# Universal enveloping algebra for vertex algebras $\mathbb{C}[JX]$.

This is about a special case of the completed universal enveloping algebra $\tilde{U}(V)$ for a vertex algebra $V$ in the case when $V = \mathbb{C}[JX]$ for an affine scheme $X/\mathbb{C}$ of finite type (recall that a commutative vertex algebra is the same thing as the commutative algebra w. a derivation & $\mathbb{C}[JX]$ comes w. a natural derivation).

**Proposition:** $\tilde{U}(\mathbb{C}[JX]) = \mathbb{C}[\mathcal{L}X]$, where $\mathcal{L}X$ is the loop space of $X$.

Proof by example. Let $X = \mathbb{A}^1$. Then $\mathbb{C}[JX] = \mathbb{C}[x_n \mid n<0]$ w. derivation $T$ sending $x_n$ to $-nx_{n-1}$. Recall that we first form the Lie algebra $F_V = V \otimes \mathbb{C}[t^{\pm 1}]/im(T\otimes 1 + 1\otimes \partial_t)$ $(V = \mathbb{C}[J\mathbb{A}^1])$. Then we form the completed universal enveloping algebra $\tilde{U}(F_V)$ and mod it out by the relations

$$(A_{-1}B)_{[k]} = \sum_{n+m=k-1} :A_{[n]}B_{[m]}:\, , \quad A,B \in V$$

where for $A \in V$, $k \in \mathbb{Z}$ we write $A_{[k]}$ for $A \otimes t^k \in V \otimes \mathbb{C}[t^{\pm 1}]$ and its image in $F_V$. Note that if $V$ is commutative as a vertex algebra, $F_V$ is an abelian Lie algebra. So $\tilde{U}(V)$ is the quotient of the usual completion of $S(V\otimes \mathbb{C}[t^{\pm 1}])$ $\left(:= \varprojlim_{N\to\infty} S(V\otimes\mathbb{C}[t^{\pm1}])/(A_{[i]} \mid i\geq N)\right)$ by the relations:

$$(TA)_{[n]} = -n A_{[n-1]}$$

$$(A\cdot B)_{[k]} = \sum_{n+m=k-1} A_{[n]} B_{[m]}$$

Here "$\cdot$" is the product in $V$. Analyzing these relations for $V = \mathbb{C}[JX]$, we see that $\tilde{U}(V)$ is the usual completion of $\mathbb{C}[X_{-1,[n]} \mid n\in\mathbb{Z}]$, which is exactly what we need to show. Compare this to Proposition 2.21 in Hamilton's talk.