1. Motivation: Hecke Algebras in Nature

$(\star)$ Let G be a finite group with the properties:

1) $\exists$ subgroups $B, N \subset G$ s.t. $B \cap N \trianglelefteq N$
2) $W := N/B\cap N$ is generated by a set of involutions $S \subset W$ s.t. (W,S) is a Coxeter system
3) Have double coset decomposition
$$G = \coprod_{w\in W} BwB$$
4) If ℓ is the length function for (W,S), have:
$$C(s)C(w) \subset \begin{cases} C(sw) & \text{if } \ell(sw) > \ell(w) \\ C(sw) \cup C(w) & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where $C(w) := BwB$.

<u>Example</u> Let G be a connected, reductive linear algebraic group over an alg closed field k of char $p>0$. A <u>Frobenius map</u> on G is an endomorphism $F: G \to G$ s.t. $\exists n \geq 1$ and an embedding $G \hookrightarrow GL_m(k)$ s.t. the diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "F^n"'] & GL_m(k) \arrow[d, "\text{standard Frobenius}"] \\ G \arrow[r, hook] & GL_m(k) \end{tikzcd}$$

commutes, where standard Frobenius is a map $(a_{ij}) \mapsto (a_{ij}^q)$ where $q = p^e$, some $e > 0$.

If B is an F-stable Borel subgroup, $T \subset B$ is an F-stable max'l torus, and $N := N_G(T)$ is the normalize of T, then (G^F, B^F, N^F) satisfy $(\star)$. G^F is finite and called a <u>finite group of Lie type</u>.

Sub-example Let $G = GL_n(\overline{\mathbb{F}_p})$, $F(a_{ij}) = (a_{ij}^q)$ w/ $q = p^e$, some $e \geq 1$. Then $G^F = GL_n(\mathbb{F}_q)$, can take $B = \uparrow\Delta$ular, $T = $ diags, so B^F has same description, and $N^F =$ monomial matrices / $\mathbb{F}_q$. $W \cong S_n$, in fact $N^F = T^F \rtimes S^n$.

Let (G, B, N) satisfy ⊛ w/ Cox system (W, S). We associate the __Hecke algebra__
$$H = \mathrm{End}_G(1_B^G).$$
We have $1_B^G = \{f : G \to \mathbb{C} : f(bg) = f(g)\}$ is the set of all left-B-invariant $\mathbb{C}$-valued functions on G. The representation is by $(gf)(x) = f(xg^{-1})$. This is the permutation representation of G on $B\backslash G$.

We recall:

__Mackey's Theorem__ Let $H_1, H_2 \subset G$ be finite groups, V_1, V_2 reps of H_1, H_2. For $\Delta : G \to \mathrm{Hom}(V_1, V_2)$ s.t. $\Delta(h_2 g h_1) = h_2 \Delta(g) h_1$, we get a map
$$V_1^G \longrightarrow V_2^G$$
$$f \longmapsto \Delta * f, \quad \text{where}$$
$$(\Delta * f)(g) = \frac{1}{|H_1|} \sum_{x \in G} \Delta(x) f(x^{-1} g)$$

Note $\Delta * f : G \to V_2$ and $(\Delta * f)(g'g) = g' \circ (\Delta * f)(g')$

Clearly $f \mapsto \Delta * f$ commutes w/ G-action on right, so $\Delta * \in \mathrm{Hom}_G(V_1^G, V_2^G)$.

__Mackey__ $\{\Delta : G \to \mathrm{Hom}(V_1, V_2) : \Delta(h_2 g h_1) = h_2 \Delta(g) h_1\}$
$$\cong \mathrm{Hom}_G(V_1^G, V_2^G)$$
$$\Delta \longmapsto \Delta *.$$

Thus $H = \mathrm{End}_G(1_B^G) \cong$ B-biinvariant $f: G \to \mathbb{C}$ as vector spaces. But have convolution

$$(\Delta_1 * \Delta_2)(g) = \frac{1}{|B|} \sum_{x \in G} \Delta_1(x) \Delta_2(x^{-1}g),$$

and w/ this product the iso is as algebras.

Thus H has a natural basis given by indicator functions of the double cosets $C(w) = BwB$. Let $\{T_w\}$ be these indicator function, so $|W| = \dim_{\mathbb{C}} H$.

We would like to understand how to multiply the T_w.

For $s \in S$, let $q_s = |BsB/B|$.

<u>Proposition</u> $|BwB/B| = q_{s_1} \cdots q_{s_k}$ when $w = s_1 \cdots s_k$ reduced.

<u>Proof</u> By induction, need to show $sw > w \Rightarrow |BsB/B||BwB/B| = |BswB/B|$. We have surjective mult. map $C(s) \times C(w) \to C(sw)$. We need to check all fibers have size $|B|$. The size of the fiber containing (x,y) is $\#\{g \in G : (xg, g^{-1}y) \in C(s) \times C(w)\}$. By Bruhat decomp, given $g \in G$ $\exists w' \in W$ w/ $g \in C(w')$. Then $xg \in C(s) \cap (C(s)C(w')) \subset C(s) \cap (C(sw') \cup C(w')) \Rightarrow s \in \{sw', w'\}$ $\Rightarrow w' \in \{1, s\}$. But can't have $w' = s$ since then $g^{-1}y \in C(w) \cap (C(s)C(w)) = C(w) \cap C(sw) = \emptyset$, so get $w' = 1$, so $g \in B$, which works. ∎

Thus $q_{s_1} \cdots q_{s_k}$ only depends on $s_1 \cdots s_k$ when reduced, so can define q_w.

Note $s \mapsto q_s$ is thus a class function on S.

<u>Proposition</u> The $\mathbb{C}$-linear map $H \longrightarrow \mathbb{C}$, $T_w \longmapsto q_w$, is a map of algebras.

<u>Proof</u> Consider $\varepsilon : H \longrightarrow \mathbb{C}$, $\varepsilon(f) = \frac{1}{|B|}\sum_{g\in G} f(g)$.

Then $\varepsilon(f * f') = \frac{1}{|B|}\sum_{x\in G}(f * f')(x)$

$$= \frac{1}{|B|}\sum_{x\in G}\frac{1}{|B|}\sum_{y\in G} f(y)f'(y^{-1}x)$$

$$= \left(\frac{1}{|B|}\sum_{x\in G} f(x)\right)\left(\frac{1}{|B|}\sum_{y\in G} f'(y)\right) = \varepsilon(f)\varepsilon(y).$$

But $\varepsilon(T_w) = \frac{1}{|B|}|BwB| = q_w$. ∎

<u>Prop</u> If $sw > w$, $T_sT_w = T_{sw}$.

<u>Proof</u> Note $(T_sT_w)(g) = \frac{1}{|B|}\sum_{x\in G} T_s(x)T_w(x^{-1}g)$.

But $T_s(x)T_w(x^{-1}g) \neq 0$

$\Rightarrow x \in C(s),\ x^{-1}g \in C(w)$

$\Rightarrow g \in C(s)C(w) = C(sw)$,

so T_sT_w is supported on $C(sw)$

$\Rightarrow T_sT_w = c(s,w)T_{sw}$

for some $c(s,w) \in \mathbb{C}$. Applying ε, get

$q_sq_w = c(sw)q_{sw} \Rightarrow c(s,w) = 1$. ∎

<u>Prop</u> $T_s^2 = q_sT_1 + (q_s - 1)T_s$.

<u>Proof</u> Since $C(s)C(s) \subset C(1) \cup C(s)$ as above, get $\exists \lambda, \mu \in \mathbb{C}$ s.t. $T_s^2 = \lambda T_1 + \mu T_s$. Evaluating at 1, get

$\lambda = T_s^2(1) = \frac{1}{|B|}\sum_{x\in G} T_s(x)T_s(x^{-1}) = \frac{1}{|B|}|C(s)| = q_s$, so

$T_s^2 = q_sT_1 + \mu T_s$. Applying ε, get

$q_s^2 = q_s + \mu q_s \Rightarrow \mu = q_s - 1$. ∎

2. Formal Parameters + Freeness

Note that for the algebras H we just constructed, H depended only on the Coxeter system (W,S), and we have, by the relations we found, a surjection of algebras

$$H' := \mathbb{C}\langle T_w : T_sT_w = \begin{cases} T_{sw} & \text{if } sw>w \\ a_sT_{sw}+b_sT_w & \text{if } sw<w \end{cases}\rangle \twoheadrightarrow H$$

We see H is spanned by the T_w, so $\dim H' \leq \dim H$ and hence the above is an iso, and H' is free on the T_w'.

We can replicate this result abstractly for any Coxeter system (W,S). Let A be a commutative ring, and let $a,b : S \to A$ be class functions, and write $a_s := a(s)$, $b_s := b(s)$. We then define the <u>generic algebra</u> $H(W,S,a,b)$ to be the A-algebra generated by $\{T_w : w\in W\}$ with relations

1) $T_s^2 = a_s 1 + b_s T_s$
2) $T_sT_w = T_{sw}$ when $sw>w$.

Note then if $sw<w$, we have

$$T_sT_w = T_sT_{s(sw)} = T_s^2T_{sw} = (a_s+b_sT_s)T_{sw} = a_sT_{sw}+b_sT_w,$$

so we could have as well said

$$T_sT_w = \begin{cases} T_{sw} & \text{if } sw>w \\ a_sT_{sw}+b_sT_w & \text{if } sw<w. \end{cases}$$

Note if we take W as before, $a_s = q_s$, $b_s = q_s - 1$, $A = \mathbb{C}$, we get the algebras we already saw.

Theorem $H(W,S,a,b)$ is free on $\{T_w\}$ over A.

Proof Consider the free A-module $E := \bigoplus_{w \in W} AT_w$. Define for $s \in S$ the operators $\lambda_s, \rho_s \in \mathrm{End}_A(E)$ by

$$\lambda_s(T_w) = \begin{cases} T_{sw} & \text{if } sw > w \\ a_s T_{sw} + b_s T_w & \text{if } sw < w \end{cases} \qquad \rho_s(T_w) = \begin{cases} T_{ws} & \text{if } ws > w \\ a_s T_{ws} + b_s T_w & \text{if } ws < w. \end{cases}$$

These are <u>optimistic</u> left- and right-multiplication operators.

Let $L \subset \mathrm{End}_A(E)$ be the A-algebra generated by the λ_s. Suppose we knew $[\lambda_s, \rho_t] = 0$ $\forall s,t$. We have the evaluation-at-T_1 map

$$ev : L \longrightarrow E.$$

It is clearly surjective, since if $w = s_1 \cdots s_k$ reduced then $ev(\lambda_{s_1} \cdots \lambda_{s_k}) = T_w$ by induction on $l(w)$. I claim it is also injective. Let $f \in \ker(ev)$. Then $f(T_1) = 0$. But if $f(T_w) = 0$ and $ws > w$, we have $f(T_{ws}) = f(\rho_s T_w) = \rho_s f(T_w) = 0$ since $[L, \rho_s] = 0$. So by induction on $l(w)$, $f(T_w) = 0$ $\forall w \Rightarrow f = 0$ $\Rightarrow$ ev is an isomorphism.

I claim $\lambda_s^2 = a_s + b_s \lambda_s$. We show this by evaluating on the T_w. Suppose first $sw > w$. Then $\lambda_s^2 T_w = \lambda_s T_{sw} = a_s T_w + b_s T_{sw} = (a_s + b_s \lambda_s) T_w$. If $sw < w$, $\lambda_s^2 T_w = \lambda_s (a_s T_{sw} + b_s T_w) = a_s T_w + b_s T_{sw} = (a_s + b_s \lambda_s) T_w$, as needed.

Since ev is an isomorphism, L has a basis given by the $\lambda_w := \lambda_{s_1} \cdots \lambda_{s_k}$ where $w = s_1 \cdots s_k$ is <u>any</u> reduced expression. But then if $sw > w$ obviously $\lambda_s \lambda_w = \lambda_{sw}$.

Thus, we have a map $H(W,S,a,b) \longrightarrow L$ of A-algebras which is surjective, $T_w \longmapsto \lambda_w$. But the T_w span $H(W,S,a,b)$ and the λ_w are A-linearly independent in L, so this is injective too. Thus $H(W,S,a,b)$ is free on the T_w.

It remains to check that $[\lambda_s, \rho_t] = 0 \ \forall s,t$. This is by comparing the action of $\lambda_s \rho_t$ and $\rho_t \lambda_s$ on the T_w. We need a lemma:

<u>Lemma</u> If $l(swt) = l(w)$, $l(sw) = l(wt)$, we have $swt = w$, $sw = wt$.

<u>Proof</u> The numbers $l(swt) = l(w)$, $l(sw) = l(wt)$ differ by 1. We can assume $l(w)$ is the small one, since we have symmetry by setting $w' = sw$, so $wt = sw't$, $swt = w't$, $w = sw'$, and $sw' = w't \Leftrightarrow w = swt$.

Thus $l(sw) > l(w)$, so can write $sw = ss_1 \cdots s_k$ reduced. But $swt < sw$, so by the exchange lemma swt has a reduced expression by deleting a term from $ss_1 \cdots s_k$. But it can't be an s_i since then we'd have $l(wt) < l(w)$, so it's s. But then $w = swt$. ∎

Back to the proof now.

1, 4

We do 6 cases:

1) $\ell(w) < \ell(sw) = \ell(wt) < \ell(swt)$.

Then $\lambda_s \rho_t T_w = T_{swt} = \rho_t \lambda_s T_w$.

2) $\ell(swt) < \ell(sw) = \ell(wt) < \ell(w)$.

Then $\lambda_s \rho_t T_w = \lambda_s(a_t T_{wt} + b_t T_w) = a_t(a_s T_{swt} + b_s T_{wt}) + b_t(a_s T_{sw} + b_s T_w)$

$= a_s(a_t T_{swt} + b_t T_{sw}) + b_s(a_t T_{wt} + b_t T_w)$

$= a_s \rho_t(T_{sw}) + b_s \rho_t(T_w)$

$= \rho_t(a_s T_{sw} + b_s T_w) = \rho_t \lambda_s T_w$.

3) $\ell(sw) < \ell(w) = \ell(swt) < \ell(wt)$

$\lambda_s \rho_t T_w = \lambda_s T_{wt} = a_s T_{swt} + b_s T_{wt}$

$\rho_t \lambda_s T_w = \rho_t(a_s T_{sw} + b_s T_w) = a_s T_{swt} + b_s T_{wt}$.

4) $\ell(wt) < \ell(w) = \ell(swt) < \ell(sw)$

$\lambda_s \rho_t(T_w) = \lambda_s(a_t T_{wt} + b_t T_w) = a_t T_{swt} + b_t T_{sw}$

$\rho_t \lambda_s T_w = \rho_t T_{sw} = a_t T_{swt} + b_t T_{sw}$.

5) $\ell(sw) = \ell(wt) < \ell(swt) = \ell(w)$

Then $\rho_t \lambda_s T_w = \rho_t(a_s T_{sw} + b_s T_w) = a_s T_{swt} + b_s(a_t T_{wt} + b_t T_w)$

while

$\lambda_s \rho_t T_w = \lambda_s(a_t T_{wt} + b_t T_w) = a_t T_{swt} + b_t(a_s T_{sw} + b_s T_w)$.

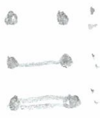
But also $sw = wt$, $swt = w$, so $s = wtw^{-1}$ so $a_s = a_t$, $b_s = b_t$, and we have equality.

6) $\ell(w) = \ell(swt) < \ell(sw) = \ell(wt)$:

$\rho_t \lambda_s T_w = \rho_t T_{sw} = a_t T_{swt} + b_t T_{sw}$ while

$\lambda_s \rho_t T_w = \lambda_s T_{wt} = a_s T_{swt} + b_s T_{wt}$. But

$sw = wt$, $a_s = a_t$, $b_t = b_s$.



$: (1+q)^8$

$: -2^4 q^2 (1+q)^2 (1+q+q^2)^6$

$: -2^{10} q^6 (1+q)^8 (1+q^2)^6$

3. Specializations and Semisimplicity

Let (W,S) be a finite Coxeter system, and let $H = H(W,S,a_s,b_s)$ be the generic Hecke algebra over $A := \mathbb{C}[\{a_s, b_s\}]$ so we have one formal variable for each conjugacy class of S. Given a $\mathbb{C}$-algebra homomorphism $\sigma : A \to \mathbb{C}$, which amounts to choosing $a_s', b_s' \in \mathbb{C}$, we have the specialization by σ defined by

$$H_\sigma := H \otimes_{A \xrightarrow{\sigma}} \mathbb{C}$$

This is a $\mathbb{C}$-algebra, and from our freeness results we see it is a $|W|$-dim'l $\mathbb{C}$-algebra with generators and relations as in section (2), i.e. $H_\sigma \cong H(W,S,a_s',b_s')$.

Basic Question What does H_σ look like? When/how often is it semisimple?

We've seen something already. When W is a Weyl group and we set $a_s' = p_s$, $b_s' = p_s - 1$, we get an endomorphism algebra of a representation of a finite group, hence is semisimple.

For general finite W, setting $a_s' = 1$, $b_s' = 0$ we get $\mathbb{C}W$, which is semisimple.

Proposition H_σ is semisimple for generic $a_s', b_s' \in \mathbb{C}$.

Proof Given a finite dimensional $\mathbb{C}$-algebra and a basis $\{b_i\}$, one can consider the discriminant $\det(\mathrm{Tr}(b_i b_j))$. This is well-defined up to nonzero multiple. I claim semisimple $\Leftrightarrow$ disc $\neq 0$. If x is in the

radical, then xy is in the radical $\forall y$, so mult by xy is a nilpotent operator, so $Tr(xy)=0$ $\forall y$. Thus x is in the kernel of the trace form. But disc $\neq 0 \Rightarrow$ Trace form nondegen $\Rightarrow x=0 \Rightarrow$ semisimplicity.
Conversely, if an algebra is semisimple it is a product of complete matrix algebras over $\mathbb{C}$, and you can check the discriminant is nonzero.

The discriminant of H is a poly in the a_s, b_s, and the disc of H_σ is the specialization of this poly at a_s', b_s'. Thus H_σ semisimple $\Leftrightarrow$ a_s', b_s' outside of a Zariski closed set. But H_{σ_1} where $\sigma_1(a_s)=1, \sigma_1(b_s)=0$ is $\cong \mathbb{C}W$ so is semisimple, so H_σ semisimple at a nonempty Zariski open set. ∎

So what do these semisimple specializations look like? Let's check out type A.

Set $W = S_n$, $q=p^k$, $G=GL_n(\mathbb{F}_q)$, $B = \uparrow\Delta$ular's, and $H = End_G(1_B^G)$. For $\lambda \vdash n$, let P_λ be the corresponding standard parabolic. Let
R = Groth group of cat of $<\infty$-dim'l $\mathbb{C}$-reps of S_n,
$R(q)$ = Groth " " " " " " " " $GL_n(\mathbb{F}_q)$.
For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \vdash n$, let $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \subset S_n$.
Then from symmetric function theory we know the $h_\lambda := Ind_{S_\lambda}^{S_n} 1$ form a $\mathbb{Z}$-basis for R. So define the linear operator $R \to R(q)$ by $h_\lambda \mapsto Ind_{P_\lambda}^G 1$.
By Mackey and Bruhat decomposition, we have
$\langle Ind_{P_\lambda}^G 1, Ind_{P_\mu}^G 1\rangle = \# P_\lambda \backslash G / P_\mu = \# S_\lambda \backslash S_n / S_\mu = \langle h_\lambda, h_\mu \rangle$.

Thus $R \to R(q)$ is an isometry. It follows it sends irreps to $\pm$irreps, and since every irrep occurs in some h_λ w/ positive multiplicity, we see it is irrep $\mapsto$ irrep. But we then see that if $\mathrm{Ind}_B^G 1 = \bigoplus n_i V_i(q)$ is the decomp into irreps, then $\mathrm{Ind}_1^{S_n} 1 = \bigoplus n_i V_i$ (with $V_i \mapsto V_i(q)$ irreps). Thus $H = \mathrm{End}_G(1_B^G) \cong \mathrm{End}_{S_n}(1_1^{S_n}) = \mathrm{End}_{S_n}(\mathbb{C}S_n) \cong \mathbb{C}S_n$. So all these H for different $q = p^k$ are not only all semisimple, but also isomorphic!

This is a general thing:

__Theorem (Tits')__ If $\sigma, \sigma' : A \to \mathbb{C}$ are so that $H_\sigma, H_{\sigma'}$ are semisimple, then $H_\sigma \cong H_{\sigma'}$. So all semisimple specializations of H are isomorphic.

__Proof__ Let $F = \mathrm{Frac}(A)$, and $\bar{F}$ be an algebraic closure. Let $H_{\bar{F}} = H \otimes_A \bar{F}$. We've seen $\mathrm{disc}(H_{\bar{F}})$ is a nonzero poly in the a_s, b_s's, so $H_{\bar{F}}$ is semisimple. So it is a product of complete matrix algebras $/\bar{F}$ of sizes $n_1, \dots, n_k$. Call these the __numerical invariants__ of $H_{\bar{F}}$. It suffices to show if $\sigma : A \to \mathbb{C}$ is such that H_σ is semisimple, then H_σ has numerical invariants $n_1, \dots, n_k$ also.

Now we adjoin more formal variables x_w for $w \in W$, and consider $H_{\bar{F}} \otimes_{\bar{F}} \bar{F}(x_w : w \in W)$ and consider the "generic element" $a = \sum_w x_w T_w$. If $P(t)$

is the characteristic poly for left mult by a, we can factor

$$P(t) = \prod P_i(t)^{e_i}$$

in $\overline{F}(x_w)[t]$ where the $P_i(t)$ are distinct irreds and the $e_i \geq 1$.

But $H_F \otimes_F \overline{F}(x_w)$ also has a direct sum decomp as $\bigoplus_i M_{n_i}(\overline{F}(x_w))$, so has a basis $\{E^l_{ij} : 1 \leq l \leq k, 1 \leq i,j \leq n_l\}$. So we can write

$$a = \sum_{i,j,l} y^l_{ij} E^l_{ij}.$$

The change of basis matrix between the T_w and E^l_{ij} has coefs in $\overline{F}$ (since we can do all this over $\overline{F}$), so we conclude $\overline{F}(x_w) = \overline{F}(y^l_{ij})$ so the y^l_{ij} are alg ind by transcendence degree reasons. But working in the E^l_{ij} basis we see

$$P(t) = \prod_l \det(t1 - y^l_{ij})^{n_l}$$

Any monic poly $g(t)$ of degree n_l is the determinant of some matrix

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ c_1 & c_2 & \cdots & 1 \end{pmatrix}$$

Since $\exists$ ireds of all degrees and the y^l_{ij} are alg ind, we can specialize y^l_{ij} s.t. $\det(t1 - y^l_{ij})$ is irred. Thus $\det(t1 - y^l_{ij})$ is irred. Clearly they are distinct for distinct l, so we conclude $P_l(t) = \det(t1 - y^l_{ij})$ and $e_l = n_l = \deg P_l(t)$.

Consider the coefs of $P_\ell(t)$. They are polys in the roots of $P_\ell(t)$, so polys in the roots of $P(t)$, which are hence integral over the coefs of $P(t)$, which lie in $A[x_\omega]$.

So, the coefs of $P_\ell(t)$ lie in the integral closure of $A[x_\omega]$ in $\overline{F}(x_\omega)$. If I is the integral closure of A in $\overline{F}$, from commutative algebra we have $\overline{A[x_\omega]} = I[x_\omega]$, so coefs of $P_\ell(t)$ lie in $I[x_\omega]$.

I claim $\sigma: A \longrightarrow \mathbb{C}$ can be extended to a algebra hom $\sigma: I \longrightarrow \mathbb{C}$. By Zorn it suffices to add one element at a time. But since $\mathbb{C}$ is alg closed we just send this elt to a root of its minimal polynomial.

Now consider the specialized algebra H_σ. Consider the generic element $\alpha = \sum x_\omega T_\omega \in H_\sigma \otimes_{\mathbb{C}} \mathbb{C}(x_\omega)$. Let $P_\sigma(t)$ be its characteristic poly. Clearly this is the specialisation of $P(t)$ by σ. Thus we have, using the extension $\sigma: I \longrightarrow \mathbb{C}$,

$$P_\sigma(t) = \prod_\ell P_{\ell,\sigma}(t)^{n_\ell}$$

where $P_{\ell,\sigma}$ is the specialization of P_ℓ by σ. But since H_σ is semisimple, we know each irred factor of $P_\sigma(t)$ occurs w/ multiplicity = degree, by 1st argument. Since $n_\ell = \deg P_{\ell,\sigma}$, the $P_{\ell,\sigma}(t)$ must therefore be irred and distinct. Thus the n_ℓ are the numerical invariants for H_σ and we win. $\blacksquare$

4. Hecke Algebras as Symmetric Algebras

Let (W,S) be a finite Coxeter system, and let A be a commutative ring, and let $a,b: S \to A$ be class functions so that $a_s \in A^\times$ $\forall s$. We will see now that $H := H_A(W,S,a,b)$ admits a non-degenerate symmetric bilinear form $(\cdot,\cdot): H \otimes_A H \to A$ such that $(xy,z) = (x,yz)$. This gives H the structure of a <u>symmetric algebra</u>.

Let $\tau: H \to A$ be the A-linear functional defined by $\tau(T_1) = 1$, $\tau(T_w) = 0$ for $w \neq 1$. So τ gives the coefficient of T_1 in an expression in the T_w basis.

Define $(\cdot,\cdot): H \otimes_A H \to A$ by $(x,y) = \tau(xy)$.

<u>Theorem</u> $(\cdot,\cdot)$ is symmetric nondegenerate bilinear form with $(xy,z) = (x,yz)$. We have explicitly

$$\tau(T_w T_{w'}) = \begin{cases} a_w & \text{if } w' = w^{-1} \\ 0 & \text{———} \end{cases},$$

where $a_w = a_{s_1} \cdots a_{s_k}$ wherever $s_1 \cdots s_k$ is a redex for w — this is well-defined since a is a class function. The dual basis is $T_w^\vee = a_w^{-1} T_{w^{-1}}$. In particular $\tau(xy) = \tau(yx)$ so τ is a <u>trace function</u>.

Proof Clearly $(xy,z)=(x,yz)$. The rest follows immediately from the explicit formula. We prove that by induction on $l(w)$.

If $l(w)=0$, $w=1$ so nothing to prove.

Let $l(w)>0$. Then $\exists s\in S$ s.t. $ws<w$. We then have for $w'\in W$

$$\tau(T_wT_{w'})=\tau(T_{ws}T_sT_{w'}).$$

Case 1 $sw'>w'$. Then $\tau(T_wT_{w'})=\tau(T_{ws}T_{sw'})$

Now $sw'=(ws)^{-1}\iff w'=w^{-1}$. But then $ws<w$ $\Rightarrow sw'=sw^{-1}<w^{-1}=w'$, contradiction, so $w'\neq w^{-1}$ and $\tau(T_wT_{w'})=\tau(T_{ws}T_{sw'})=0$ by induction, so the formula holds.

Case 2 $sw'<w'$. Then

$$\tau(T_wT_{w'})=\tau(T_{ws}T_sT_{w'})=\tau(T_{ws}(a_sT_{sw'}+b_sT_{w'}))$$
$$=a_s\tau(T_{ws}T_{sw'})+b_s\tau(T_{ws}T_{w'}).$$

Note $w'=(ws)^{-1}\iff w'=sw^{-1}<w^{-1}=sw'$, contradiction, so 2^{nd} term $=0$ by induction.

As before $ws=(sw')^{-1}\iff w'=w^{-1}$, so if $w'\neq w^{-1}$ get 0, and otherwise get $a_sa_{ws}=a_w$ since $ws<w$. ∎