Rouquier, Lecture 2

$W \curvearrowright V$, $S \subset W$ - refl-ns, $\mathfrak{A} = \{\text{Maps } S/W \to \mathbb{C}\}$

H = Cherednik alg at $t=0$

$\check{Z} = Z(H)$

$\check{P} = A \otimes \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$.

Special case: $\dim V = 1$, $W = \langle s \rangle$, s acting by $\zeta = \exp(2\pi i/d)$

$V = \mathbb{C}y$, $V^* = \mathbb{C}x$, $\langle x, y \rangle = 1$

$A = \mathbb{C}[c_1 \dots c_{d-1}]$, $\mathbb{C}[V]^W = \mathbb{C}[\underbrace{x^d}_{X}]$, $\mathbb{C}[V^*] = \mathbb{C}[\underbrace{y^d}_{Y}]$

$\mathbb{C}[V \times V^*]^{\Delta W} = \mathbb{C}[X, Y, eu_0]/(eu_0^d - XY)$, $eu_0 = xy$

$H = A\langle x, y, s \mid sxs^{-1} = \zeta^{-1}x,\ sys^{-1} = \zeta y,\ s^d = 1,\ [y,x] = \sum_{i=1}^{d-1} (\zeta^i - 1) c_i s^i \rangle$

Gen-l case: Euler element $eu = \sum_{i=1}^{n} y_i x_i + \sum_{s \in S} c_s s$

$K = \operatorname{Frac} P \subset L = \operatorname{Frac} Z = K(eu)$ b/c $\mathbb{C}(V \times V^*/\Delta W) = \mathbb{C}(V/W \times V^*/W)(eu))$

Back to special case: $eu = yx + \sum_{i=1}^{d-1} c_i s^i$

$\mathbb{C}W = \bigoplus_{i=1}^{d} \mathbb{C}s_i = \bigoplus_{i=1}^{d} \mathbb{C}\varepsilon_i$, $\varepsilon_i = \frac{1}{d} \sum_j \zeta^{ij} s^j$

Define $K_0, \dots K_{d-1}$ by $d \sum_{j=0}^{d-1} \zeta^{i(j-1)} K_j = c_i$

$Z = A[X, Y, eu] / \left(\prod_{i=1}^{d} (eu - K_i) - XY \right)$

$\mathcal{Z} = \operatorname{Spec}(Z)$

$\downarrow$

$\mathcal{P} = \operatorname{Spec}(P) = \underset{\substack{\uparrow \\ \text{space of} \\ \text{param-s}}}{\mathbb{A}^{d-1}} \times \underset{\substack{\parallel \\ \operatorname{Spec}(\mathbb{C}[X,Y])}}{\mathbb{A}^2}$

General V

$P \subset Z \subset R$ L/K is not Galois

$\cap \quad \cap \quad \cap$

$K \subset L \subset M$ - Galois closure

R - integr. closure of Z in M

G - Galois group $Gal(M/K)$

$\cup$

$H = Gal(M/L)$

$P = R^G,\ Z = R^H.$

Example: $W = A_2$; $[L:K] = |W| = 6$, $G = \mathfrak{S}_6 \supset H = \mathfrak{S}_5$

$W = B_2$: $G = Weyl(D_4) \supset H = Weyl(A_3)$

The cyclic case $W = \mu_\ell$, $\dim V = 1$, $G = \mathfrak{S}_\ell \supset H = \mathfrak{S}_{\ell-1}$

Rem: everything is bi-graded; in particular, $\deg V = 1 = \deg V^*$, $\deg \varsigma = 2$

R is positively graded $\rightsquigarrow R_+/(R_+)^2$ - reflection rep-n in examples

General situation:

$$\begin{tikzcd}
\mathcal{R} \arrow[d, "\varphi"] & (V\times V^*)/\Delta Z(W) \arrow[d] \\
\mathcal{Z} \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & V\times V^*/W \arrow[l, hook'] \arrow[d] \\
\mathcal{P} \arrow[d] \arrow[dr, phantom, "\square"] & V/W \times V^*/W \arrow[l, hook'] \arrow[d] \\
\mathcal{A} & 0 \arrow[l, hook']
\end{tikzcd}$$

- intersection of the conjugates of ΔW.

$\varphi^{-1}((V\times V^*)/\Delta W)$ is not irreducible

Choose $\mathcal{R}_0$ - irred. comp-t.

Then $V\times V^*/\Delta Z(W) \longrightarrow \mathcal{R}_0$ is the normalization morphism

$$\begin{tikzcd}
\mathcal{R} \arrow[d] & \mathcal{R}_0 \arrow[l] & (V\times V^*)/\Delta Z(W) \arrow[l] \arrow[d] \\
\mathcal{Z} & & (V\times V^*)/\Delta W \arrow[ll]
\end{tikzcd}$$

$\mathcal{D}_0 = \mathrm{Stab}_G(R_0) \supset I_0 = \mathrm{Fix}_G(R_0)$

$$\begin{tikzcd} \mathcal{D}_0/I_0 \arrow[r, "\sim"] & (W\times W)/\Delta Z(W) \ni (w,1) \\ & W \ni w \arrow[u] \end{tikzcd}$$

$G = \mathcal{D}_0 H$, $I_0 \triangleleft \mathcal{D}_0$, $I_0 \subset H \Rightarrow I_0 \subset \mathcal{D}_0 \cap H \overset{?}{\Rightarrow} I_0 = \{1\}$

$\mathcal{D}_0 \cap H/I_0 = \Delta W/\Delta Z(W)$.

So $G/H \longleftarrow W$ (induced by the choice of R_0)

diff choices differ by G-action

$\dim V = 1$: description of R: | ith elementary symm. function

$$\lambda_\alpha \in R = A \otimes \mathbb{C}[\lambda_1, \dots \lambda_\alpha, X, Y] \Big/ \begin{array}{l} \sigma_i(\lambda) = \sigma_i(K) \quad i = 1, \dots \alpha-1 \\ \sigma_\alpha(\lambda) = \sigma_\alpha(K) + (-1)^\alpha XY \end{array}$$

$$\uparrow$$

$$eu \in Z = A \otimes \mathbb{C}[eu, X, Y] \Big/ \prod_{i=1}^{\alpha} (eu - K_i) = XY$$

$G = \mathfrak{S}_\alpha$ acts on λ's

H = stabilizer of λ_α

| complete inters + ⇓ smooth in codim 1

Sketch of proof: need R is normal domain - using Serre's criterion

then need to prove the claim on the level of fraction fields

To prove that R is smooth in codim 1

ram $R \to P$ - codim 1

ram $P \to$ [forget K's] - codim 1

intersection of these loci has codim 2 + use normality of the two bases $\square$

$$\begin{tikzcd}
\mathcal{R} \arrow[r, hook] \arrow[d, "\varphi"'] & \mathcal{R}_c \arrow[dd, "\mathcal{D}_c / I_c = \mathcal{D}_c \subset G"] \\
\mathcal{Z} \arrow[d, "\pi"'] & \\
\mathcal{P} \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{C} \times V/W \times V^*/W \arrow[d] \\
\mathcal{A} & c \arrow[l, hook']
\end{tikzcd}$$

$\mathcal{R}_c$ - irr. comp

$W = \mu_d$ can get $\mathcal{D}_c = A_9,\ \mathrm{Aut}(A_6),\ \mathbb{Z}/23 \times \mathbb{Z}/2$

$\mathcal{D}_c = \mathrm{Gal}\left(\prod_{i=1}(T - K_i) - XY\right)$

Cells $X \subset \mathcal{R}$ - closed irreducible variety

Def: X-cells of W are orbits of $\mathcal{D}_X = \mathrm{Fix}_G(X)$ acting on W

Can take: • $\pi\varphi(X) = c \times V/W \times 0$ (irred. comp X of the preimage, need to choose it in a way compatible w the initial choice of $\mathcal{R}_0$) $\rightsquigarrow$ left cells.

• $\pi\varphi(X) = c \times 0 \times V^*/W \rightsquigarrow$ right cells

• $\pi\varphi(X) = c \times 0 \times 0 \rightsquigarrow$ 2-sided cells

Example: $\dim V = 1$: $R = \mathbb{C}[K_1, \ldots K_d, \lambda_1, \ldots \lambda_d, X, Y] \Big/ \begin{array}{l} \sigma_i(K) = \sigma_i(\lambda),\ i < d \\ \sigma_d(\lambda) = \sigma_d(K) + (-1)^d XY \end{array}$

$\mathcal{R}_0 = \mathrm{Spec}(R/\mathfrak{p}_0)$
$= \{ K_i = 0,\ \lambda_i = \zeta^i \lambda_d$

Left cells: K_i's are $\underbrace{\text{fixed}}_{\text{choice of } c}$, $Y = 0$, $\lambda_i = K_i$

Right cells: $X = 0$, same as before

Two-sided cells: $X = Y = 0$

$\mathcal{D}_{\text{left-cell}} = \{g \in \mathfrak{S}_2 \mid K_{g(i)} = K_i\}$ so left cells are
$s^i \sim_L s^j \iff s^i \sim_R s^j \iff s^i \sim_{LR} s^j \iff K_i = K_j$

Prop: All cells have one element $\iff \mathcal{R} \to \mathcal{P}$ is unramified at gen. pt of X.

Prop: $\{X\text{-cells}\} \xrightarrow{\sim}$ blocks of $\mathbb{C}(X)\otimes_P H$ $\mathbb{C}(X)$-

so $w \sim_{X,L} w' \iff L_w, L_{w'}$ in same block, where $\mathrm{Irr}(M\otimes_P H) = \{L_w\}_{w\in W}$.

$M\otimes_P H \underset{\text{Morita}}{\sim} M\otimes_P Z = M\otimes_k L \xrightarrow{\sim} M^{G/H} \xrightarrow{\sim} M^{(W)}$

Note: X-cells $\leftrightarrow$ irred. comp. of $X\times_{\mathcal{P}} Z$

~~Cor~~ Prop: $X\subset X' \Rightarrow X$-cells are unions of X'-cells

$\Rightarrow$ two-sided cells are union of left cells

Conj W-fin. Coxeter group (w. some suitable choices)
then KL-cells = CM-cells
($c_s\in\mathbb{Q}$)

Prop: If $\varphi(X) \subset Z^{sing} \Rightarrow \mathbb{C}(X)\otimes_P H$-blocks have only one simple module (~~abs. irreducible~~)

Cor: X corresponds to left cell, then $\mathbb{C}(X)\otimes_P H$-blocks have only one simple module