Kostant slices.

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transitive closure of the pre-order given by the arrows is the scheme of dependence.

1) Principal $\mathfrak{sl}_2$-triple.

In what follows $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ w. triangular decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$. Let α_i, $i=1,\dots r$, be the simple roots, $e_i\in\mathfrak{g}_{\alpha_i}$, $f_i\in\mathfrak{g}_{-\alpha_i}$ be the root vectors normalized so that $[e_i,f_i]=\alpha_i^\vee$, the simple coroot. We consider $\rho^\vee\in\mathfrak{h}$, the sum of fundamental coweights.

Let $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}$ denote the standard Borel. By G we denote a connected alg. group w. $\mathrm{Lie}(G)=\mathfrak{g}$, it acts on $\mathfrak{g}$ via the adjoint action.

1.1) Construction

Set $e=\sum_{i=1}^r e_i$, $h=2\rho^\vee$ & $f=\sum_{i=1}^r m_i f_i$, where $m_i\in\mathbb{Z}$ are determined by $2\rho^\vee = \sum_{i=1}^r m_i \alpha_i^\vee$ (makes sense b/c $2\rho^\vee = \sum_{\alpha>0} \alpha^\vee$ lies in the coroot lattice).

Exercise: The elements $e, h, f \in \mathfrak{g}$ satisfy the $\mathfrak{sl}_2$-relations: $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$.

Example: For $\mathfrak{g}=\mathfrak{sl}_n$, $h=\mathrm{diag}(n-1, n-3, \dots, n-1-2i, \dots, 1-n)$, hence $m_i=(n-i)i$. So e,h,f are the images of the corresponding elements of $\mathfrak{sl}_2$ under the n-dimensional irreducible representation (in its standard basis).

1.2) $\mathfrak{g}$ as $\mathfrak{sl}_2$-representation

We can view $\mathfrak{g}$ as an $\mathfrak{sl}_2$-representation via operators $\mathrm{ad}(e)$, $\mathrm{ad}(h)$, $\mathrm{ad}(f)$. Let $\mathfrak{g}_i$ be the i-th weight space, i.e.

$$\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h,x]=ix\}$$

Note that $\mathfrak{h} \subset \mathfrak{g}_0$, while for a root α, we have $\mathfrak{g}_\alpha \subset \mathfrak{g}_{2\rho^\vee(\alpha)}$. It follows that $\mathfrak{h}=\mathfrak{g}_0$,

(1) $$\mathfrak{g}_i = \{0\} \text{ for } i \text{ odd}$$

& $\mathfrak{g}_i = \bigoplus \mathfrak{g}_\alpha$, where the sum is over the roots α w. $\rho^\vee(\alpha) = i/2$ for nonzero even i. In particular,

(2) $\mathfrak{b} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$, $\mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}_i$.

We will also need a more subtle fact (about root systems).

Fact: Let $d_1, \ldots d_r$ be the degrees of free homogeneous generators of the algebra of invariants $\mathbb{C}[\mathfrak{g}]^G$. Then
$\mathfrak{g} \underset{\mathfrak{sl}_2}{\simeq} \bigoplus_{i=1}^{r} V(2d_i - 2)$, where $V(m)$ denotes the irreducible $\mathfrak{sl}_2$-module w. highest weight m.

Exercise: Prove this for $\mathfrak{g} = \mathfrak{sl}_n$ (hint: compute the centralizer of e & the eigenvalues of $\mathrm{ad}(h)$ there).

1.3) Some corollaries

Let $\mathcal{N} \subset \mathfrak{g}$ denote the nilpotent cone of $\mathfrak{g}$, i.e. the subvariety of all nilpotent elements. In the next proposition, the closures are taken in the Zariski topology.

Proposition: 1) $\overline{\mathbb{B}e} = \mathfrak{n}$,

2) $\overline{Ge} = \mathcal{N}$.

Proof:

1): Combining (1) & (2) in Sec 1.2 we see that $[\mathfrak{b}, e] = \mathfrak{n}$. The l.h.s. is $T_e \mathbb{B}e$. Hence $\mathbb{B}e$ is dense in $\mathfrak{n}$ & we are done.

2): follows from 1) & the observation that $Gx \cap \mathfrak{n} \neq \emptyset$ $\forall$ nilpotent element $x \in \mathfrak{g}$. $\square$

Exercise: $Gf = Ge$.

2) Slices & their basic properties.

2.1) Construction.

Our goal now is to construct a transverse slice to Gf inside of $\mathfrak{g}$. By this we mean an affine subspace S w. $f \in S$ & $T_f S \oplus T_f(G) = T_f \mathfrak{g} (= \mathfrak{g})$. Note that

$$T_f(Gf) = [\mathfrak{g}, f] = \operatorname{im} \operatorname{ad}(f).$$

By the representation theory of $\mathfrak{sl}_2$ we know that

(1) $\ker \mathrm{ad}(e) \oplus \mathrm{im}\, \mathrm{ad}(f) = \mathfrak{g}$ so we set

$S := f + \ker \mathrm{ad}(e)$. This is the Kostant slice.

2.2) Properties

An awesome feature of S is that it comes equipped w. a contracting $\mathbb{C}^\times$-action. Namely, assume that G is simply connected. Then $\exists!$ algebraic group homomorphism

$$\gamma: \mathbb{C}^\times \longrightarrow G$$

w. $d_1\gamma = \rho^\vee \left(= \frac{h}{2}\right)$. Note that $\mathrm{Ad}(\gamma(t))x = t^i x$ for $x \in \mathfrak{g}_{2i}$, in particular, $\mathrm{Ad}\,\gamma(t) f = t^{-1} f$ & $\mathrm{Ad}\,\gamma(t)$ acts on $\ker \mathrm{ad}(e)$ by non-negative powers of t. By Fact in Sec 1.2, these powers are $d_i - 1$, $i = 1, \ldots r$.

Consider the following action of $\mathbb{C}^\times$ on $\mathfrak{g}$

$$t \cdot x = t\, \mathrm{Ad}(\gamma(t))x.$$

We see that $\mathbb{C}^\times$ fixes f and acts on $\ker \mathrm{ad}(e)$ by positive powers of t. So it restricts to S and moreover,

(1) $$\lim_{t \to 0} t \cdot s = f \;\; \forall\, s \in S.$$

Remark: the following observation will be very important in Sec 4: $\mathbb{C}[S]$ acquires a grading from the $\mathbb{C}^\times$-action and it's isomorphic to $\mathbb{C}[\mathfrak{g}]^G$ as a graded algebra.

Here's another nice application of the contracting $\mathbb{C}^\times$-action:

Exercise: Show that $Gf \cap S = \{f\}$.

3) Kostant slice vs $f+\mathfrak{b}$.

By (2) in Sec 1.2, $\ker \operatorname{ad}(e) \subset \mathfrak{b}$. The maximal unipotent subgroup $N \subset G$ acts on $\mathfrak{g}$ via the adjoint action. Observe that $Nf \subset f+\mathfrak{b}$ (hint: $N = \exp(\mathfrak{n})$) so $f+\mathfrak{b}$ is N-stable. The following important result of Kostant relates the slice S (that lies in $f+\mathfrak{b}$) to the action of N on $f+\mathfrak{b}$.

Proposition: The map $\alpha: N\times S \longrightarrow f+\mathfrak{b}$, $(n,s)\mapsto \operatorname{Ad}(n)s$ is an isomorphism.

Proof:

Step 1: we claim that $d_{(1,f)}\alpha: T_1 N \oplus T_f S \xrightarrow{\sim} T_f(f+\mathfrak{b}')$

Indeed, $T_1 N \xrightarrow{\sim} \mathfrak{n}$, $T_f S \xrightarrow{\sim} \ker ad(e)$, $T_f(f+\mathfrak{b}') \xrightarrow{\sim} \mathfrak{b}'$. Under these identifications, $d_{(1,f)}\alpha$ becomes $(x,y) \mapsto [x,f]+y$. Now recall ((1) & (2) in Sec 1.2) that $\mathfrak{g}_i = \{0\}$ for odd i, $\mathfrak{b}' = \bigoplus_{i\geqslant 0} \mathfrak{g}_i$, $\mathfrak{n} = \bigoplus_{i\geqslant 0} \mathfrak{g}_i$. The claim that $d_{(1,f)}\alpha$ is an isomorphism is an exercise (in the representation theory of $\mathfrak{sl}_2$).

Step 2: To deduce that α is an isomorphism we use suitable $\mathbb{C}^\times$-actions. Namely, consider the actions

$t\cdot(n,s) = (\gamma(t)n\gamma(t)^{-1}, t\cdot s)$ on $N\times S$

$t\cdot x = t\gamma(t)x$ on $f+\mathfrak{b}'$, $t\in\mathbb{C}^\times$, $n\in N$, $s\in S$, $x\in f+\mathfrak{b}'$.

Exercise: • α is $\mathbb{C}^\times$-equivariant

• the $\mathbb{C}^\times$-actions contract $N\times S$ to $(1,f)$ & $f+\mathfrak{b}'$ to f (cf. (1) in Sec 2.2).

Step 3: The claim that α is an isomorphism now follows from combining Steps 1 & 2 w. the following general claim.

Exercise: Let $\varphi: X \to Y$ be a morphism of two affine spaces. Suppose that

(i) $d_x\varphi: T_xX \xrightarrow{\sim} T_{\varphi(x)}Y$ for some $x \in X$.

(ii) $\mathbb{C}^\times \curvearrowright X, Y$ contracting X, Y to x, y respectively & φ is $\mathbb{C}^\times$-equivariant.

Show that φ is an isomorphism $\square$

4) Kostant slice vs $\mathfrak{g}/\!/G$.

We write $\mathfrak{g}/\!/G$ for the categorical quotient for the action of G on $\mathfrak{g}$, i.e. $\mathfrak{g}/\!/G = \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G)$. The inclusion $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ gives rise to the dominant morphism $\pi: \mathfrak{g} \to \mathfrak{g}/\!/G$ called the quotient morphism. The following important result is due to Kostant.

Theorem: $\pi|_S: S \to \mathfrak{g}/\!/G$ is an isomorphism

Proof:

Step 1: We claim that $\pi|_S$ is dominant. Consider the action map $\beta: G\times S \to \mathfrak{g}$, $(g,s)\mapsto \mathrm{Ad}(g)s$. By (1) in Sec 2.1, the map $d_{(1,f)}\beta: \mathfrak{g}\times \mathfrak{z}_{\mathfrak{g}}(e) \to \mathfrak{g}$, $(x,y)\mapsto [x,f]+y$, is surjective. So β is dominant $\Leftrightarrow \mathbb{C}[\mathfrak{g}]\hookrightarrow \mathbb{C}[G\times S] \Rightarrow \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[G\times S]^G =$ $= \mathbb{C}[S] \Leftrightarrow \pi|_S$ is dominant.

Step 2: The usual $\mathbb{C}^\times$-action on $\mathfrak{g}$ by dilations gives rise to an action of $\mathbb{C}^\times$ on $\mathfrak{g}/\!/G$. Notice that the latter coincides w. the action induced by $\mathbb{C}^\times \curvearrowright \mathfrak{g}/\!/G$: $(t,x)\mapsto t\cdot x$ (b/c π is G- and hence $\mathrm{Ad}(\gamma(t))$-invariant). It follows that $\pi|_S$ is $\mathbb{C}^\times$-equivariant.

Step 3: Consider the homomorphism $\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[S]$ induced by $\pi|_S$. By Step 1, it's injective. By Step 2, it's graded. But thx to Remark in Sec 2.2, the algebras in question are graded polynomial algebras w. generators of the same positive degrees. Any injective graded algebra homomorphism between such graded algebras is an isomorphism. $\square$

5) Kostant slice vs regular elements.

Recall that an element $x\in\mathfrak{g}$ is regular if

$$\dim \ker \operatorname{ad}(x) = \operatorname{rk}\mathfrak{g} \Leftrightarrow \dim Gx = \dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$$

It's known that regular semisimple elements form a non-empty open subset. It follows that the set $\mathfrak{g}^{reg}$ of regular elements in $\mathfrak{g}$ is Zariski open & non-empty.

Example: we claim that e (equiv, f) is regular. An easy way to see this as follows: $h=2\rho^\vee$ is regular and since $\mathfrak{g}$ is the sum of irreducible $\mathfrak{sl}_2$-representations w. even weights we have $\dim\ker\operatorname{ad}(e) = \dim\ker\operatorname{ad}(h)$. So e is regular as well.

Note that thx to 2) of Proposition in Sec 1.3,

$$\mathfrak{g}^{reg}\cap\mathcal{N} = Ge.$$

The main result of this section is as follows:

Proposition: 1) For a G-orbit $\mathbb{O}\subset\mathfrak{g}$ TFAE:

a) $\mathbb{O}$ consists of regular elements

b) $\mathbb{O}\cap S\neq\emptyset$.

2) $\mathbb{O}\cap S$ is a single point $\forall$ regular orbit $\mathbb{O}$.

3) Each fiber of $\pi:\mathfrak{g}\to\mathfrak{g}/\!/G$ contains a unique regular orbit.

Proof: 1):

b) $\Rightarrow$ a): Note that $\mathfrak{g}^{reg}$ is stable w.r.t. the adjoint action of G & the dilation action of $\mathbb{C}^\times$. So $\mathfrak{g}^{reg}$ is stable under $\mathbb{C}^\times\curvearrowright\mathfrak{g}$, $(t,x)\mapsto t.x$. Since $\mathfrak{g}^{reg}$ is open & contains f (see Example), it contains all points contracted to f by the $\mathbb{C}^\times$-action. By (1) in Sec 2.2, $S\subset\mathfrak{g}^{reg}\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Consider the action map $\beta: G\times S\longrightarrow\mathfrak{g}$, $(g,s)\mapsto Ad(g)s$. It's smooth at $(1,f)$ (see Step 1 of the proof of Thm in Sec 4). The map β is $G\times\mathbb{C}^\times$-equivariant, where $(g,t).(g',s)=(gg'\gamma(t)^{-1}, t\cdot s)$ & $(g,t).x=t\,Ad(g)x$ $(g,g'\in G,\ s\in S,\ t\in\mathbb{C}^\times,\ x\in\mathfrak{g})$. So the locus, where β is smooth is open, $G\times\mathbb{C}^\times$-stable & contains $(1,f)$. Such a locus must coincide w. $G\times S$ (exercise), so β is smooth. In particular, $\operatorname{im}\beta\subset\mathfrak{g}$ is open & G-stable.

Now let $\mathbb{O} \subset \mathfrak{g}^{reg}$ be an orbit. Consider the subvariety $X := \overline{\mathbb{C}^\times \mathbb{O}} \subset \mathfrak{g}$. It's G- & $\mathbb{C}^\times$-stable. We claim that $\mathcal{N} \subset X$.

Since X is G-stable & closed, the general properties of the quotient morphisms (deduced from the complete reducibility of the rational representations of G) tell us that $\pi(X) \subset \mathfrak{g}/\!/G$ is closed. Also $\pi(X)$ is $\mathbb{C}^\times$-stable. Note that $\mathbb{C}^\times$ contracts $\mathfrak{g}/\!/G$ to $\pi(0)$. So $\pi(0) \in \pi(X)$. So $\mathcal{N} \cap X = \pi^{-1}(\pi(0)) \cap X$ has dimension that is $\geqslant \dim \pi^{-1}(x) \cap X$ for $x \in \mathbb{O}$ (by semi-continuity of dimensions of fibers).

Since $\mathbb{O} \subset \pi^{-1}(X) \cap X$, we see that $\dim \pi^{-1}(x) \cap X \geqslant \dim \mathbb{O} = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$. On the other hand, $\mathcal{N} =$ [2) of Prop. in Sec 1.3] $= \overline{Ge}$ has $\dim = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$. So

$$\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} = \dim \mathcal{N} \geqslant \dim (X \cap \mathcal{N}) \geqslant \dim \mathbb{O} = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}.$$

Since $\mathcal{N}$ is irreducible, we see that $\mathcal{N} = X \cap \mathcal{N} \Leftrightarrow \mathcal{N} \subset X$.

To see that $\mathbb{O} \cap S \neq \emptyset$ we observe that this is equivalent to $\mathbb{O} \subset \mathrm{im}\, \beta$. Assume the contrary: $\mathbb{O} \not\subset \mathrm{im}\, \beta$; since $\mathrm{im}\, \beta$ is G-stable this is equivalent to $\mathbb{O} \cap \mathrm{im}\, \beta = \emptyset \Leftrightarrow X \cap \mathrm{im}\, \beta = \emptyset$.

Since $G e \subset \operatorname{im} \beta$, we arrive at a contradiction w. $N \subset X$.

2) & 3) : exercuses - use Theorem from Sec 4. $\square$

Corollary (of the proof)

$\pi|_{\mathfrak{g}^{reg}} : \mathfrak{g}^{reg} \to \mathfrak{g}/\!/G$ is smooth.

Proof: Consider the following diagram, it's commutative

$$\begin{tikzcd} G \times S \arrow[r, "pr_2"] \arrow[d, "\beta"] & S \arrow[d, "\pi|_S", "s"'] \\ \mathfrak{g}^{reg} \arrow[r, "\pi|_{\mathfrak{g}^{reg}}"'] & \mathfrak{g}/\!/G \end{tikzcd}$$

pr_2 is clearly smooth, so is $\pi|_S \circ pr_2 = \pi|_{\mathfrak{g}^{reg}} \circ \beta$. Since β is surjective (by 1) of Proposition), we see that $d_x \pi$ is surjective $\forall\, x \in \mathfrak{g}^{reg}$. This means that $\pi|_{\mathfrak{g}^{reg}}$ is smooth. $\square$