MATH 603, PROBLEM SET 1, DUE FEB 17

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There are six problems worth 25 points total. You need to score 15 points to get the maximal score. You can use previous problems (or previous parts) in your solutions of the subsequent problems (or subsequent parts of the same problem) and get full credit even if you haven't solved the problems/parts you have used. Partial credit is given. The italized text serves as comments to a problem, but it is not a part of the problem.

The solutions need to be submitted via Canvas. Hand-written solutions are accepted but please make sure they are readable.

Problem 1, 3pts. Use Remark 2.12 in [RT1] (the claim that the algebra $Z_{n-1}(n)$ is generated by $Z_n(n)$ and J_n) as well as other results from Section 2 to prove the following claim. Let V be an irreducible $\mathbb{C}S_n$-module. Every eigenspace for J_n in V is an irreducible $\mathbb{C}S_{n-1}$-module (*where by an eigenspace we mean the span of all eigenvectors with a given eigenvalue*), equivalently, J_n acts by different scalars on different irreducible $\mathbb{C}S_{n-1}$-submodules. *Hint: do revisit the proof of Theorem 3.6; note that the result of the problem follows from the final classification, but you are not supposed to use that.*

The remaining problems in this problem set concern various aspects of the structure and the representation theory of the degenerate affine Hecke algebras.

Problem 2, 4pts total. *The goal of this problem is to construct a suitable representation of $\mathcal{H}(2)$ (and also of $\mathcal{H}(d)$). We start with constructing a representation of $\mathcal{H}(2)$ in $\mathbb{C}[x_1, x_2]$.*

a, 3pts) Consider the representation of S_2 in $\mathbb{C}[x_1, x_2]$, where the non-unit element $s \in S_2$ acts by $[s.f](x_1, x_2) = f(x_2, x_1)$. Show that the assignment

$$X_1.f := x_1 f, X_2.f := x_2 f, T.f := sf + \frac{sf - f}{x_1 - x_2}$$

extends to a representation of $\mathcal{H}(2)$ in the space $\mathbb{C}[x_1, x_2]$.

b, 1pt) Generalize this construction and construct a representation of $\mathcal{H}(d)$ in $\mathbb{C}[x_1, \ldots, x_d]$. *Yes, you should check the details. You can use Remark 4.4 in the lecture.*

Problem 3, 4pts total. *The goal of this problem is to establish a vector space basis in $\mathcal{H}(2)$ – and in more general $\mathcal{H}(d)$.*

a, 3pts) Show that the monomials $X_1^{d_1} X_2^{d_2} \sigma$ with $d_1, d_2 \geqslant 0, \sigma \in S_2$, form a basis in $\mathcal{H}(2)$. *Hint: to prove the linear independence show that the images of these monomials in* $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[x_1, x_2])$ *are linearly independent. This is one of many examples how looking at representations allows to determine vector space bases for algebras given by generators and relations.*

b, 1pt) *This is harder.* State and prove the direct analog of a) for $\mathcal{H}(d)$.

Problem 4, 5pts. *Now we construct a central subalgebra of $\mathcal{H}(2)$ – and of the general $\mathcal{H}(d)$. Notice that we have a natural algebra homomorphism $\mathbb{C}[X_1, X_2] \to \mathcal{H}(2)$, and Problem 3 implies it is injective.*

a, 3pts) Show that the symmetric polynomials in X_1, X_2 are central in $\mathcal{H}(2)$.

b, 1pt) State and prove a direct analog of a) for $\mathcal{H}(d)$.

Hint: it's not so hard to prove a) computationally with some basic knowledge about symmetric polynomials, while for b) this won't really work. What will work is again considering the representation in $\mathbb{C}[x_1, \ldots, x_d]$.

c, 1pt) Use b) to show that the center of $\mathbb{C}S_d$ is spanned as a vector space by the symmetric polynomials in $J_1, \ldots, J_d$. *Hint: use a homomorphism $\mathcal{H}(d) \to \mathbb{C}S_d$.*

Problem 5, 6pts total. *The goal of this problem is to establish some basic results about the representations of $\mathcal{H}(d)$. We'll need the following well-known fact that you don't need to prove. Let $\mathbb{C}[x_1, \ldots, x_d]_{sym}$ denote the subalgebra of symmetric polynomials inside $\mathbb{C}[x_1, \ldots, x_d]$. Then $\mathbb{C}[x_1, \ldots, x_d]$ is a free $\mathbb{C}[x_1, \ldots, x_d]_{sym}$-module of rank $d!$. We'll discuss possible bases in an "Aside" to this problem set, after Problem 6.*

a, 1pt) Prove that all irreducible representations of $\mathcal{H}(d)$ are finite dimensional. *You can use Proposition 2.10 in [RT0].*

b, 1pt) Fix an element $\alpha := (a_1, \ldots, a_d) \in \mathbb{C}^d$. Further, consider the maximal ideal $\mathfrak{m}_\alpha \subset \mathbb{C}[x_1, \ldots, x_d]_{sym}$ consisting of all symmetric polynomials vanishing at α. Finally, set $\mathcal{H}(d)_\alpha := \mathcal{H}(d)/\mathcal{H}(d)\mathfrak{m}_\alpha$. Prove that $\mathcal{H}(d)\mathfrak{m}_\alpha$ is a 2-sided ideal in $\mathcal{H}(d)$ and, moreover, $\dim \mathcal{H}(d)_\alpha = (d!)^2$. Further, show that all irreducible representations of $\mathcal{H}(d)$ have dimension at most $d!$.

In fact, most irreducible representations have dimension exactly $d!$. Here are some partial results in this direction.

c, 2pts) Consider the $\mathcal{H}(d)$-module

$$M(\alpha) := \mathcal{H}(d) \otimes_{\mathbb{C}[X_1,\ldots,X_n]} \mathbb{C},$$

where $\mathbb{C}$ is the $\mathbb{C}[X_1, \ldots, X_d]$-module with X_i acting by multiplication with a_i. Assume that $a_i - a_j \notin \{0, \pm 1\}$ for all $i \neq j$. Show that the module $M(\alpha)$ has the following properties:

1. The elements $X_1, \ldots, X_d$ act on $M(\alpha)$ by diagonalizable operators.
2. Each simultaneous eigenspace is 1-dimensional, and the d-tuples of eigenvalues are exactly the permutations of α.
3. $M(\alpha)$ is irreducible.

Hint: you are allowed to use a variation of b) in Problem 3. You could also try to prove it by constructing a suitable anti-automorphism of $\mathcal{H}(d)$.

d, 1pt) Show that under the assumptions of c), $\mathcal{H}(d)_\alpha$ is the matrix algebra of size $d!$.

Here is a related – and deeper – fact: if $a_1 = \ldots = a_d$, then $M(\alpha)$ is irreducible – and $\mathcal{H}(d)_\alpha$ is the matrix algebra of size $d!$. This holds over any algebraically closed field. This result is very important, say, in the study of modular representations of symmetric groups.

e, 1pt) Show that the center of $\mathcal{H}(d)$ coincides with the subalgebra $\mathbb{C}[X_1, \ldots, X_d]_{sym}$.

Problem 6, 3pts total. *The goal of this (harder) problem is to justify Remark 4.3 in [RT1]. We will concentrate on the case when $d = 2$, although the general case is completely analogous. We note that for any f in the center $\mathbb{C}[X_1, \ldots, X_d]_{sym}$ of $\mathcal{H}(d)$, the localization $\mathcal{H}(d)[f^{-1}]$ of the $\mathbb{C}[X_1, \ldots, X_d]_{sym}$-module $\mathcal{H}(d)$ has the natural algebra structure. You do not need to prove this.*

a, 1pt) Let $f = (X_1 - X_2)^2(X_1 - X_2 - 1)(X_2 - X_1 - 1) \in \mathbb{C}[X_1, X_2]_{sym}$. Show that $\mathcal{H}(2)[f^{-1}]$ is isomorphic to $\mathrm{Mat}_2(\mathbb{C}[X_1, X_2]_{sym}[f^{-1}])$ as a $\mathbb{C}[X_1, X_2]_{sym}[f^{-1}]$-algebra.

b, 1pt) Use a) to show that every two-sided ideal in $\mathcal{H}(2)$ has a nonzero intersection with the subalgebra $\mathbb{C}[X_1, X_2]_{sym}$.

c, 1pt) Use b) to show that the intersection of the kernels of the homomorphisms $\mathcal{H}(2) \to Z_{i-1}(i+1)$ from Section 4.1 in [RT1] (for all $i \geqslant 2$) is zero. *In this sense the relations of the algebra $\mathcal{H}(2)$ are exhaust the defining relations between $J_{i-1}, J_i, (i-1, i)$ that are independent of i.*

Aside. Here we discuss the following result:

$\mathbb{C}[x_1, \ldots, x_d]$ is a free rank $d!$ module over $\mathbb{C}[x_1, \ldots, x_d]_{sym}$, the subalgebra of symmetric polynomials.

Here is an easy basis: $x_1^{n_1} \ldots x_{d-1}^{n_{d-1}}$ with $0 \leqslant n_i \leqslant d - i$. You could try to prove this is indeed a basis – this is an elementary combinatorial problem.

There is also another, "better" and "more intelligent" choice of a basis: of S_d*-harmonic* polynomials. Note that S_d acts on $\mathbb{C}[x_1, \ldots, x_d]$ by permuting the variables, $\mathbb{C}[x_1, \ldots, x_d]$ is nothing else but the subalgebra of all invariant elements, hence the more common notation, $\mathbb{C}[x_1, \ldots, x_d]^{S_d}$. One can ask to find an S_d-submodule $V \subset \mathbb{C}[x_1, \ldots, x_d]$ such that the multiplication map gives rise to an isomorphism

$$\mathbb{C}[x_1, \ldots, x_d]^{S_d} \otimes V \xrightarrow{\sim} \mathbb{C}[x_1, \ldots, x_d]. \tag{1}$$

Note that in this case we automatically have $\dim V = d!$.

We say that an element $f \in \mathbb{C}[x_1, \ldots, x_d]$ is S_n*-harmonic* if $g(\partial_1, \ldots, \partial_d)f = 0$ for all symmetric polynomials g with $g(0) = 0$, where $\partial_1, \ldots, \partial_d$ denote the partials. For example, for $d = 2$, the polynomials $1, x_1 - x_2$ form a basis in the space of all S_d-harmonic polynomials. And $V := \mathrm{Span}_{\mathbb{C}}(1, x_1 - x_2)$. A deeper general result says that the space V of all S_d-harmonic polynomials satisfies (1).