Equivariant Algebraic K-Theory

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Contents

1 Motivation

The goal of these lectures is to give an introduction to equivariant algebraic K-theory. Out motivation will be to provide a proof of the classical Weyl character formula using a localization result. Our primary reference is the book of Chriss-Ginzburg [1], chapters 5 and 6.

Proposition 1.1 (Weyl)**.** *Let L_λ be the irreducible representation of G, of highest weight λ. Then the following formula holds*

$$Ch(L_\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})} : \mathfrak{h} \to \mathbb{C}^* \tag{1.1}$$

where $\rho = \frac{1}{2}\sum\alpha$ is half the sum of the positive roots, and W is the Weyl group of G.

Our ground field is always $\mathbb{C}$, and unless otherwise specified, G is a complex reductive algebraic group. The action of G on varieties X is always assumed to be algebraic.

2 Basic Definitions

For any abelian category $\mathcal{C}$, we can form its Grothendieck group $K(\mathcal{C})$, also known as the K-theory of $\mathcal{C}$. This is defined to be the free abelian group generated by isomorphism classes of objects in $\mathcal{C}$, modulo the relations that whenever we have an exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of objects in $\mathcal{C}$, we have have the relation $[V_1] - [V_2] + [V_3] = 0$ in $K(\mathcal{C})$.

Some classic examples:

- $\mathcal{C} = Vect_G(X)$, the category of G-equivariant vector bundles on a topological space X. $K(Vect_G(X))$ is known as equivariant (topological) K-theory.
- $\mathcal{C} = Coh(X)$, the category of coherent sheaves on an algebraic variety X. This is called algebraic K-theory

If we wish to generalize this last example to the equivariant setting, we have to be careful about what it means for a sheaf to be equivariant.

2.1 Equivariant Sheaves

Let X be a G-variety. This means that we have an action map $a : G \times X \to X$, which satisfies the property

$$a \circ (m \times \mathrm{Id}_X) = a \circ (\mathrm{Id}_G \times a) : G \times G \times X \to X, \tag{2.1}$$

where $m : G \times G \to G$ is the group multiplication.

Definition 2.1 (5.1.6)**.** *A sheaf $\mathcal{F}$ on a G-variety X is called equivariant if*

1) *There is a given isomorphism I of sheaves on $G \times X$,*

$$I : a^*\mathcal{F} \simeq p^*\mathcal{F} \tag{2.2}$$

At the level of open sets, this gives isomorphisms $I_{(g,U)} : \mathcal{F}(gU) \simeq \mathcal{F}(U)$.

2) *We want the isomorphisms I_g to satisfy the cocycle conditions*

$$I_{(gh,U)} = I_{(h,U)} \circ I_{(g,hU)} : \mathcal{F}(ghU) \simeq \mathcal{F}(U).$$

- *The isomorphism I restricted to $\{e\} \times X$ gives the identity.*

Two basic examples of equivariant sheaves we will be considering:

- The sheaf of G-invariant functions on X.
- $\mathcal{O}(V)$, the sheaf of sections of an equivariant vector bundle $V \to X$. These are precisely the locally free sheaves. This is equivalent to a bundle equipped with an action $G \times V \to V$, which is fiberwise linear, and covers the action of G on X.

- $i_*\mathcal{O}_Y$, where $i : Y \to X$ is the inclusion of the G-stable subvariety Y. In particular, we'll be focusing on the case $Y = X^G$, the fixed point locus. We'll come back to this example later.

It will also be important to note that is $\mathcal{F}$ is an equivariant coherent sheaf on X, then the space $\Gamma(X, \mathcal{F})$ of global sections has a natural structure of a G-module. This can be seen by the following sequence of ismorphisms

$$\Gamma(X, \mathcal{F}) \xrightarrow{a^*} \Gamma(G \times X, a^*\mathcal{F}) \xrightarrow{I} \Gamma(G \times X, p^*\mathcal{F}) = \mathbb{C}[G] \otimes \Gamma(X, \mathcal{F}) \tag{2.3}$$

2.2 Structure of K-theory

We want to show that G-equivariant coherent sheaves have resolutions consisting of *equivariant* locally free sheaves. First, we need to know that there are "sufficiently many" G-equivariant line bundles on any smooth G-variety.

Proposition 2.2. *If X is a smooth G-variety, where G is a connected linear algebraic group, and $\mathcal{L}$ is an arbitrary line bundle over X, then some positive power $\mathcal{L}^{\otimes n}$ admits a G-equivariant structure.*

Proof. See [1] Thm 5.1.9 □

Proposition 2.3. *If X is a smooth quasi-projective G-equivariant variety, then*

1) Any G-equivariant coherent sheaf is the quotient of a G-equivariant locally free sheaf.

2) Any G-equivariant sheaf on X has a finite, locally free, G-equivariant resolution.

Outline. We assume X is projective, the quasi-projective case follows by standard embedding arguments.

- Show that there exists an equivariant projective embedding $(X, G) \to (\mathcal{P}(V), GL(V))$. Do this by taking the space of sections of an ample equivariant line bundle $\mathcal{L}$ on X. For $x \in X$, let $H_x \subset V = \Gamma(X, \mathcal{L})$ be the hyperplane of functions vanishing at X. These maps give an equivariant embedding $X \to \mathcal{P}(V^*)$.

- Let $\mathcal{F}$ be an equivariant sheaf. Let $\mathcal{L}$ be an equivariant line bundle. For n large enough, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\times n}$ is generated by global sections. Then the map $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes(-n)} =: \mathcal{F}_1 \to \mathcal{F}$ is surjective, yielding 1).

- Iterate the previous argument on $\mathcal{F}^1$, to yield a an equivariant resolution $\cdots \to \mathcal{F}^2 \to \mathcal{F}^1 \to \mathcal{F}$. By Hilbert's syzygy theorem, this resolution can be terminated at $\ker(\mathcal{F}^n \to \mathcal{F}^{n-1})$.

□

With this, we define $K_G(X) = K(Coh_G(X))$, where $Coh_G(X)$ is the abelian category of G-equivariant coherent sheaves on X.

Consider $X = *$. Then a G-equivariant sheaf on X is just a G-vector space, i.e $Coh_G(*) = Rep(G)$, and $K_G(*) = R(G)$, the representation ring of G.

2.3 Functorality of K-theory

Given a G-equivariant map $f : Y \to X$, we will now define various morphisms between K-groups of X and Y. If a functor on the category of coherent sheaves is exact, then it automatically descends to K-theory.

2.3.1 Tensor Products

There is an exact functor $\boxtimes : Coh_G(X) \times Coh_G(Y) \to Coh_G(X \times Y)$, which maps

$$\boxtimes : (\mathcal{F}, \mathcal{E}) \mapsto p_Y^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} p_Y^* \mathcal{E}$$

This descends to K theory, and is called the external tensor product, and defines a $R(G) = K_G(*)$ module structure on $K_G(X)$.

If X is a smooth variety, the diagonal embedding $\Delta : X \to X \times X$ gives an exact functor $\Delta^* : Coh_G(X \times X) \to Coh_G(X)$. Combining with the external tensor product, the map $\otimes : K_G(X) \otimes K_G(X) \to K_G(X)$, given by

$$\otimes : \mathcal{F} \otimes \mathcal{F}' \mapsto \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$$

turns $K_G(X)$ into a commutative, associative $R(G)$ algebra.

2.3.2 Flat Morphism

If f is flat morphism of varieties, e.g. an open embedding, then we consider the usual sheaf theoretic inverse image functor

$$f^* : Coh^G(X) \to Coh^G(Y), \qquad \mathcal{F} \mapsto f^*\mathcal{F} := \mathcal{O}_Y \otimes_{f^*\mathcal{O}_X} f^*\mathcal{F}$$

Since f was a flat morphism this functor is exact, so it descends to K-theory. Thus we get a pullback map

$$f^* : K_G(X) \to K_G(Y)$$

2.3.3 Closed Embedding

Now, suppose instead we are given a G-equivariant closed embedding $f : Y \to X$, e.g, inclusion of the fixed point locus $Y = X^G$. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the defining ideal of Y inside X. The restriction map $f^* : Coh(X) \to Coh(Y)$, given by $\mathcal{F} \mapsto \mathcal{F}/\mathcal{I}_Y\mathcal{F} \cong f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}$ is not (in general) an exact functor, so it does not descend to a map in K-theory. We avoid this complication, by only considering the case where both X and Y are smooth varieties, and proceed as follows. Pick a locally free resolution $E^\bullet$ of $f_*\mathcal{O}_Y$ (or, instead a resolution of $\mathcal{F}$)

$$\cdots \to E^1 \to E^0 \to f_*\mathcal{O}_Y \to 0$$

Thus, for each i, the sheaf $E^i \otimes_{\mathcal{O}_X} \mathcal{F}$ is coherent on X. Furthermore, the cohomology of the complex

$$\cdots \to E^1 \otimes_{\mathcal{O}_X} \mathcal{F} \to E^0 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

denoted by $\mathcal{H}^i(E^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$, may be viewed as coherent $f_*\mathcal{O}_Y$ modules, and hence as $\mathcal{O}_Y$ modules. We define the class

$$f^*[\mathcal{F}] = \sum(-1)^i[\mathcal{H}^i(E^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})] = \sum(-1)^i[Tor_i^{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{F})] \in K_G(Y) \tag{2.4}$$

Obviously, the r.h.s. is independent of the choice of equivariant resolution.

2.3.4 Pushforward

Let $f : X \to Y$ be a proper G-equivariant morphism, and we no longer require X and Y to be smooth. We have the natural direct image functor $f_* : Coh(X) \to Coh(Y)$. This functor is left exact, but not right exact. For a short exact sequence of coherent sheaves $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ on X, we get a long exact sequence of G-equivariant coherent sheaves on Y

$$0 \to f_*\mathcal{E} \to f_*\mathcal{F} \to f_*\mathcal{G} \to R^1f_*\mathcal{E} \to R^1f_*\mathcal{F} \to R^1f_*\mathcal{G} \to R^2f_*\mathcal{E} \to \cdots \tag{2.5}$$

which terminates at finite length. Thus, if we define $f_*[\mathcal{F}] = \sum(-1)^i[R^if_*\mathcal{F}] \in K_G(X)$, we have $f_*([\mathcal{E}]-[\mathcal{F}]+[\mathcal{G}]) = (f_*[\mathcal{E}]-f_*[\mathcal{F}]+f_*[\mathcal{G}])$, and thus f_* descends to a well defined map $f_* : K_G(X) \to K_G(Y)$. If we are pushing forward to a point, $f : X \to *$, then this map is

$$f_*[\mathcal{F}] = \sum(-1)^i[H^i(X, \mathcal{F})]$$

2.3.5 Induction

Let $H \subset G$, be a closed subgroup, and X and H-variety. Then there is an isomorphism

$$K_H(X) \cong K_G(G \times_H X). \tag{2.6}$$

Here we note that $G \times_H X$ can be given the structure of an algebraic variety. Firstly, the projection $G \times X \to G$, gives a flat map $G \times_H X \to G/H$, with fiber X. Thus, given a G-equivariant sheaf $\mathcal{F}$ on $G \times_H X$, we can restrict it to X over the basepoint $eH \in G/H$. This restriction map $res : K_G(G \times_H X) \to K_H(X)$, has an inverse given as follows. Consider the projection onto the second factor $p : G \times X \to X$, and let $\mathcal{F}$ be a H-equivariant sheaf on X. Then the pullback $f^*\mathcal{F}$ is H equivariant with respect to the diagonal H action on $G \times X$. Then we use equivariant descent in the étale topology to show that this sheaf descends to a G-equivariant sheaf given by an induction functor $ind_H^G\mathcal{F}$ on $G \times_H X$. In particular, if $X = *$, then we get an isomorphism

$$R(H) = K_H(*) \cong K_G(G/H) \tag{2.7}$$

which is given by the usual induction and restriction maps.

2.4 Example: The K-theory of the flag variety

Recall the flag variety. Given a connected complex semisimple group G, with Lie algebra $\mathfrak{g}$, we consider Borel subgroups B, i.e. a maximal solvable subgroups, $B \subset G$, with Lie algebra $\mathfrak{b} \subset \mathfrak{g}$. Denote the space of all such Borel algebras $\mathcal{B}$. It is easy to see that G acts transitively on B by conjugation, and the stabilizer of any such group is isomorphic to some fixed B, hence $\mathcal{B} \cong G/B$. This space is called the *flag variety* of G. Here G now acts on G/B by left multiplication. Our immediate goal is to work out the equivariant K-theory $K_G(\mathcal{B})$.

First, some conventions. For any particular Borel $\mathfrak{b}$, we get a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let us make the following (rather unusual) choice of positive roots $R^+ \subset \mathrm{Hom}(T, \mathbb{C}^*)$: we declare the weights of the adjoint T action on $\mathfrak{b}$ to be the *negative* roots. We do this, for the following reason. The tangent space $T_\mathfrak{b}\mathcal{B}$ is given by $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^-$. Thus the weights of the T action on the tangent space $T_\mathfrak{b}\mathcal{B}$ are precisely given by the positive roots.

For any character $\lambda \in \mathrm{Hom}(T, \mathbb{C}^*)$, we can form an equivariant line bundle $L_\lambda \to \mathcal{B}$ as follows. Since $B/[B,B] \cong T$, we can extend λ to a character of B. Set $L_\lambda = G \times_B \mathbb{C}_\lambda$, where B acts on $\mathbb{C}_\lambda$ with character lambda. The map $(g, z) \to g/B \in G/B$ gives L_λ the structure of a G-equivariant line bundle over $\mathcal{B}$. Furthermore, any equivariant line bundle on $\mathcal{B}$ is isomorphic to some L_λ, by looking at the fiber above some particular Borel. Extending this map by linearity we get a map $R(T) \to K_G(\mathcal{B})$, which we will show is an isomorphism of $R(G)$-modules. Since $\mathcal{B} \cong G/B$, we use 2.7 to get

$$K_G(\mathcal{B}) \cong K_G(G/B) \cong K(B) \cong R(B) \cong R(T) \tag{2.8}$$

Note that $R(G) = R(T)^W$, so in particular $K_G(\mathcal{B})$ is a free $R(G)$ module of rank $|W|$.

3 The Thom Isomorphism

The Thom isomorphism relates the K-theory of a smooth variety X, with the K-theory of the total space of a vector bundle V over X.

3.1 The Koszul Resolution

Let $\pi : V \to X$ be a (G-equivariant) vector bundle, and $i : X \to V$ be the inclusion of the zero section. We we will construct a resolution of the sheaf $i_*\mathcal{O}_X$. Consider the following complex of vector bundles on the total space V

$$\cdots \to \pi^*(\Lambda^2 V^\vee) \to \pi^*(\Lambda^1 V^\vee) \to \mathcal{O}_V \to 0 \tag{3.1}$$

The differentials at over a point $v \in V$, is given by contraction with v, ie.

$$f_1 \wedge \cdots \wedge f_j \mapsto \sum (-1)^k \langle v, f_k \rangle f_1 \wedge \cdots \wedge \check{f}_k \wedge \cdots \wedge f_j \tag{3.2}$$

We can easily check that this complex is exact everywhere in each fiber in V, except at the origins $0 \in V_x$. Thus this complex has one dimensional cohomology in degree 0, supported precisely at the image of the zero section. Thus this complex is a resolution of $i_*\mathcal{O}_X$. We define the element

$$\lambda(V^\vee) = \sum (-1)^j [\Lambda^j (V^\vee)] \in K_G(X) \tag{3.3}$$

so that the relation $i_*\mathcal{O}_X = \pi^*\lambda(V^\vee)$ holds in $K_G(V)$. Now, on the other hand, since $\pi^*\lambda(V^\vee)$ is a resolution of the zero section, we can use it to define a restriction map from sheaves on V, to sheaves supported on the zero section. For $\mathcal{F}$ a sheaf on V, following 2.4, we have

$$[i^*\mathcal{F}] = \sum(-1)^i H^i(V, \pi^*\Lambda V^\vee \otimes \mathcal{F}) \in K_G(X) \tag{3.4}$$

We now have the following important K-theoretic version of the Thom isomorphism.

Proposition 3.1. *Let $V \to X$ be a bundle as before, and $\mathcal{F} \in K_G(X)$. Then we have the following equalities:*

$$i^*(\pi^*\mathcal{F}) = \mathcal{F}, \qquad i^*(i_*\mathcal{F}) = \lambda(E^\vee) \otimes \mathcal{F}, \tag{3.5}$$

Proof. For the first statement, we want to compute the cohomology of $\pi^*\Lambda^\bullet(V^\vee) \otimes \pi^*\mathcal{F} = \pi^*(\Lambda^\bullet(V^\vee) \otimes \mathcal{F})$ over V. This complex is exact everywhere except at the $i = 0$ term, since $\pi^*\mathcal{F}$ is constant along fibers of V, and the result follows. For the second statement, we want to compute the cohomology of $\pi^*\Lambda^\bullet(V^\vee) \otimes i_*\mathcal{F} = i_*(\Lambda^\bullet(V^\vee) \otimes \mathcal{F})$, a sheaf supported on the zero section. Now in K-theory we have

$$\begin{aligned}\sum_i(-1)^i[\mathcal{H}^i(X, \Lambda^\bullet(V^\vee) \otimes \mathcal{F})] &= \sum_i(-1)^i[\Lambda^\bullet(V^\vee) \otimes \mathcal{F}] \\ &= \lambda(V^\vee) \otimes [\mathcal{F}]\end{aligned}$$

□

4 The Localization Theorem

The localization theorem will tell us what happens when we restrict equivariant bundles over X to the fixed point locus X^G.

4.1 Fixed Point Loci

We begin with a well-known result.

Proposition 4.1. *Let G be reductive, acting on smooth X, then the fixed point set X^G is a smooth subvariety of X.*

Proof. See [1][5.11.1] □

From now on, let $T \subset G$ be an abelian reductive subgroup. Since X is smooth, we may consider the normal bundle $N = N_{M^T}M$. Since T acts trivially on M^T, it induces a linear action on the fibers of this normal bundle, and so we get decomposition $N = \oplus_{\alpha \in R(T)} N_\alpha$, and thus $N \in K^T(M^T)$. Let $i : X^T \to X$ be the inclusion of the fixed point set. We now have the following extension of 3.1.

Lemma 4.2. *For all $\beta \in K^T(M^T)$, we have*

$$i^* i_* \beta = \lambda(N^\vee) \otimes \beta \tag{4.1}$$

We will primarily be interested in the following question: under what circumstances can we invert the operation i^*i_*, i.e. when does $\lambda(N^\vee)^{-1}$ exist? The answer is that it is invertible, as long as we avoid certain irregular points in t.

4.2 Localization

Consider $R(T)$ the representation ring of T. We can think of this ring as a subring of regular functions on T, by mapping a representation V to the function $f_V : a \mapsto \mathrm{Tr}_V(t)$. For any point $t \in T$, we consider representations that do not vanish at t. These form a multiplicative set, at which we can localize $R(T)$, to form the ring R_t. Likewise any $R(T)$ module can be localized $M_t := R_t \otimes_{R(T)} M$. For our purposes, we will be localizing the $K_A(*) = R(T)$ module, $K_T(X)$.

Now, consider a variety X equipped with the *trivial* T action, and a vector bundle E over it equipped with a fiberwise linear action of T. The bundle E has a decomposition into characters of T,

$$E = \bigoplus_{\mu \in Sp(E)} E_\mu \tag{4.2}$$

where $Sp(E)$ is the set of all characters $\mu : T \to C^*$ which appear in any fiber of E. The bundle $E_\mu \subset E$ is the eigenbundle of E corresponding to the weight μ. Since X has the trivial A action, the bundle E gives us a class $[E] \in K^T(X) \cong R(T) \otimes K(X)$. Under this isomorphism we have $[E] = \sum_\mu \mu \otimes [E_\mu] \in R(T) \otimes K(X)$, where E_μ is thought of as a non-equivariant vector bundle. We now consider the Koszul bundle 3.3 for equivariant E, and investigate it's invertibility.

First of all, consider the case where $Sp(E) = \{\mu\}$, and E_μ is the trivial rank 1 bundle. Then $\lambda(E) = \Lambda^0(E) - \Lambda^1(E) = 1 - \mu$. Clearly this function on A is invertible at all points except those $t \in T$ at which $\mu(t) = 1$. Thus, if we work at a point at which $\mu(t) \neq 1$, then $1 - \mu$ is an invertible function in the localized ring R_t. So now we have

Proposition 4.3. *let $t \in T$ be element such that $\mu(t) \neq 1$ for all $\mu \in Sp(E)$. Then multiplication by $\lambda(E)$ induces an automorphism of the localized K-group $K_T(X)_t$.*

Note that we can't localize at *all* weights $\mu \in R(G)$, since this would result in removing too many points from T.

Proof. First, we have the weight decomposition, $E = \sum_{\mu \in Sp(E)} \mu \otimes E_\mu$. Now consider $E_0 = \sum_{\mu \in Sp(E)} \mu \otimes \mathcal{O}_X^{\mathrm{rk}\, E_\mu}$. This bundle has the same characters as E, but none of the topology. We now need a lemma.

Lemma 4.4. *Let E be a rank d vector bundle on a variety X, and $\mathcal{O}_X^r$ the trivial rank r bundle, then the operation of multiplication by $E - \mathcal{O}_X^r$ is a nilpotent operator on $K(X)$. Specifically,*

$$(E - \mathcal{O}_X^r)^{\dim X + 1} = 0 \tag{4.3}$$

Thus we see that $\lambda(E) - \lambda(E_0)$ acts nilpotently in $K_T(X)$. However, $\lambda(E_0) = \prod_{\mu \in Sp(E)} (1 - \mu)$, and thus

$$\lambda(E) = \prod_{\mu \in Sp(E)} (1 - \mu) + \text{nilpotent} \tag{4.4}$$

as operators on $K_T(X)$. Hence if $t \in T$ satisfies $\mu(t) \neq 1$ for all μ, $\lambda(E)$ is invertible. ☐

So want to consider those points $t \in T$ that aren't in the vanishing locus of some set of functions. These points are called regular.

For fixed $t \in T$, we say that X is t-regular if $X^t = X^T$. This generalizes the notion of a regular semisimple element t in a maximal torus, as such an elements generate a dense set inside T. As before, let $N = N_{X^t}X$ be the normal bundle to the inclusion $X^t \to X$, and let $N = \oplus_{\mu \in Sp(N)} N_\mu$ be its weight decomposition. Note that the statement that X is t-regular is equivalent to $\mu(t) \neq 1$ for all $\mu \in Sp(N)$, since the normal bundle $N_{X^A}X^t$ has precisely those weights for which $\mu(t) = 1$.

Thus, we arrive at

Corollary 4.5. *If $i : X^T \to X$ is the inclusion, and t is X-regular, then the induced map*

$$i^* i_* : K_T(M^T)_t \to K_T(M^T)_t$$

(which is given by multiplication by $\lambda(N^\vee)$) is an isomorphism.

In fact, the *localization theorem* due to Thomason, says that once we localize, all K-groups of X are concentrated at the fixed point locus X^T.

Proposition 4.6 (Thomason)**.** *For an arbitrary T-variety X, which is t-regular, the induced map $i_* : K_T(X^t)_t \to K_T(X)_t$ is an isomorphism.*

This powerful theorem allows us to compute the K-groups of various spaces (up to localization), just by studying the fixed point loci. We won't prove this, but a short proof in the case of cellular fibrations is found in [1].

4.3 Pushforwards and Restriction

Assume t is X-regular. Denote $K(X) \otimes \mathbb{C}$ as $K(X, \mathbb{C})$, i.e. non-equivariant K-theory. Now consider the following map $res_t : K_T(X) \to K(X^T, \mathbb{C})$, given by

$$res_t : \mathcal{F} \to ev_t \left(\lambda(N^\vee_{X^T} X)^{-1} \otimes i^* \mathcal{F} \right)$$

where ev_t is the evaluation map $R(T) \to \mathbb{C}$, and we have used the isomorphism $K_T(X^T) \cong R(T) \otimes K(X^T)$. If $X = *$, and $V = \alpha \otimes \mathbb{C}^n \in K_T(*) = R(T) \otimes K(*)$ is a T-bundle over it, then $res_t(V) = \alpha(t) \otimes \mathbb{C}^n \in K(*, \mathbb{C})$.

With all that we have constructed before, we can find the inverse of res_t, once we have complexified.

Proposition 4.7. *The map $i_* : K(X^T, \mathbb{C}) \to K_T(X, \mathbb{C})$ is the inverse to res_t*

Proof. We have

$$res_t i_* \mathcal{F} = ev_t(\lambda(N^\vee)^{-1} \otimes \lambda(N^\vee) \otimes \mathcal{F}) = ev_t(\mathcal{F}) \tag{4.5}$$

☐

Proposition 4.8. *Let $f : X \to Y$ be an T-equivariant proper morphism of smooth T-varieties, for which both of X and Y are t-regular. Then the following diagram commutes*

$$\begin{array}{ccc} K_T(X) & \xrightarrow{f_*} & K_T(Y) \\ {\scriptstyle res_t}\downarrow & & \downarrow{\scriptstyle res_t} \\ K(X^T, \mathbb{C}) & \xrightarrow[f_*]{} & K(Y^T, \mathbb{C}) \end{array}$$

Proof. We first consider the inverses to res_t, in the following diagram,

$$\begin{array}{ccc} K_T(X, \mathbb{C})_t & \xrightarrow{f_*} & K_T(Y, \mathbb{C})_t \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\ K(X^T, \mathbb{C}) & \xrightarrow[f_*]{} & K(Y^T, \mathbb{C}) \end{array}$$

This commutes with due to functoriality of the push-forward. However, this diagram consists of the vertical inverses of the diagram we want, after we complexify and localize at t in the top row.

$$\begin{array}{ccc} K_T(X, \mathbb{C})_t & \xrightarrow{f_*} & K_T(Y, \mathbb{C})_t \\ {\scriptstyle res_t}\downarrow & & \downarrow{\scriptstyle res_t} \\ K(X^T, \mathbb{C}) & \xrightarrow[f_*]{} & K(Y^T, \mathbb{C}) \end{array}$$

□

Thus push forwards commute with restriction. We now apply this lemma to the case in which we push-forward to a point, $f : X \to *$, and we get the following Lefschetz fixed point theorem.

Proposition 4.9. *Let X be a smooth, t-regular, compact T-variety. Then for any vector bundle $V \in K^T(X)$, we have*

$$\sum (-1)^i \mathrm{Tr}\left(t; H^i(X, V)\right) = \sum (-1)^i \mathrm{Tr}\left(t; H^i(X^t, \lambda(N_{X^t}^\vee)^{-1} \otimes V|_{X^t})\right) \tag{4.6}$$

Proof. The map on the top row is $f_*[V] = \sum(-1)^i[H^i(X, V)]$, write this as $\sum \alpha \otimes \mathbb{C}^{m(\alpha)} \in R(T) \otimes K(*, \mathbb{C})$, and $m(\alpha)$ is the multiplicity of α in the virtual representation $f_*[V]$. Thus

$$\begin{aligned} res_t f_*[V] &= \sum \alpha(t) \otimes \mathbb{C}^{m(\alpha)} \\ &= \sum (-1)^i \mathrm{Tr}\left(t; H^i(X, V)\right) \end{aligned}$$

In other direction, we write $res_t[V] = \sum \beta(t) \otimes V_\beta \in \mathbb{C} \otimes K(X^T)$, where β are the weights of T appearing in $\lambda(N_{X^t}^\vee)^{-1} \otimes V|_{X^t}$.

$$\begin{aligned} f_* res_t[V] &= \sum \beta(t) \otimes f_* V_\beta \\ &= \sum \beta(t) \otimes \mathbb{C}^{m(\beta)} \\ &= \sum (-1)^i \mathrm{Tr}\left(t; H^i(X^t, \lambda(N_{X^t}^\vee)^{-1} \otimes V|_{X^t})\right) \end{aligned}$$

□

5 The Weyl Character Formula

The Weyl character formula gives a very useful way to compute the characters of finite-dimensional irreducible representations of G. Here we will show that it follows quite naturally from the localization formula applied to line bundles on the flag variety.

5.1 The Character Formula

Let $[L_\lambda] \in K_G(\mathcal{B})$ be the class of the line bundle. Pushing forward along the map $p : \mathcal{B} \to *$, we get

$$p_*[L_\lambda] = \sum_i (-1)^i [H^i(\mathcal{B}, L_\lambda)] \in K^G(*) \cong R(G) \tag{5.1}$$

We will work at a regular element $t \in T$, so that $\mathcal{B}^t = \mathcal{B}^T$, and also, B^T is the set of Borel subalgebras containting $\mathfrak{t}$. These are precisely given by $B^T = \{\mathfrak{b}_w := w(\mathfrak{b})\}_{w\in W}$. Since the fixed points are isolated, the normal bundle to such a point is it's tangent bundle, and we can easily see that $T_{\mathfrak{b}_w}\mathcal{B} = \mathfrak{g}/\mathfrak{b}_w \cong n_w^-$, and thus the dual to the normal bundle is the cotangent bundle is $\mathfrak{n}_w^+ = w(\mathfrak{n}^+)$. The characters of T that appear in $N_{\mathfrak{b}_w}\mathcal{B}$ are precisely $e^{w\alpha}$, for $\alpha \in R^+$ the positive weights.

Thus we see that

$$\lambda(N_{\mathfrak{b}_w}^\vee \mathcal{B}) = \prod_{\alpha\in R^+} (1 - e^{-w\alpha}) \in R(T)$$

Now, since all the fixed points are isolated, $\mathcal{B}^t = \{\mathfrak{b}_w : w \in W\}$, the Lefschetz fixed point formula (4.6) gives us the character of the virtual representation

$$\sum_i (-1)^i \mathrm{Tr}(t, H^i(\mathcal{B}, L_\lambda)) = \sum_{w\in W} \mathrm{Tr}\left(t; \lambda(N_{\mathfrak{b}_w}^\vee)^{-1} \otimes L_\lambda|_{\mathfrak{b}_w}\right) \tag{5.2}$$

Now by construction, the element $L_\lambda|_{\mathfrak{b}_w} \in R(T)$ is given by $e^{w\lambda}$. Thus, the character is

$$\sum_{w\in W} \frac{e^{w\lambda}}{\prod_{\alpha\in R^+}(1 - e^{-w\alpha})}(t) \tag{5.3}$$

Now, we perform the standard tricks. Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha$, then we have the identity

$$\prod_{\alpha\in R^+} (1 - e^{-\alpha}) = e^{-\rho} \prod_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2}) =: e^{-\rho}\Delta \tag{5.4}$$

It is easy to check that if s is a simple reflection in W, then $s\Delta = -\Delta$ (s flips the sign of one positive root, and permutes the rest). Hence $w\Delta = (-1)^{\ell(w)}\Delta$, and so the denominator in our character formula is

$$\prod_{\alpha \in R^+} (1 - e^{-w\alpha}) = w(e^{-\rho}\Delta) = e^{-w\rho}(-1)^{\ell(w)}\Delta = e^{-w\rho}(-1)^{\ell(w)}e^{\rho} \prod_{\alpha \in R^+} (1 - e^{-\alpha}) \qquad (5.5)$$

Thus we arrive at

$$\sum_i (-1)^i \mathrm{Tr}(t, H^i(\mathcal{B}, L_\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R^+}(1 - e^{-\alpha})}(t) \qquad (5.6)$$

The finish off the proof of our main claim, we now result to the classical

Proposition 5.1 (Borel-Weil-Bott). *If λ is a dominant weight, then*

- *the space $H^0(\mathcal{B}, L_\lambda)$ is a simple G-module with highest weight $w_0(\lambda)$, i.e $H^0(\mathcal{B}, L_\lambda) = V_{w_0(\lambda)}$, where w_0 is the longest element in W.*
- *All higher cohomologies vanish, i.e. $H^i(\mathcal{B}, L_\lambda) = 0$, for $i > 0$.*

However, with our choice of positive roots, irreducible representations are classified by *anti-dominant* weights λ, this is equivalent to $w_0(\lambda)$ being dominant. So using the geometric choice of roots, we see that $H^0(\mathcal{B}, L_\lambda) \cong V_\lambda$.

References

[1] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry.* Birkhäuser Boston, Inc., Boston, MA, 1997.