MATH 603, PROBLEM SET 2, DUE MAR 15

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There are five problems worth 25 points total. You need to score 15 points to get the maximal score. You can use previous problems (or previous parts) in your solutions of the subsequent problems (or subsequent parts of the same problem) and get full credit even if you haven't solved the problems/parts you have used. Partial credit is given. The italized text serves as comments to a problem, but it is not a part of the problem.

The solutions need to be submitted via Canvas. Hand-written solutions are accepted but please make sure they are readable.

Problem 1, 4pts total. *The purpose of this problem is to prove the PBW theorem for* $\mathfrak{sl}_2$ *over a characteristic* 0 *field.*

a, 3pts) Let $\operatorname{char}\mathbb{F} = 0$. Prove that the monomials $f^k h^\ell e^m$ are linearly independent by checking that their nontrivial linear combination acts by a nonzero operator on the Verma module $\Delta(\lambda)$ for some $\lambda \in \mathbb{F}$.

b, 1pt) Explain why this fails in characteristic $p > 0$.

Problem 2, 5pts total. *This problem concerns the classical harmonic polynomials and shows the ubiquity of* $\mathfrak{sl}_2$ *in Math!* Consider the Laplace operator $\Delta := \sum_{i=1}^n \partial_i^2$, the Euler operator $\mathsf{eu} = \sum_{i=1}^n x_i\partial_i$, and the operator F of multiplication by $-\frac{1}{4}\sum_{i=1}^n x_i^2$ acting on the space $\mathbb{C}[x_1, \ldots, x_n]$.

a, 1pt) Show that the assignment $e \mapsto \Delta, f \mapsto F, h \mapsto -\mathsf{eu} - \frac{n}{2}$ defines a representation of $\mathfrak{sl}_2$ in $\mathbb{C}[x_1, \ldots, x_n]$.

b, 2pts) Set $\Lambda_n := -\frac{n}{2} - \mathbb{Z}_{\geqslant 0}$. Consider an $\mathfrak{sl}_2$-module M, where h acts diagonalizably with finite dimensional eigenspaces and eigenvalues in Λ_n. Show that M is isomorphic to the direct sum of (irreducible) Verma modules $\Delta(\lambda)$ with $\lambda \in \Lambda_n$.

c, 1pt) Deduce that every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ can be uniquely written as $\sum_{k=0}^\infty F^k h_k$, where h_k is a *harmonic* polynomial, i.e., a polynomial killed by Δ.

d, 1pt) Prove that the projection $\mathbb{R}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \ldots + x_n^2 - 1)$ restricts to an isomorphism between the subspace of all harmonic polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ and $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \ldots + x_n^2 - 1)$. *This is an algebraic counterpart of the theorem in PDE that a solution to the Laplace equation in a domain with fixed boundary condition exists and is unique; the domain of interest is a ball.*

In the three problems below we use the following notation: $G = \mathrm{SL}_2(\mathbb{F}), \mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, *where* $\operatorname{char}\mathbb{F} = p > 2$, *and, as usual,* $\mathbb{F}$ *is algebraically closed. Generally, a fairly careful reading of Lectures 8-11, and occasionally, previous lectures is a prerequisite.*

Problem 3, 5pts total. *As we have seen in Lecture 9, the representations of* $\mathfrak{g}$ *in characteristic* p *are not completely reducible, in general. Here are some sufficient conditions to get completely reducible representations. Note that the Casimir element is still central in* $U(\mathfrak{g})$.
Let M be a finite dimensional representation of $\mathfrak{g}$.

a, 2pts) Suppose that there is a nonzero element $a \in \mathbb{F}$ such that $e^p, h^p - h - a, f^p$ act by zero on M. Show that M is completely reducible. *Hint: look at the algebra U^χ from the complement section in Lecture 9.*

b, 3pts) Suppose $e^p, h^p - h, f^p$ act by 0 on M, while C acts on M with a single eigenvalue $-1/2$. Show that M is the direct sum of several copies of the irreducible representation $M(p-1) = \underline{\Delta}^0(p-1)$.

Problem 4, 6pts total. *One can show that the Casimir element C is not only central, but also G-invariant for the adjoint action of G on $U(\mathfrak{g})$. This will be verified later in the characteristic 0 setting, but the proof will carry over to characteristic p as long as $p > 2$. In this problem we will investigate the decomposition of the category of rational representations of G into "infinitesimal blocks" and also the "singular block". These constructions are important throughout the representation theory of semisimple algebraic groups and their Lie algebras.*

Let M be a rational representation of G. Show that

1, 1pt) The elements $x^p - x^{[p]}, x \in \mathfrak{g}$, act by 0 on M (*nothing specific about $\mathfrak{sl}_2$ here*).

2, 1pt) Each generalized eigenspace for C in M is a G-subrepresentation. Moreover, any homomorphisms of rational G-representations preserves the decomposition into the direct sum of generalized eigenspaces.

We will study the representations M where the only eigenvalue for C is $-\frac{1}{2}$. For this we will need the following claim:

3, 1pt) Let N, M be arbitrary rational representations of G so that $\operatorname{Hom}_{\mathbb{F}}(N, M) = N^* \otimes M$ is also a rational representation. Show that $\operatorname{Hom}_{\mathfrak{g}}(N, M)$ is a subrepresentation of $\operatorname{Hom}_{\mathbb{F}}(N, M)$, and the action of $\mathfrak{g}$ on $\operatorname{Hom}_{\mathfrak{g}}(N, M)$ induced by the G-action is trivial (*nothing specific about $\mathfrak{sl}_2$ here as well*).

4, 2pts) Suppose now that M satisfies the property that the only eigenvalue of C on M is $-\frac{1}{2}$. Construct a natural morphism

$$M(p-1) \otimes \operatorname{Hom}_{\mathfrak{g}}(M(p-1), M) \xrightarrow{\sim} M$$

and show it is an isomorphism.

5, 1pt) Let M' be a representation of G, where the induced action of $\mathfrak{g}$ is zero. Construct a natural morphism $M' \mapsto \operatorname{Hom}_{\mathfrak{g}}(M(p-1), M(p-1) \otimes M')$ and show that it is an isomorphism.

Here is the significance of parts 4 and 5. Consider two categories of rational representations of G. First, $\operatorname{Rep}_{sing}(G)$, the "singular block" consisting of all rational representations, where C acts with eigenvalue $-\frac{1}{2}$. Second, $\operatorname{Rep}(G/\mathfrak{g})$ of all rational representations of G, where $\mathfrak{g}$ acts by 0. Parts 4 and 5 show that these categories are equivalent.

In fact, $\operatorname{Rep}(G/\mathfrak{g})$ is equivalent to $\operatorname{Rep}(G)$, again. In more detail, every representation in $\operatorname{Rep}(G/\mathfrak{g})$ is the pullback of a rational representation under $\operatorname{Fr} : G \to G$. It should be clear that $\mathfrak{g}$ acts by 0 on the pullback under Frobenius. The opposite implication is more subtle and is expected to be addressed in one of the bonus lectures. So we have an equivalence $\operatorname{Rep}(G) \xrightarrow{\sim} \operatorname{Rep}(G/\mathfrak{g})$, where the source is the category of all rational representations, given by Fr^*.

So $\operatorname{Rep}(G)$ is realized as a direct summand of itself...

Problem 5, 5 points total. *In this problem we will examine two classes of representations of G: the dual Weyl modules $M(n)$, and their duals, Weyl modules, $W(n) := M(n)^*$. Namely, we will examine homomorphisms and extensions between such modules.*

1, 1pt) Show that $\mathrm{Hom}_G(M(k), M(\ell))$ is zero for $k < \ell$ and is 1-dimensional for $k = \ell$. State and prove an analog of this result for the Weyl modules $W(n)$.

2, 1pt) Show that every SES $0 \to M(\ell) \to V \to M(k) \to 0$ of rational representations of G splits if $k \leqslant \ell$. State and prove an analogous result for the Weyl modules.

3, 1pt) Show that $\dim \mathrm{Hom}_G(W(k), M(\ell)) = \delta_{k,\ell}$.

4, 2pt) Show that every SES $0 \to M(k) \to V \to W(\ell) \to 0$ of rational representations of G splits.

One can give an abstract definition of a category, where "highest weight theory works". We get so called "highest weight categories". In such categories we have a bunch of families of objects, including "standard" and "costandard" objects. It turns out that in the category of rational representations of $\mathrm{SL}_2(\mathbb{F})$ *these objects are the Weyl and dual Weyl modules. The properties in the problem are general properties of standard and costandard objects.*