MATH 603, PROBLEM SET 4, DUE APR 28

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There are four problems worth 25 points total. You need to score 15 points to get the maximal score. You can use previous problems (or previous parts) in your solutions of the subsequent problems (or subsequent parts of the same problem) and get full credit even if you haven't solved the problems/parts you have used. Partial credit is given. The italized text serves as comments to a problem, but it is not a part of the problem.

The solutions need to be submitted via Canvas. Hand-written solutions are accepted but please make sure they are readable.

Problem 1, 6pts. *This problem provides examples of dual Weyl modules.* Let $\mathbb{F}$ denote an algebraically closed field of characteristic p. Let $n > 1$ and $G := \mathrm{SL}_n(\mathbb{F})$. We write $\omega_1, \ldots, \omega_{n-1}$ for the fundamental weights (*Section 1 of Lecture 13*). *In your solutions you are allowed to use all facts mentioned in the main body of the lectures (excluding the complement sections). And it's not confined to Lecture 17!*

1, 3pts) Show that, for $k = 1, \ldots, n-1$, we have $M(\omega_k) \cong \Lambda^k \mathbb{F}^n$ (constructed as the quotient of $(\mathbb{F}^n)^{\otimes k}$) and this module is irreducible.

2, 3pts) *This is a harder part.* Show that, for $d > 0$, we have $M(d\omega_1) \cong \mathbb{F}[x_1, \ldots, x_n]_d$, the space of homogeneous degree d polynomials, where G acts by linear changes of variables. *Hint: let P denote the subgroup of all matrices $(x_{ij}) \in G$ with $x_{n,1} = x_{n,2} = \ldots = x_{n,n-1} = 0$. Observe that $\pi_{dw_0\omega_1}$ is restricted from an algebraic group homomorphism $P \to \mathbb{F}^\times$ that we also denote by $\pi_{dw_0\omega_1}$. Show that $M(d\omega_1) \cong \mathrm{Ind}_P^G \pi_{dw_0\omega_1}$ – by stating and proving a transitivity property for induction in this setting– and use this to show that $M(d\omega_1) \cong \mathbb{F}[x_1, \ldots, x_n]_d$.*

Problem 2, 7pts. Let $\mathbb{F}_q$ be a finite field, $G = \mathrm{GL}_n(\mathbb{F}_q)$, B the subgroup of all upper triangular matrices, and T the subgroup of all diagonal matrices. We have the projection $B \twoheadrightarrow T$ and so can view every T-representation as a B-representation. *In Lecture 18 we have studied the structure of $\mathrm{Ind}_B^G \mathrm{triv}$. In this problem our task is to understand the inductions from general irreducible representations of T (viewed as representations of B).* We write $H_n(q)$ for the Hecke algebra $(\mathbb{C}[B\backslash G/B], *)$ with $G = \mathrm{GL}_n(\mathbb{F}_q)$. The set of irreducible representations $\mathrm{Hom}(T, \mathbb{C}^\times)$ of T is identified with $\mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)^n$ (where Hom means the set of group homomorphisms) with $\chi = (\chi_1, \ldots, \chi_n)$ sending $t = \mathrm{diag}(t_1, \ldots, t_n)$ to $\chi_1(t_1) \ldots \chi_n(t_n)$. So $W = S_n$ acts on this set of irreducible T-representations by permuting the components.

1, 1pt) Show that $\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi')$ has basis labelled by elements $w \in W$ such that $w\chi = \chi'$.

2, 1pt) Deduce that if the entries of χ are pairwise distinct, then $\mathrm{Ind}_B^G \chi$ is irreducible.

3, 1pt) Deduce that if χ' is not conjugate to χ under the W-action, then $\mathrm{Ind}_B^G \chi$ and $\mathrm{Ind}_B^G \chi'$ have no common irreducible summands.

4, 2pts) Prove that if $\chi' \in W\chi$, then $\mathrm{Ind}_B^G \chi \cong \mathrm{Ind}_B^G \chi'$. *Hint: handle the case of $n = 2$ first. Then prove the isomorphism for $\chi' = s\chi$ for $s = (i, i+1)$ using the transitivity of induction with intermediate subgroup P_s, see Section 3 of Lecture 18. Conclude the proof.*

5, 2pts) Suppose that χ has n_1 entries χ_1, n_2 entries χ_2, ..., n_k entries χ_k with $n_1 + \ldots + n_k = n$ and pairwise distinct $\chi_1, \ldots, \chi_k \in \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$. Identify the algebra $\mathrm{End}_G(\mathrm{Ind}_B^G \chi)$ with $\bigotimes_{i=1}^k H_{n_i}(q)$.

If we believe in a bijection between irreducible representations and conjugacy classes – and we shouldn't really – then this problem yields a classification of all G-irreps corresponding to conjugacy classes with eigenvalues in $\mathbb{F}_q^\times$ (and not in a field extension).

Problem 3, 6pts. *In this problem we investigate the Okounkov-Vershik story for the Hecke algebra of S_n.* Consider the Hecke algebra $\mathcal{H}_{t^{\pm 1}}(S_n) := \mathcal{H}_t(S_n)[t^{-1}]$. Note that we have the chain of embeddings $\mathcal{H}_{t^{\pm 1}}(S_1) \subset \mathcal{H}_{t^{\pm 1}}(S_2) \subset \ldots \subset \mathcal{H}_{t^{\pm 1}}(S_n)$. *We start with an analog of the JM element.*

1, 1pt) Consider the element $J_n(t) := T_{n-1}T_{n-2}\ldots T_2T_1^2T_2\ldots T_{n-1} \in \mathcal{H}_{t^{\pm 1}}(S_n)$. Show that it commutes with $\mathcal{H}_{t^{\pm 1}}(S_{n-1})$.

2, 1pt) Show that $J_n(t) - 1 \in (t-1)\mathcal{H}_{t^{\pm 1}}(S_n)$. Compute the image of $(J_n(t)-1)/(t-1)$ in $\mathcal{H}_{t^{\pm 1}}(S_n)/(t-1)\mathcal{H}_{t^{\pm 1}}(S_n) = \mathbb{Z}S_n$.

Now we proceed to the (actual) affine Hecke algebra $\mathcal{H}^{aff}_{t^{\pm 1}}(n)$. Consider the $\mathbb{Z}[t^{\pm 1}]$-algebra $\mathcal{H}^{aff}_{t^{\pm 1}}(n)$ generated by the elements $X_1^{\pm 1}, \ldots, X_n^{\pm 1}, T_1, \ldots, T_{n-1}$ with the following relations:

- The elements $X_1, \ldots, X_n$ pairwise commute and $X_iX_i^{-1} = 1$.
- The elements $T_1, \ldots, T_{n-1}$ satisfy the defining relations of generators of $\mathcal{H}_{t^{\pm 1}}(S_n)$.
- We have $T_iX_j = X_jT_i$ if $j - i \neq 0, 1$, and $T_iX_iT_i = X_{i+1}$.

3, 1pt) Construct an algebra isomorphism $\mathcal{H}^{aff}_{t^{\pm 1}}(n)/(X_1 - 1) \cong \mathcal{H}_{t^{\pm 1}}(S_n)$ *(we mod out the two-sided ideal).*

4, 1pt) Construct an algebra isomorphism $\mathcal{H}^{aff}_{t^{\pm 1}}(n)[t']/(X_1^2 - (t'-1)X_1 - t') \cong \mathcal{H}_{t,t'}(B_n)[t^{-1}, t'^{-1}]$ of algebras over $\mathbb{Z}[t^{\pm 1}, t'^{\pm 1}]$, where before the localization we have the generic Hecke algebra for the Weyl group of type B_n.

5, 2pts) *Now we relate $\mathcal{H}^{aff}_{t^{\pm 1}}(n)$ with the degenerate affine Hecke algebra $\mathcal{H}(n)$ that was the main hero of HW1. Similarly to that homework, one can show that the monomials $T_wX_1^{d_1}\ldots X_n^{d_n}$ with $d_i \in \mathbb{Z}$ form a basis of $\mathcal{H}^{aff}_{t^{\pm 1}}(n)$, you can use this.* Consider the localization $\mathcal{H}^{aff}_{t^{\pm 1}}(n)[(t-1)^{-1}]$ and the $\mathbb{Z}[t^{\pm 1}]$-subalgebra $\tilde{\mathcal{H}}_{t^{\pm 1}}(n)$ generated by the $T_i's$ and the elements $(X_i - 1)/(t-1)$. Identify $\tilde{\mathcal{H}}_{t^{\pm 1}}(n)/(t-1)\tilde{\mathcal{H}}_{t^{\pm 1}}(n)$ with $\mathcal{H}(n)$. *In this sense, $\mathcal{H}(n)$ is a degeneration of the affine Hecke algebra.*

Problem 4, 6pts. *This problem deals with the Kazhdan-Lusztig basis $C_w \in \mathcal{H}_v(W), w \in W := S_n$.*

1, 2pts) Let M be a free $\mathbb{Z}[v^{\pm 1}]$-module with a finite basis H'_b, where b is in an labeling set $\mathcal{B}$. Suppose that $\mathcal{B}$ comes with a partial order, $\leqslant$. Also suppose that M comes with a $\mathbb{Z}[v^{\pm 1}]$-semilinear involution $\bar{\bullet}$ satisfying $\bar{H}'_b \in H'_b + \mathrm{Span}_{\mathbb{Z}[v^{\pm 1}]}(H'_{b'}|b' < b)$. Show that, for each $b \in \mathcal{B}$, there is a unique element $C'_b \in H'_b + v\,\mathrm{Span}_{\mathbb{Z}[v]}(H'_{b'}|b' < b)$ such that $\bar{C}'_b = C'_b$.

2, 2pts) Fix a simple reflection s. Consider the $\mathbb{Z}[v^{\pm 1}]$-module $C_s\mathcal{H}_v(S_n)$. Show that the elements $H'_w := C_sH_w$, where w runs over $\mathcal{B} := \{w \in W | \ell(sw) > \ell(w)\}$, form a basis.

Further, show that the bar-involution restricts to $C_s\mathcal{H}_v(S_n)$ and the corresponding element C'_w equals C_{sw} for all $w \in \mathcal{B}$.

3, 2pts) Use 2) to show that $C_{w_0} = \sum_{w\in W} v^{\ell(w)} H_{ww_0}$.

Aside. We have the affine Hecke algebra $\mathcal{H}^{aff}_{t^{\pm 1}}(n)$ and also the Hecke algebra $\mathcal{H}_{aff}(W(\tilde{A}_n))$. The two are closely related. Namely, inside $\mathbb{Z}^n$ we can consider the "root lattice" Λ_0 consisting of all n-tuples with zero sum of entries. A description of $W(\tilde{A}_n)$ in Section 2.1 of Lecture 20 shows that $W(\tilde{A}_n) \xrightarrow{\sim} W \ltimes \Lambda_0$, where $W = S_n$. There is a deformed version of this result. The $\mathbb{Z}[t^{\pm 1}]$-span of the elements of the form $T_w X_1^{d_1} \dots X_n^{d_n}$ with $\sum d_i = 0$ is a subalgebra. It turns out that this subalgebra is isomorphic to $\mathcal{H}_{t^{\pm 1}}(\tilde{A}_n)$ (a.k.a. the Bernstein isomorphism). This is a general feature of the affine types: the objects (Kac-Moody algebras, Weyl groups, Hecke algebras) have the general Kac-Moody realization as well as the "loop realization", compare to Section 3.1 of Lecture 20. This has various important consequences, in particular, for the Langlands program...