over $\mathbb{C}(Y)$ $\square$

2.2) Birational invariant & separation of orbits

If $G \curvearrowright X$, then it acts on $\mathbb{C}(X)$ by field autom'ms. So we can take invariant elements, they form a subfield $\mathbb{C}(X)^G \subset \mathbb{C}(X)$.

Thm (Rosenlicht) $\exists$ G-stable open $X' \subset X$ and $f_1, \ldots f_k \in \mathbb{C}[X']^G$ s.t for $y_1, y_2 \in U$ TFAE (a) $Gy_1 = Gy_2$

(b) $f_i(y_1) = f_i(y_2)$ $\forall$ $i = 1..k$

Rem: $f_1 \ldots f_k$ as in Thm generate $\mathbb{C}(X)^G$. Indeed, let $A \subset \mathbb{C}[X']^G$ be gen'd by the f_i's and let Y be the affine variety corresp to A. Let $\varphi: X' \to A$ be the morphism ind'd by $A \hookrightarrow \mathbb{C}[X']$. Let $f \in \mathbb{C}(X)^G$. By Thm, $\varphi(y_1) = \varphi(y_2) \Rightarrow Gy_1 = Gy_2 \Rightarrow f(y_1) = f(y_2)$ $\forall$ $y_1, y_2 \in X' \cap D_f$ (note that D_f is automat. G-stable). Applying Lem 1, we get $f \in \operatorname{Frac}(A)$

2.3) Proof of Thm

Step 1 (reduce to the case when all G-orbits have the same dim'n). $\forall$ $m \Rightarrow$ $\{x \in X \mid \dim Gx \leq m\} \subset X$ is closed. Let $d = \max_{x \in X} \dim Gx$. Replace X w. open G-stable subvar'y $X \setminus \{x \in X \mid \dim Gx < d\}$. Hence all orbits are of same dim $\Rightarrow$ closed

Step 2 (reduce to the case when the graph of the action is closed). Graph $\Gamma := \{(x_1, x_2) \in X \times X \mid Gx_1 = Gx_2\}$ = image of $G \times X$ in $X \times X$ under $(g, x) \to (x, gx)$. Γ is image of a morphism $\Rightarrow$ $\exists$ $U \subset \Gamma$ w $\Gamma \subset \overline{U}$ & U is loc. closed in X. $\Gamma \subset \overline{\Gamma} = \overline{U}$ are $G \times G$ stable so can replace U w $(G \times G)U$ & assume U is $G \times G$-stable. Consider projection $\pi_1: \overline{U} \to X$, we claim that $\pi_1(\overline{U} \setminus U)$ isn't dense then for $X^0 = X \setminus \overline{\pi_1(\overline{U} \setminus U)}$ we have $\Gamma \cap (X^0 \times X^0)$ is closed. Since X^0 is open & G-stable, we will replace X w X^0 and achieve our goal. So assume the contrary: $\pi_1(\overline{U} \setminus U)$ is dense. Note that if $x \in \pi_1(U) \Rightarrow U \cap \pi_1^{-1}(x)$ (**) $\Rightarrow \dim U \cap \pi_1^{-1}(x) = d$. Also $\pi_1^{-1}(x)$ is G-stable $\forall x \in X$ (here we consider the action of G on the second factor in $X \times X$). Since all orbits in X have dim'n d, we see that $\dim((\overline{U} \setminus U) \cap \pi_1^{-1}(x)) \geq d$. If $\pi_1(\overline{U} \setminus U)$ is dense in X, then $\dim \overline{U} \setminus U \geq \dim X + d =$ $= \dim U$ - contradiction

(**) $\in \Gamma \cap \pi^{-1}(x) = \{x\} \times Gx \Rightarrow$ [U is $G \times G$-stable] $U \cap \pi^{-1}(x) = \{x\} \times Gx$

Step 3: So now the dimensions of all orbits in X are the same and $\Gamma = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X$ is closed. Let $Y \subset X$ be an open affine subvariety. Let I be the ideal of $\Gamma \cap (Y \times Y)$ in $\mathbb{C}[Y \times Y] = \mathbb{C}[Y] \otimes \mathbb{C}[Y]$. Set $\mathbb{K} = \mathbb{C}(X)$ $(= \mathbb{C}(Y))$ and consider the $\mathbb{K}$-algebra $\mathbb{K}[Y] = \mathbb{K} \otimes \mathbb{C}[Y]$ (containing $\mathbb{C}[Y \times Y]$). Let J be the ideal of $\mathbb{K}[Y]$ gen'd by I. $G \curvearrowright \mathbb{K}[Y]$ via action on $\mathbb{K}$. Note that J consists of all $f \in \mathbb{C}[U] \otimes \mathbb{C}[Y]$ ($U \subset Y$ open) s.t. $f|_\Gamma = 0$. Since Γ is G-stable (for the G-action on 1st copy) J is G-stable.

Step 4: We claim that J is gen'd by G-invariant elements. In fact, this follows from the claim that $\forall$ vect. space $V / \mathbb{K}^G$ (inf. dim in gen'l) any G-stable $\mathbb{K}$-subspace $V_0 \subset \mathbb{K} \otimes_{\mathbb{K}^G} V$ (w. G-action on 1st factor) is generated by inv't el'ts.(***) The proof of the claim easily reduces to the case when $V_0^G = \{0\}$ (♤).
Pick $v \in V_0 \setminus \{0\}$, $v = \sum_{i=1}^{k} \alpha_i \otimes v_i$ ($\alpha_i \in \mathbb{K}$, $v_i \in V$) w. min'l k. Can assume $\alpha_1 = 1$. Then $gv - v$ has a similar expression w. less than k summands $\Rightarrow$ is zero.

Step 5. $\mathbb{K}[Y]$ is Noetherian as it's finitely gen'd over $\mathbb{K}$. So let F_i, $i = 1..l$ be G-inv't generators of J. Then $F_i = \sum_j f_{ij} \otimes h_{ij}$, $f_{ij} \in \mathbb{K}^G$, $h_{ij} \in \mathbb{C}[Y]$. We claim that the f_{ij}'s are the functions we need.

Step 6: Can shrink Y (e.g. by replacing it w. principal affine open subset) so that $f_{ij} \in \mathbb{C}[Y]$, $F_1 \ldots F_l$ lie in I and generate it as an ideal in $\mathbb{C}[Y \times Y]$. So $(y_1, y_2) \in \Gamma \cap Y \times Y \Leftrightarrow F_i(y_1, y_2) = 0$ $\forall i = 1..l$ $\Leftrightarrow \sum_j f_{ij}(y_1) h_{ij}(y_2) = 0$ $\forall i$. So if $f_{ij}(y_1) = f_{ij}(y_1')$ for some $y_1' \in Y$, then $(y_1, y_2) \in \Gamma \Leftrightarrow (y_1', y_2) \in \Gamma \Leftrightarrow Gy_1 = Gy_2 = Gy_1' \Rightarrow Gy_1 = Gy_1'$. Now take $X' = GY$ - open in X. The functions f_{ij} are G-inv't so $f_{ij} \in \mathbb{C}[X']$. If $f_{ij}(x_1) = f_{ij}(x_2)$ $\forall i, j$, but $Gx_1 \neq Gx_2$ we take $y_1 \in Gx_1 \cap Y$, $y_2 \in Gx_2 \cap Y$ and get contr'n. $\square$

Cor 2: $\operatorname{tr.deg} \mathbb{C}(X)^G = \dim X - d$, where $d = \max_{x \in X} \dim Gx$. In part'l: $\mathbb{C}(X)^G = \mathbb{C} \Leftrightarrow X$ has (an automatically unique) open dense orbit)

This follows from the Thm and Remark after it.

(***) We take $V = \mathbb{K}^G[Y]$, $V_0 = J$

(♤) Indeed replace V w. compl't V' to V_0^G & V_0 w. $V_0 \cap (\mathbb{K} \otimes_{\mathbb{K}^G} V')$