KAZHDAN-LUSZTIG CELLS

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ABSTRACT. These are notes for a talk on Kazhdan-Lusztig Cells for Hecke Algebras. In this talk, we construct the Kazhdan-Lusztig basis for the Hecke algebra associated to an arbitrary Coxeter group, in full multiparameter generality. We then use this basis to construct a partition of the Coxeter group into the Kazhdan-Lusztig cells and describe the corresponding cell representations. Finally, we specialize the construction to the case of the symmetric group. The main references for the talk are [Lus14, GJ11, Wil].

CONTENTS

1. Hecke Algebra associated to a Weighted Coxeter Group

We begin by defining the notion of a weighted Coxeter group.

Definition 1.1. Let W, S be a Coxeter system. Let $l : W \to \mathbb{Z}$ be the length function of the Coxeter group. Then, a weight function on W is a map $L : W \to \mathbb{Z}$ such that

$$l(ww') = l(w) + l(w') \Rightarrow L(ww') = L(w) + L(w').$$

We call the pair (W, L) a weighted Coxeter group.

Remark. Note that the additivity condition on the weight function is equivalent to the statement that a weight function is additive on reduced decompositions in W and is hence determined by its values on S. In fact, a weight function can be specified by giving arbitrary weights to elements in S subject to the sole condition that if m_{st} is odd, then $L(s) = L(t)$.

Remark. Because reduced decompositions for w^{-1} are obtained by reversing reduced decompositions for w, we have $L(w) = L(w^{-1})$.

Throughout the rest of the talk, let us fix a weighted Coxeter group W, L. Fix some field k of characteristic 0. We now define the generic Iwahori-Hecke algebra associated to W, L.

Definition 1.2. Let $A = k[q, q^{-1}]$ be the algebra of Laurent polynomials over k and for $s \in S$, let $q_s = q^{L(s)}$. Then, the (generic) Iwarhori-Hecke algebra $\mathcal{H}$ associated to W, is the A-algebra with generators $\{T_s : s \in S\}$ and relations

1. Eigenvalue Relation: $(T_s - q_s)(T_s + q_s^{-1}) = 0$
2. Braid Relation: $T_sT_t \cdots = T_tT_s \cdots$ (with m_{st} many factors on each side).

As a consequence of the defining relations we have

Proposition 1.3. $\mathcal{H}$ is free over A with basis T_w. In this basis, the multiplication formula can be described as follows. For $s \in S, w \in W$

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w) \\ T_{sw} + (q_s - q_s^{-1})T_w & \text{if } l(sw) < l(w) \end{cases}.$$

2. The Bar Involution and the Kazhdan-Lusztig Basis

Let $a \mapsto \bar{a} : A \to A$ be the k-algebra involution defined by sending q to q^{-1}. $a \mapsto \bar{a}$ extends to a semilinear involution on $\mathcal{H}$ as follows:

Proposition 2.1. There is a unique $(A, \bar{\ })$-semilinear ring homomorphism $x \mapsto \bar{x} : \mathcal{H} \to \mathcal{H}$ defined by sending $T_s \mapsto T_s^{-1}$. This homomorphism is involutive and sends T_w to $T_{w^{-1}}^{-1}$ for each $w \in W$.
This map is known as the bar involution on $\mathcal{H}$.

Definition 2.2. For $w, y \in W$ we define $r_{w,y} \in A$ by

$$\bar{T}_w = \sum_{y \in W} \bar{r}_{y,w} T_y.$$

Remark. Note that $r_{w,w} = 1$.

Using the bar involution, we can now construct the Kaszhdan-Lusztig basis for $\mathcal{H}$.

Definition 2.3. For an integer n, define

$$A_{\leq n} = \bigoplus_{m \leq n} kq^m.$$

Similarly define $A_{\geq n}$, $A_{<n}, A_{>n}$. With this definition in hand, define

$$\mathcal{H}_{\leq 0} = \bigoplus_w A_{\leq 0} T_w$$

and

$$\mathcal{H}_{<0} = \bigoplus_w A_{<0} T_w.$$

Theorem 2.4. (Kazhdan-Lusztig Basis) Let $w \in W$. There exists a unique element $C_w \in \mathcal{H}_{\leq 0}$ such that

$$\bar{C}_w = C_w \text{ and } C_w \equiv T_w \mod \mathcal{H}_{<0}.$$

Additionally, $C_w \in T_w + \sum_{y<w} A_{<0} T_y$ (where $y < w$ is in the Bruhat-Chevalley order on W) and $\{C_w : w \in W\}$ is an A-basis for $\mathcal{H}$.

Proof. To prove the theorem, we need to prove the following Lemma regarding $r_{w,y}$. Before stating the lemma, recall the Bruhat-Chevally order on W: $x \leq y$ if x can be obtained from a reduced expression for y by removing some of the elements of S. Note that $x \leq y$ implies that $l(x) \leq l(y)$ with equality if and only if $x = y$. Now,

Lemma 2.5. The following two properties hold:

1. For any $x, z \in W$,

$$\sum_{y \in W} \bar{r}_{x,y} r_{y,z} = \delta_{x,z}.$$

2. For any $x, y \in W$, let $s \in S$ be such that $y > sy$. Then,

$$r_{x,y} = \begin{cases} r_{sx,sy} & \text{if } sx < x \\ r_{sx,sy} + (v_s - v_s^{-1}) r_{x,sy} & \text{if } sy > y \end{cases}.$$

3. If $r_{x,y} \neq 0$, then $x \leq y$.

Proof of Lemma. Property 1 follows from the fact that $^-$ is an involution. Property 2 follows from the formula for $T_s T_w$ using the fact that $^-$ is multiplicative. To prove property 3, we induct on the length of y. The case of $l(y) = 0$ is obvious. So suppose $l(y) > 0$. Choose some s such that $sx < x$. Suppose first that $sx < x$. Then, by property 2,

$$r_{sx,sy} = r_{x,y} \neq 0$$

and hence by induction $sx \leq sy$ which implies that $x \leq y$. On the other hand, if $sx > x$, then by property 2, either $r_{sx,sy} \neq 0$ or $r_{x,sy} \neq 0$. In the first case, by induction, $x \leq sx \leq sy < y$ and in the second case $x \leq sy < y$. This proves the Lemma.

□

We now return to the proof of the existence and uniqueness of the Kazhdan-Lusztig basis. We first prove existence. Fix $w \in W$. For any $x \leq w$, we construct an element $u_x \in A_{\leq 0}$ such that

1. $u_w = 1$.
2. for $x < w$, $u_x \in A_{<0}$ and

$$\bar{u}_x - u_x = \sum_{y : x < y \leq w} r_{x,y} u_y.$$

We induct on $l(w) - l(x) \geq 0$. For 0, $x = w$ and hence $u_x = u_w$. By the inductive hypothesis, u_y is defined for all $y \leq w$ such that $l(y) > l(x)$ and satisfies the above properties. Hence, the term

$$a_x = \sum_{y : x < y \leq w} r_{x,y} u_y$$

is defined. We show that $a_x + \bar{a}_x = 0$. This follows from the previous Lemma and the following computation:

$$\begin{aligned} a_x + \bar{a}_x &= \sum_{y : x < y \leq w} r_{x,y} u_y + \bar{r}_{x,y} (u_y + \sum_{z : y < z \leq w} r_{y,z} u_z) \\ &= \sum_{z : x < z \leq w} r_{x,z} u_z + \sum_{z : x < z \leq w} \bar{r}_{x,z} u_z + \sum_{z : x < z \leq w} \sum_{y : x < y < z} \bar{r}_{x,y} r_{y,z} u_z \\ &= \sum_{z : x < z \leq w} r_{x,z} u_z + \sum_{z : x < z \leq w} \bar{r}_{x,z} u_z + \sum_{z : x < z \leq w} \delta_{x,z} u_z - r_{x,z} u_z + \bar{r}_{x,z} u_z = 0 \end{aligned}$$

Hence, $a_x = \sum_{n \in Z} c_n q^n$ where $c_n + c_{-n} = 0$. Define

$$u_x := -\sum_{n < 0} c_n q^n.$$

Then, u_x satisfies properties 1 and 2, as desired. Now, define the Kazhdan-Lusztig element associated to w as

$$C_w := \sum_{y:y\le w} u_y T_y \in \mathcal{H}_{\le 0}.$$

Clearly, C_w satisfies the properties stated in the theorem, apart perhaps from invariance under the bar involution. This we verify with the following calculation:

$$\begin{aligned}\bar{C}_w &= \sum_{y:y\le w} \bar{u}_y \bar{T}_y = \sum_{y:y\le w} \bar{u}_y \sum_{x:x\le y} \bar{r}_{x,y} T_x = \sum_{x:x\le w} \left(\sum_{y:x\le y\le w} \bar{r}_{x,y}\bar{u}_y \right) T_x \\ &= \sum_{x:x\le w} (\bar{a}_x + u_x) T_x = \sum_{x:x\le w} u_x T_x = C_w\end{aligned}$$

This completes the proof of existence. To prove uniqueness, it suffices to prove that if $h \in \mathcal{H}_{<0}$ satisfies $\overline{h} = h$, then $h = 0$. Since $h \in \mathcal{H}_{<0}$, we can write h uniquely as $\sum_{y\in W} f_y T_y$, where $f_y \in A_{<0}$. Suppose for contradiction that not all $f_y = 0$. Choose y_0 with $f_{y_0} \ne 0$ maximal among such in the Bruhat-Chevalley order. Then, since h is bar invariant, we have

$$\sum_y f_y T_y = \sum_y \bar{f}_y \bar{r}_{x,y} T_x.$$

Since $r_{y_0,y_0} = 1$ and $r_{y_0,y} = 0$ for all $y < y_0$, we see that the coefficient of T_{y_0} on the left is f_y and on the right is $\bar{f}_y$, which are not equal. This gives us a contradiction. Hence, $h = 0$ and we have uniqueness.

The last statement of the theorem is obvious. By construction and uniqueness, C_w has the desired form and by upper triangularity (with respect to the Bruhat-Chevally order), $\{C_w : w \in W\}$ is a basis for $\mathcal{H}$ over A.

□

3. Cells and Cell Representations

The Kazhdan-Lusztig basis of a Hecke algebra can be computed recursively but is difficult to compute. However, we can now use this basis to construct cells on the Coxeter group which has a much nicer description. We begin with an abstract definition of cells.

Definition 3.1. Let $\mathcal{A}$ be an associative algebra with a basis $\{a_w : w \in W\}$ indexed by a weighted Coxeter group W, L. We say that an ideal in $\mathcal{A}$ is based if it is spanned by basis elements a_w. For, $x \in W$ we define three ideals $I_{x,L}, I_{x,R}, I_{x,LR}$ which are respectively the left, right and two-sided based ideals generated by a_x.

Define the preorder $\le_L$ (resp. $\le_R, resp. \le_{LR}$) as $x \le_L y$ if $a_x \in I_{y,L}$ (resp. $a_x \in I_{y,R}$, resp. $a_x \in I_{y,LR}$.) Let $\sim_L, (resp. \sim_R, resp. \sim_{LR})$ be the corresponding equivalence relations. Then, we call the corresponding equivalence classes the left cells (resp. right cells, resp. two-sided cells) of W (with respect to $\mathcal{A}$ and its chosen basis).

Remark. Note that $x \sim_L y$ if and only if they generate the same based left ideal (and similarly for the other two relations).

We now apply this definition to $\mathcal{A} = \mathcal{H}$. If we use the standard basis, however, we only get one left, right or two sided cell (because the basis elements T_w are all invertible). Instead, we apply the definition to the Kazhdan-Lusztig basis of $\mathcal{H}$. The resulting cells are called the (left, right, two-sided) Kazhdan-Lusztig cells, which we will abbreviate as KL cells. In the case of $\mathcal{H}$, we will also use $\mathcal{H}_{\le_L x}$ to denote $I_{x,L}$ and similarly for the right and two-sided ideals.

Remark. The map $w \mapsto w^{-1}$ carries left cells to right cells and vice versa. This is because the map $C_w \mapsto C_{w^{-1}}$ defines an anti-involution on $\mathcal{H}$.

From now on, the preorders and cells are defined with respect to the Kazhdan-Lusztig basis. We now use cells to construct representations of $\mathcal{H}$. We begin by introducing some notation:

Definition 3.2. Let $w \in W$. Define

$$\mathcal{H}_{<Lw} = \bigoplus_{x \leq w} AC_x$$

and define similar notions for the right and two-sided relations.

Note that all of the above constructions depend only on the cell of w and hence we also use the notation $\mathcal{H}_{\leq_L \mathcal{C}}$ where $\mathcal{C}$ is the cell corresponding to w. Additionally, both $\mathcal{H}_{\leq_L \mathcal{C}}$ and $\mathcal{H}_{<_L \mathcal{C}}$ are left ideals in $\mathcal{H}$. Hence, we have the following defintion:

Definition 3.3. Define the left cell module associated to $\mathcal{C}$ as $L_{\mathcal{C}} = \mathcal{H}_{\leq_L \mathcal{C}}/\mathcal{H}_{<_L \mathcal{C}}$. Similarly, define $R_{\mathcal{C}}$ and $LR_{\mathcal{C}}$. Note that the above proposition shows that these are respectively left, right and two-sided $\mathcal{H}$-modules.

Finally, note that the definition of cells immediately implies the following decomposition.

Proposition 3.4. As a left $\mathcal{H}$-module (after base changing to $k(q)$), we have

$$\mathcal{H} \cong \oplus_{\mathcal{C}} L_{\mathcal{C}}.$$

We have similar decompositions over $R_{\mathcal{C}}$ and $LR_{\mathcal{C}}$.

Proof. This follows from the fact that $L_{\mathcal{C}}$ has $\{C_w : w \in \mathcal{C}\}$ as a basis and that $W = \sqcup \mathcal{C}$ with disjoint union taken over all cells. □

4. Examples of Cells: The case of Type A

We end the talk by describing the left, right and two-sided cells (and the corresponding modules) in Type A i.e. when $W = S_n$ for some n. Before giving this description, we have to recall the RSK algorithm.

Definition 4.1. (Row Bumping Algorithm) Let T be a semistandard Young tableau and let i be a positive integer. We describe a new semistandard Young tableau denoted $T \leftarrow i$ as follows:

If i is greater than or equal to every element in row 1, then i is added in a new box at the end of row 1. Otherwise, i replaces the leftmost number greater than i. This new number, i_2, is then added to row 2 in the same manner. The process continues until one of the numbers is added at the end of a row (which may have been of length 0 in T)

This algorithm is called the Row Bumping Algorithm.

Definition 4.2. (RSK Correspondence) Let $w \in S_n$ and let the one-line notation of w be $w_1 \cdots w_n$, where $w_i = w(i)$. The RSK algorithm inductively defines a pair of standard Young tableau, P_i, Q_i as follows:

1. $P_0 = Q_0 = \phi$.
2. $P_{i+1} = P_i \leftarrow w_{i+1}$.
3. Q_{i+1} adds a box labelled with $i+1$ in the location of $P_{i+1} \backslash P_i$.

Let $P(w) = P_n, Q(w) = Q_n$. Then the map $w \mapsto (P(w), Q(w))$ is called the RSK correspondence.

Remark. It is well-known that the RSK correspondence establishes a bijection between S_n and the set of pairs of standard Young tableau of the same shape.

The RSK correspondence can now be used to describe the left, right and two sided cells in S_n. We omit the details and simply give the description. Details can be found in [Wil].

Proposition 4.3. For $x, y \in S_n$,

1. $x \sim_R y \Leftrightarrow P(x) = P(y)$.
2. $x \sim_L y \Leftrightarrow Q(x) = Q(y)$.
3. $x \sim_{LR} y \Leftrightarrow P(x)$ has the same shape as $P(y)$.

We finish by describing the cell modules.

Proposition 4.4. Let $\mathcal{C}$ be a cell (left, right or two-sided determined by context). Then,

1. The left cell module associated to $\mathcal{C}$ is the Specht module associated to the Young diagram determined by $P(x)$ for any $x \in \mathcal{C}$.
2. The corresponding right cell module is the dual of the Specht module (viewed as a right module over the algebra $\mathcal{H}$).
2. The corresponding two sided cell module is the endomorphism algebra of the Specht module.

References

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