Def: Let G be alg'c group. By a character of G we mean alg. group homom'm $G\to\mathbb{C}^\times$. Characters form a group denoted by $\mathfrak{X}(G)$.

We've seen that for unipotent G, we have $\mathfrak{X}(G)=\{e\}$, while for $G=(\mathbb{C}^\times)^n$, we have $\mathfrak{X}(G)=\mathbb{Z}^n$. Exer: $\mathfrak{X}(G_1\times G_2)=\mathfrak{X}(G_1)\times\mathfrak{X}(G_2)$

Lem 3: Every alg'c subgroup $T_0\subset$ torus T is a direct product of a torus and finite comm've group. Moreover, $\mathfrak{X}(T)\twoheadrightarrow\mathfrak{X}(T_0)$

Proof: There is a bij'n between subgroups $T_0\subset T$ and $\Lambda\subset\mathfrak{X}(T)$: $T_0\rightsquigarrow\underbrace{\{\chi\mid\chi|_{T_0}=1\}}_{=\mathfrak{X}(T/T_0)}$. Lem follows from class'n of sublattices of $\mathbb{Z}^n$ □

3.1) Structure of conn'd solvable groups

Thm 4: Let G be conn'd solvable group, and U its unipotent radical. Then there is a subtorus $T\subset G$ w. $G=T\ltimes U$. Moreover for any subgroup $A\subset G$ that's a product of a torus & finite comm've group $\exists u\in U$ s.t. $uAu^{-1}\subset T$.

Proof: Recall $G\hookrightarrow B_n$, hence $U=G\cap U_n$ (*). Now the proof is by ind'n on dim U. If $\exists$ norm'l in G subgr'p $U_0\subset U$, then we can choose $T_1\subset G/U_0$ w. required properties and let G_1 to be the preimage of T_1 in G. Once we know our statements in G/U_0 & in G_1 we are done.

So suppose no such non-triv. U_0 exists. We claim that $\dim U=1$ (hence U is comm've) and G/U acts on $U(\simeq\mathbb{C})$ by mult'g w. character. Recall the normal in B_n subgroups $U_{n,k}$. Pick minimal k s.t. $U\subset U_{n,k}$. Then $U\cap U_{n,k+1}$ is normal in G and is diff't from $U\Rightarrow U\cap U_{n,k+1}=\{e\}$. So $U\hookrightarrow U_{n,k}/U_{n,k+1}=\mathbb{C}^{k+1}$ and conj. action of G/U on U is by alg'c rep'n (b/c B_n acts on $U_{n,k}/U_{n,k+1}$ this way). The action is diagon'le and every stable subspace is normal in G. So $\dim U=1$.

Exist'e of T_1: Let χ be the character of action of G/U on U.

Case 1: $\chi\neq 0$. Let $g\in G$ be Weil gen'c (◇), and let $T=\overline{\{g^n\mid n\in\mathbb{Z}\}}\subset G$. The image of T in G/U is dense $\Rightarrow$ coincides w. G/U, T is closure of comm've $\Rightarrow$ comm've $\Rightarrow T\neq G$. If $T\cap U\neq\{e\}$, then $T\supset U$. So $T\cap U=\{e\}$ & $G=T\ltimes U$

(*) and $G/U\twoheadrightarrow T_n$. Since G/U is conn'd, by Lem 3 G/U is a torus.

(◇) Outside of countable union of subvarieties

Case 2: $\chi=0$. If G isn't comm've we are done like in Case 1 (but in fact $\chi=0 \Rightarrow G$ is comm've). Finite order el'ts are dense in any torus $\rightsquigarrow$ $T:=\{$fin. order el'ts of $G\}$ - subgroup; $T \curvearrowright \mathbb{C}^n$ diag'ly, so $T\cap U=\{e\}$
Now let us show $T \twoheadrightarrow G/U$. Let $\underline{h}$ be a fin. order el't in G/U. $T\twoheadrightarrow G/U$ will follow from $\exists$ fin. order $s\in \underline{h}U$. Let $h\in \underline{h}U$
$h=h_s h_u$ be the mult've Jordan decomp in $GL_n(\mathbb{C})$ (h_s is diag'le in $GL_n(\mathbb{C})$, h_u unip, $h_s h_u = h_u h_s$). h_s is of finite order $\Rightarrow \exists\, n>0$ $h^n = h_u^n$. In fact, $U=\{\exp(tA) \mid t\in\mathbb{C}\}$, where $A\in Mat_n(\mathbb{C})$ is nilpotent. So $h_u^n = h^n \in U$ $\Rightarrow h_u \in U \Rightarrow h_s = G$, it lifts $\underline{h}$. The proof of existence of T is done.

Now let's show existence of $u\in U$ w. $uAu^{-1}\subset T$. In case 2, $A\subset T$ by the proof. In Case 1, we need to consider two cases.

Case 1.1 $A\not\subset \ker\chi$. Pick $g\in A\setminus\ker\chi$. To show $\exists\, u \mid ugu^{-1}\in T$ is an exercise. Let $Z=Z_G(ugu^{-1})$, the centralizer. Then $T\subset Z$, $Z\cap U=\{e\}$. Since $uAu^{-1}\subset Z$, we are done.

Case 1.2. $A\subset\ker\chi$. Then similarly to case 2, we see $A\subset T$. (exer.) $\square$

Rem: The same argument works for any alg subgr'p of B_n, now we have $G=T\ltimes U$, where T doesn't need to be conn'd.

Cor (of Thms 3,4) Any two max'l (w.r.t. inclusion) tori in an alg. gr'p G are conj'te. (Thm 3 reduces to the case of solvable group & Thm 4 handles that case)