Def: Let $G$ be algebraic group. By a character of $G$, we mean algebraic group homomorphism $G \to \mathbb{C}^\times$. Characters form a group denoted by $X(G)$.

We've seen that for unipotent $G$, we have $X(G) = \{0\}$. Also, for $G = \mathbb{G}_m$, we have $X(G) = \mathbb{Z}$. Example: $X(G \times \mathbb{G}_m) = X(G) \times X(\mathbb{G}_m)$.

Lemma 3: Every algebraic subgroup $T < \text{torus } T$ is a direct product of $T$ and a finite commutative group. Moreover, $X(T) \to X(T')$.

Proof: There is a bijection between subgroups $T < T$ and $A < X(T)$. Hence, follows from classification of sublattices of $\mathbb{Z}$.

3.5: Structure of connected solvable groups

Theorem 4: Let $G$ be connected solvable group, and $U$ its unipotent radical. Then there is a subtorus $T < G$ with $G = T \times U$. Moreover, for any subgroup $A < G$, there's a product of a torus and finite commutative group $A = U \times T$.

Proof: Recall $G \cong B_n$, hence $U = G/U_0$. Now, the proof is by induction on $\dim U$.

If $T$ is normal in $G$, subgroup $U < T$, then we can choose $T < G/U$ with required properties and let $G'$ to be the preimage of $T$ in $G$. Once we know our statement in $G/U$, and in $G$, we are done.

So suppose no such non-trivial $U_0$ exists. We claim that $\dim U = 1$.

Glue acts on $U(= \mathbb{C})$ by multiplication with character. Recall the normal in $B_n$ subgroups $U_{kn}$. Pick minimal $k$ such that $U < U_{kn}$. Then $U_{kn}/U_{kn}$ is normal in $G$ and is different from $U \to U_{kn}/U_{kn} = \mathbb{C}^{(k)}$ and conjugation of $G_{kn}/U$ is by algebraic return ($k < B_n$ acts on $U_{kn}/U_{kn}$ this way).

So the action is diagonalizable over a stable subspace is normal in $G$. So $\dim U = 1$.

Existence of $T$

Let $X$ be the character of action of $G_{\nu}$ on $U$.

Case 1: $X \neq 0$. Let $g \in G$ be Verma gens, and let $T = G^g \subseteq G$.

The image of $T$ in $G_{\nu}$ is dense $\Rightarrow$ comes from $G_{\nu}$, $T$ is closure of commutative subgroup $\Rightarrow T 

\Rightarrow T = G$. If $T \cap U = \{e\}$, then $T \cap U = \{e\}$ and $G = T \cup U$.

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Case 2: \(x=0\). If \(G\) isn't commutative we are done like in Case 1 (but in fact \(G = \mathbb{G}^m\text{ is commutative}\.). Finite order elements are dense in many tori -

\[ T := \{ \text{finite order elements of } G^m \text{-subgroup}, T \text{ acts diagonally, so } T \cap U = \{e\} \}
\]

Now let us show \(T \rightarrow G/U\). Let \(h\) be a finite order element in \(G/U\), \(T \rightarrow G/U\) will follow from \(\exists \text{ finite order } s \in h U\).

Let \(h \in h (U)\), \(h = h_s h_n\) be the multiplicative Jordan decomposition in \(G^m (C)\) (\(h_s\) is diagonal in \(G^m (C)\), \(h_n\) is of finite order \(\Rightarrow \exists \text{ some } n \geq 0\) \(h_n = h_s\)). In fact, \(U := \{ \exp(tA) \mid t \in C\}\), where \(A \in \text{Mat}_n (C)\) is nilpotent. So \(h_n = h_n \in U\) \(\Rightarrow h_n \in U \Rightarrow h_n = h_s\), it lefts \(h\). The proof of existence of \(T\) is done.

Now let's show existence of \(u \in U\) \(u h U = T\). In case 2, \(A \subset T\) by the proof. In Case 1, we need to consider two cases:

Case 1: \(A \cap ker x\). Pick \(g \in A \setminus ker x\). To show \(\exists u \in U\) such that \(g u \in T\) is an exercise. Let \(Z = \mathbb{Z} g (u g u^{-1})\), the center of \(T\) then \(T \subset Z, Z \cap U = \{e\}\).

Since \(u h U = Z\), we are done.

Case 2: \(A \subset ker x\). Then similarly to case 2, we see \(A \subset T\) (exercise) \(\square\).

Remark: The same argument works for any nilpotent subgroup of \(B_0\); now we have \(C \text{ is } T\), where \(T\) doesn't need to be normal.

Corollary (of Thms 74): Any two maximal (not necessarily tori) in an abelian group \(G\) are conjugate: (Thm 3 reduces to the case of solvable group & Thm 6 handles that case.)