2.2) Birational invariant I separation of orbits

If \( G \leq X \), then it acts on \( C(x) \) by field automorphisms. So we can take invariant elements, they form a subfield \( C(x)^G \subseteq C(x) \).

Thm (Rosenlicht) \( \exists \) \( G \)-stable open \( X' \subseteq X \) and \( f_1, f_2 \in C[X]^G \) s.t.

\[ f_i(y) = f_i(y_+) \quad \forall \quad i = 1, 2 \]

Rem: \( f_1, f_2 \) as in Thm generate \( C(x)^G \). Indeed, let \( A \subseteq C(X)^G \) be gen'd by the \( f_i \)'s and let \( Y \subseteq \text{affine variety} \) correspond to \( A \). Let \( \phi: X' \to A \) be the morphism induced by \( A \to C[X] \). Let \( \overline{f} \in C(x)^G \). By Thm,

\[ \phi(y) = \phi(y_+) \Rightarrow y_1 = y_2 \Rightarrow \phi(y) = \phi(y_+) \quad \forall \quad y_1, y_2 \in X' \setminus D_f \text{ (note that } D_f \text{ is automot } \subseteq \text{stable }) \]

Applying Lem 1, we get \( f \in \text{Frac}(A) \).

2.3) Proof of Thm

Step 1 (reduce to the case when all \( G \)-orbits have the same dimension).

\[ \forall \quad m \in \mathbb{N} \quad \{ x \in X \mid \dim G_x \leq m \} \subseteq X \] is closed. Let \( d = \max \dim G_x \). Replace \( X \) w/ open \( G \)-stable subvariety \( X' \subseteq X \) \( \dim G_x < d \). Hence, all orbits are of same dim \( \Rightarrow \) closed.

Step 2 (reduce to the case where the graph of the action is closed).

Graph \( \Gamma = \{ (x, y) \in X \times X \mid G_x = G_y \} \subseteq \text{image of } G \times X \in X \times X \) under \( (g, y) \to (x, gy) \). \( \Gamma \) is image of a morphism \( \exists U \subseteq \Gamma \) w/ \( \Gamma \subseteq U' \subseteq U \) for closed \( m \times X \leq \). \( \Gamma \subseteq \Gamma' \subseteq U \) \( \Rightarrow \text{ } G \times G \text{-stable } \Rightarrow \text{ can replace } U \text{ w/ } (G \times G) U \). Assume \( U \) is \( G \times G \)-stable. Consider projection \( \pi_i: U \to X \), we claim that \( \pi_i(U \cup U) \) isn't dense then for \( X' = X \setminus \pi_i(U \cup U) \) we have \( \Gamma \cap (X' \times X') \) is closed. Since \( X' \) is open, \( \Gamma \)-stable, we will replace \( X \) w/ \( X' \) and achieve our goal. So assume the contrary: \( \pi_i(U \cup U) \) is dense. Note that if \( x \subseteq \pi_i(U) \Rightarrow U \cap \pi_i^{-1}(x) \) \( \quad (**) \)

\[ \Rightarrow \dim (U \cap \pi_i^{-1}(x)) = d \quad \Rightarrow \forall \quad x \subseteq \pi_i(U) \Rightarrow U \text{ is } \pi_i \text{-closed and } \pi_x \text{-closed } \Rightarrow \dim (U \cup U) \cap \pi_i^{-1}(x) \geq d \]. If \( \pi_i(U \cup U) \) is dense in \( X \), then \( \dim U \cup U \geq \dim X + d = \dim U - \text{contradiction} \)
Step 3: So now the dimensions of all orbits in $X$ are the same and $\Gamma = \{f(x,y) \mid x \in X, y \geq 0, \exists \gamma \in X \times X \text{ is closed} \} \subset X \times X$ is closed. Let $Y \subset X$ be an open affine subvariety. Let $\mathcal{I}$ be the ideal of $\Gamma \cap (Y \times Y)$ in $\mathcal{O}(Y \times Y) = \mathcal{O}(Y) \otimes \mathcal{O}(Y)$. Set $K = \mathcal{O}(X) = \mathcal{O}(Y)$ and consider the $K$-algebra $K[\mathcal{Y}] = K \otimes \mathcal{O}(Y)$ (containing $\mathcal{O}(Y \times Y)$). Let $\mathcal{I}$ be the ideal of $K[\mathcal{Y}]$ generated by $I$. Get $K[\mathcal{Y}]$ via action on $IK$. Note that $\mathcal{I}$ consists of all $f \in \mathcal{O}(Y \times Y)$ ($Y \subset Y$ open) such that $f \mid_p = 0$. Since $\Gamma$ is $G$-stable (for the $G$-action on 1st comp.) $\mathcal{I}$ is $G$-stable.

Step 4: We claim that $\mathcal{I}$ is generated by $G$-invariant elements. In fact this follows from the claim that for any space $V/K[G]$ (int dim in gen it)
any $G$-stable $K$-subspace $V^G \subset V \subset K[G]$ (w $G$-action on 1st factor) is generated by $\nu^G$-elt's. The proof easily reduces to the case when $V^G = \{0\}$.

Picking $v \in V$, $\nu^G(v) = \sum a_i \nu^G(v_i)$ ($v_i \in V, \nu^G(v_i) \in V$) we min $\nu^G K$ Can assume $a_i = 1$.

Then $\nu^G(v) = \nu^G(v - \sum a_i \nu^G(v_i))$. If $\nu^G(v) = 0$ it concludes. Otherwise, the $\nu^G(v_i)$ are linearly independent.

Step 5 $K[\mathcal{Y}]$ is Noetherian as it's finitely gen'd over $K$. So $f_i, i = 1, l$ be $G$-invt generators of $\mathcal{I}$. Then $\mathcal{I} = \sum f_i \otimes h_i, f_i \in K[G], h_i \in \mathcal{O}(Y)$. We claim that the $f_i$'s are the $f$-functions we need.

Step 6: Can shrink $Y$ (e.g. by replacing it with principal affine open subset) so that $f_i \in \mathcal{O}(Y), f_i \notin \mathcal{I}$ and generate it as an ideal in $\mathcal{O}(Y \times Y)$. So $(y, y) \in \Gamma. Y \times Y \iff f_i(y, y) = 0 \iff \sum f_i(y) \otimes h_i(y) = 0 \iff \sum f_i(y) \otimes h_i(y) = 0 \iff $.

If $f_i(y) = f_i(y')$ for some $y', y \in Y$ then $(y, y) \in \Gamma \iff f_i(y, y) \in \Gamma \iff G_y = G_{y'}$. $G_y = G_{y'}$ for all $y \in Y$. Now take $X' = G_Y$ open in $X$. The functions $f_i$ are $G$-invt so $f_i \in \mathcal{O}(X')$. If $f_i(x') = f_i(x)$ for all $x' \in X'$ and $G_x = G_{x'}$, we take $y \in G_y$; $x, y \in G_x, G_y$ and set contr $\Gamma$.

Cor 2: $\text{tr. deg. } \mathcal{O}(X)^G = \dim X - d$, where $d =\max \dim G_x$. In part $\mathcal{O}(X)^G = \mathcal{O}(X)$ (an automorphically unique open dense orbit).

This follows from the Thm and Remark after it.