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1. Plan

We proceed to the central part of the course – representations of algebraic groups and Lie algebras. The plan is as follows:

(1) Basics on algebraic groups and Lie algebras.
(2) The representation theory of $\mathfrak{sl}_2(\mathbb{F})$ and $\mathrm{SL}_2(\mathbb{F})$, for fields of zero and positive characteristic.
(3) The representation theory of $\mathfrak{sl}_n(\mathbb{C})$ including the center of the universal enveloping algebra. A brief discussion of the Schur-Weyl duality.
(4) Generalizations: semisimple Lie algebras, Kac-Moody Lie algebras.
(5) The Weyl character formula. Category $\mathcal{O}$, blocks and simple objects. Characters.

Then we will digress to a discussion of the representation theory of finite groups of Lie type to introduce the Hecke algebra that is needed to proceed with the character formulas for the simple objects in the category $\mathcal{O}$.

2. Brief overview of the literature

The list of references contains 10 texts dealing with various aspects of the structure and the representation theory of Lie algebras and algebraic groups. Here is a brief description.

[B]: Nicholas Bourbaki wrote a lot of books on the foundations of various areas of Mathematics. Lie groups and Lie algebras are widely regarded as the masterpiece. I'd personally say that, the "algebraic parts", the Lie algebras and reflection groups (Ch. 1, 4-6, 7-8) are the best. One highlight is an excellent selection of exercises.

[E]: A very recent text.

[G]: For our purposes, the text contains a good account of the BGG category $\mathcal{O}$. It's fairly advanced.

[J]: An advanced text on the representation theory of reductive algebraic groups in positive characteristic.

[H1]: "Bourbaki light", this textbook covers Lie algebras.

[H2]: A textbook on the structure and representation theory of algebraic groups (over algebraically closed fields) in arbitrary characteristic.

[H3]: A detailed account of the BGG category $\mathcal{O}$ and related aspects of the representation theory of semisimple Lie algebras. Fairly advanced.

[K]: A fairly concise textbook covering both Lie groups and Lie algebras. It's expected to be the main reference for Lie groups and Representation theory intermediate graduate course.

[OV]: This book treats Lie groups, algebraic groups, and Lie algebras with emphasis on algebraic groups, an approach opposite, in a sense to [B] or [H1]. It only considers algebraic groups over $\mathbb{C}$ or $\mathbb{R}$ and has a lot of great problems. In fact, the exposition is based on solving problems in the text.

[S]: Another textbook on the structure of algebraic groups in arbitrary characteristic now treating the case of non-closed fields in detail.

References

[B] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 1-3 (1998), 4-6 (2002), 7-9 (2005). Elements of Mathematics (Berlin). Springer-Verlag.

[E] P. Etingof, *Lie groups and Lie algebras*. arXiv:2201.09397.

[G] D. Gaitsgory, *Geometric representation theory*. Available online.

[J] J.C. Jantzen, *Representations of algebraic groups*. Academic press, 1987.

[H1] J. Humphreys, *Introduction to Lie algebras and representation theory. Second printing, revised*. Graduate Texts in Mathematics, 9. Springer-Verlag, 1978.

[H2] J. Humphreys, *Linear algebraic groups*. GTM 21, New York-Berlin, 1995.

[H3] J. Humphreys, *Representations of semisimple Lie algebras in the BGG categories $\mathcal{O}$*. AMS, GSM 94, 2008.

[K] A. Kirillov, Jr. *An introduction to Lie groups and Lie algebras*. Cambridge Studies in Advanced Mathematics, 113. Cambridge University Press, Cambridge, 2008. Available online.

[OV] A. Onishchik, E. Vinberg, *Lie groups and algebraic groups*. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990.

[S] T.A. Springer, *Linear algebraic groups*. Birkhauser, 1998.