INVariants of jets and the center for \( \hat{\mathfrak{sl}}_2 \)

Ivan Karpov, Ivan Losev

Abstract. This is an expository talk for the student learning seminar on the representation theory of affine Kac-Moody algebras at the critical level. We develop the formalism of jet schemes and use it to compute the algebra of invariants for the action of the group \( \hat{G}[[t]] \) on its adjoint representation \( \hat{g}[[t]] \). In turn, we use this computation to show that the center of \( V_{n_c}(\hat{\mathfrak{sl}}_2) \) is the polynomial algebra freely generated by the Sugawara modes. We then identify the center of \( V_{n_c}(\hat{\mathfrak{sl}}_2) \) with the algebra of polynomial functions on the space of projective connections on the disc \( D = \text{Spec}(\mathbb{C}[[t]]) \) thus getting a coordinate free description of the center. We mostly follow [2].

1. Invariants and the center

1.1. Introduction. Throughout the talk, the base field is \( \mathbb{C} \).

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra. The corresponding connected algebraic group \( G \) acts on \( \mathfrak{g} \) (via the adjoint representation), yielding \( G \)-actions by graded algebra automorphisms on \( \mathbb{C}[\mathfrak{g}](\cong S(\mathfrak{g})) \) and by filtered algebra automorphisms on the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \).

Let \( \mathfrak{h} \subseteq \mathfrak{g} \) denote a Cartan subalgebra, and \( W \) be the corresponding Weyl group. The following is due to Chevalley:

Proposition 1.1.1. (A) We have a graded algebra isomorphism \( \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W \).

(B) The algebras in (A) are isomorphic to the polynomial algebra in \( r := \text{rk} \mathfrak{g} \) homogeneous generators, to be denoted by \( P_1, \ldots, P_r \).

It is also well-known due to Harish-Chandra (see, e.g., [4, Ch. 23]) that the center \( \mathcal{Z}(U(\mathfrak{g})) \) of \( U(\mathfrak{g}) \) is isomorphic to \( \mathbb{C}[\mathfrak{h}]^W \) as a filtered algebra. The Harish-Chandra theorem can be viewed as a finite dimensional counterpart of the main result for the seminar: a description of the center of the completed universal enveloping algebra of \( \hat{\mathfrak{g}} \) at the critical level.

We write \( \mathcal{O} \) for \( \mathbb{C}[[t]] \), \( G_{\mathcal{O}} \) for the group of \( \mathcal{O} \)-points of \( G \) and \( \mathfrak{g}_{\mathcal{O}} \) for its Lie algebra, \( \mathfrak{g} \otimes \mathcal{O} \), compare to [6, Section 3]. The main goal of the first part of the talk is to get an analog of Proposition 1.1.1 for the action of the group \( G_{\mathcal{O}} \) on \( \mathfrak{g}_{\mathcal{O}} \): we will see that the elements \( P_{i,n} \) with \( i = 1, \ldots, r \) and \( n < 0 \) introduced in [6, Section 3.4] are free generators of \( \mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}} \). We will use this to show that the Sugawara modes \( S_n|0\rangle \in V_{n_c}(\hat{\mathfrak{sl}}_2) \) (with \( n \leq -2 \)) generate the center of \( V_{n_c}(\hat{\mathfrak{sl}}_2) \).

1.2. Jet schemes. In order to compute the algebra \( \mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}} \) we will need the formalism of jet schemes (a.k.a. arc spaces).

1.2.1. Definition via functor of points. Let CommAlg denote category of commutative associative unital \( \mathbb{C} \)-algebras, its opposite category is identified with the category of affine schemes over Spec(\( \mathbb{C} \)). In particular, an arbitrary scheme \( X \) over Spec(\( \mathbb{C} \)) gives rise to its functor of points

\[ \text{Mor}(\text{Spec}(?), X) : \text{CommAlg} \to \text{Sets} \]

sending an algebra \( R \) to the set of \( R \)-points of \( X \). One recovers \( X \) uniquely from its functor of points, however, not every functor CommAlg \( \to \) Sets is representable (i.e., is a functor of points for a scheme).

Definition 1.2.1. Let \( X \) be a finite type scheme over Spec(\( \mathbb{C} \)). We define the jet functor of \( X \)

\[ J_X : \text{CommAlg} \to \text{Sets} \]

by sending \( R \) to the set of all morphisms Spec(\( R[[t]] \)) \( \to X \) (of schemes over Spec(\( \mathbb{C} \))).
Proposition 1.2.2. The functor $J_X$ is represented by a scheme to be denoted by $J_X$ and called the jet scheme (a.k.a. arc space) of $X$.

We will sketch a proof (and a construction of $JX$) below in this section.

We also note that for general Yoneda reasons, $J$ is a functor (from the category of finite type schemes to the category of schemes). For a morphism $ϕ : X → Y$ we write $Jϕ$ for the induced morphism $JX → JY$.

1.2.2. Affine case. We first give a constructive proof of Proposition 1.2.2 in the case when $X$ is affine.

Example 1.2.3. First, set $X = \mathbb{A}^m = \text{Spec}(\mathbb{C}[x_1, \ldots, x_m])$. For an arbitrary commutative $\mathbb{C}$-algebra $R$, the set of $R[[t]]$-points of $X$ is

$$\text{Hom}_{\text{Alg}}(\mathbb{C}[x_1, \ldots, x_m], R[[t]]).$$

Of course, any algebra homomorphism $ϕ : \mathbb{C}[x_1, \ldots, x_m] → R[[t]]$ is uniquely determined from the images $ϕ(x_i)$ that are formal power series

$$ϕ(x_i) = \sum_{n<0} a_{i,n} t^{-n-1}, a_{i,n} ∈ R.$$

Thus, the set of $R$-point of $JX$ is the set $\{a_{i,n} ∈ R | i = 1, \ldots, m, n < 0\}$ and hence

$$JX = \text{Spec} \mathbb{C}[x_{i,n} | i = 1, \ldots, m, n < 0].$$

Example 1.2.4. Now we consider the case when $X$ is a general finite type affine scheme over $\text{Spec}(\mathbb{C})$, it can be defined as

$$\text{Spec}(\mathbb{C}[x_1, \ldots, x_m]/(F_1, \ldots, F_k)).$$

The same reasoning as in the Example 1.2.3 shows that the set $\text{Mor}(\text{Spec}(R), JX)$ can be identified with the set of $a_i(t) := ϕ(x_i) ∈ R[[t]]$ such that

$$(1.2.1) \quad F_j(a_1(t), \ldots, a_n(t)) = 0$$

for all $j = 1, \ldots, k$.

To describe this set of formal power series, consider the algebra $R := \mathbb{C}[x_{i,n}]$ (cf. Example 1.2.3). Define a derivation $T ∈ \text{Der}_C(R)$ on the free generators by:

$$T : x_{j,n} → -nx_{j,n-1}.$$

Now, define $F_j^# := F_j(x_{i,-1})$. One can show that the system of equations (1.2.1) is equivalent to $T^ℓF_j^# = 0$ for all possible $ℓ ≥ 0$ and $j = 1, \ldots, k$. So for $JX$ we can take the closed subscheme of $J\mathbb{A}^m$ given by the equations $T^ℓF_j^#$:

$$JX = \text{Spec}(R/(T^ℓF_j^#)).$$

Remark 1.2.5. We have an algebra homomorphism $C[X] → C[JX]$ sending $F = F(x_1, \ldots, x_m)$ to $F^#$ defined by $F(x_{i, -1}, \ldots, x_{m, -1})$. It yields a scheme morphism $JX → X$.

Exercise 1.2.6. Let $X, Y$ be finite type affine schemes (over $\text{Spec}(\mathbb{C})$). Identify $J(X × Y)$ with $JX × JY$. More precisely, let $π_1 : X × Y → X$ and $π_2 : X × Y → Y$ be the projections. Then $Jπ_1 × Jπ_2 : (X × Y) → JX × JY$.

1.2.3. Gluing. Now we proceed to the case of non-affine finite type schemes $Y$. We claim that $JY$ can be glued from $JX$ for open affines $X ⊂ Y$. The key step here is to relate $JX$ and $J(X_f)$ for $f ∈ \mathbb{C}[X]$, where $X_f$ is the non-vanishing locus for $f$ (known as a principal open subset). We claim that $J(X_f)$ is naturally identified with $(JX)_f$, where $f^♯ ∈ \mathbb{C}[JX]$ is defined in Remark 1.2.5.

Indeed, recall that if $C[X] = \mathbb{C}[x_1, \ldots, x_m]/(F_1, \ldots, F_k)$, then

$$C[X_f] = \mathbb{C}[x_1, \ldots, x_m]/(F_1, \ldots, F_k, xf - 1).$$

It follows that $C[J(X_f)] = \mathbb{C}[JX]|_{x_n < 0}/(T^ℓ(xf - 1)^2)$. For $ℓ = 0$, the equation $T^ℓ(xf - 1)^2 = 0$ means that $x_{-1}f^♯ = 1$, i.e., $f^♯$ is invertible, and $x_{-1} = (f^♯)^{-1}$. The equation $T^ℓ(xf - 1)^2 = 0$ for
\( \ell > 0 \) then uniquely expresses \( x_{-\ell-1} \) as a polynomial in \( x_{-1}, \ldots, x_{-\ell}, (f^2)^{-1} \) and elements of \( \mathbb{C}[JX] \). This gives the required identification \( \mathbb{C}[J(X_p)] \cong \mathbb{C}[JX][(f^2)^{-1}] \).

This discussion finishes our sketch of proof of Proposition 1.2.2.

**Remark 1.2.7.** Note that we still have a morphism \( JY \to Y \). It is affine (of infinite type).

1.2.4. \( \text{n} \text{th order jets.} \) Let \( X \) be a finite type scheme over \( \text{Spec}(\mathbb{C}) \). It turns out that \( JX \) (which is an infinite type scheme) can be presented as the inverse limit of finite type schemes \( J_nX \) (\( \text{n} \text{-th order jet schemes} \)). By definition, \( J_nX \) represents the functor \( \text{CommAlg} \to \text{Sets} \) sending \( R \) to the set of morphisms \( \text{Spec}(R[t]/(t^{n+1})) \to X \).

For example, for \( X \) as in Example 1.2.4, we have
\[
J_nX = \text{Spec}(\mathbb{C}[JX]/(x_{i,N} | i = 1, \ldots, m, N < -n - 1)).
\]

As in the case of \( J \), \( J_n \) is a functor (in this case, from the category of finite type schemes over \( \text{Spec}(\mathbb{C}) \) to itself). The claim that \( J = \lim_{n \to \infty} J_n \) is left as an exercise (on the general categorical nonsense).

**Exercise 1.2.8.** For \( X \) smooth, show that \( J_1X \) is the tangent bundle of \( X \).

1.2.5. \( \text{Smoothness.} \) The goal of this part is to prove the following statement.

**Theorem 1.2.9.** For a smooth morphism \( \varphi : X \to Y \), the morphism \( J_n\varphi : J_n(X) \to J_n(Y) \) is smooth as well.\(^1\)

Indeed, let us recall the following criterion of smoothness ([1, Section 1.4]). If \( R \) is a commutative \( \mathbb{C} \)-algebra, then by its nilpotent extension we mean a commutative algebra \( R_1 \) equipped with an epimorphism \( R_1 \to R \) whose kernel is a nilpotent ideal.

**Proposition 1.2.10.** Suppose that \( g : A \to B \) is a morphism of schemes of finite type over \( \mathbb{C} \). Then, \( g \) is smooth if and only if for any morphism \( h : S = \text{Spec}(R) \to B \) which lifts to \( h' : S \to A \) the following holds:

- suppose that \( R_1 \) is a nilpotent extension of \( R \), that \( S_1 = \text{Spec}(R_1) \), and that \( h_1 : S_1 \to B \) is any lifting of \( h \). Then \( h_1 \) also lifts to \( h'_1 : S_1 \to A \):

\[
\begin{array}{ccc}
S & \xrightarrow{h'} & A \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{h_1} & B
\end{array}
\]

**Proof of Theorem 1.2.9.** By definition, an \( R \)-point of \( J_nA \) is an \( R[t]/(t^{n+1}) \)-point of \( A \). Now, we have the diagram
\[
\begin{array}{ccc}
\text{Spec } R[t]/(t^{n+1}) & \xrightarrow{h'} & X \\
\downarrow & & \downarrow \\
\text{Spec } R_1[t]/(t^{n+1}) & \xrightarrow{h_1} & Y
\end{array}
\]

where we need to prove the existence of \( h'_1 \). To finish the proof we combine Proposition 1.2.10 with the observation that \( R_1[t]/(t^{n+1}) \) is a nilpotent extension of \( R[t]/(t^{n+1}) \).

**Remark 1.2.11.** The similar argument proves that, for a surjective smooth morphism \( f \), the morphism \( J_nf \) is also surjective (on the level of \( \mathbb{C} \)-points) for all \( n \).

Applying Theorem 1.2.9 to \( Y = \text{pt} \), we get the following claim.

\(^1\)One can introduce the notion of “formal smoothness”. Then, the same statement would be true for the functor \( J \) itself (instead of \( J_n \)-s).
Corollary 1.2.12. For a smooth variety $X$, the scheme $J_nX$ is a smooth scheme of finite type.

The following exercise (based on the generic smoothness) will be used below.

Exercise 1.2.13. Let $\varphi : X \to Y$ be a dominant morphism to a smooth variety $Y$. Prove that $J_n\varphi : J_nX \to J_nY$ is dominant.

1.3. Jet-theoretic Chevalley theorem. Recall that we write $O$ for the algebra $\mathbb{C}[[t]]$. For an affine scheme $X$ we will often write $X_O$ for $JX$.

Let $G$ be an algebraic group. Applying the functoriality of $J_n$ and $J$ to the structure maps of $G$, we see that $J_nG$, $JG$ are group schemes over $\mathbb{C}$. In fact, $J_nG$ is an honest algebraic group with Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]/(t^{n+1}) - J_nG$ is the semi-direct product of $G$ with the unipotent group $\exp(t\mathfrak{g}[t]/t^{n+1}\mathfrak{g}[t])$. This description shows, in particular, that $J_{n+1}G \to J_nG$ for all $n$. And $JG$ is the limit $\lim_{n \to \infty} J_nG$, hence a pro-algebraic group.

Applying the functor $J$ to the action morphism $G \times \mathfrak{g} \to \mathfrak{g}$ we get the morphism $J(G \times \mathfrak{g}) \to J\mathfrak{g}$. Under the identification $JG \times J\mathfrak{g} \cong J(G \times \mathfrak{g})$ from Exercise 1.2.6, this gives an action of the pro-algebraic group $JG$ on $J\mathfrak{g}$. We want to compute the algebra of invariant polynomial functions for this action.

The following result is a jet analog of Proposition 1.1.1. Recall that $P_i, i = 1, \ldots, r$, denote free homogeneous generators of the algebra $\mathbb{C}[\mathfrak{g}]^G$. Then we can form the elements $P_{i,\ell} \in \mathbb{C}[\mathfrak{g}]^{G_O}$ for all $\ell \leq 0$ and $i = 1, \ldots, r$, see [6, Section 3.4].

Theorem 1.3.1. The algebra of invariants $\mathbb{C}[\mathfrak{g}]^{G_O}$ is identified with $\mathbb{C}[J(\mathfrak{h}/W)]$, equivalently, is freely generated by the elements $P_{i,\ell}$.

1.3.1. Preparation. We write $\mathfrak{g}/G := \text{Spec}(\mathbb{C}[\mathfrak{g}]^G)$. We have the quotient morphism $\pi : \mathfrak{g} \to \mathfrak{g}/G$ induced by the inclusion $\mathbb{C}[\mathfrak{g}]^G \subseteq \mathbb{C}[\mathfrak{g}]$. It gives rise to $J\pi : J\mathfrak{g} \to J(\mathfrak{g}/G)$. By the Chevalley theorem, $\mathfrak{g}/G$ is an affine space with coordinates $P_1, \ldots, P_r$. The polynomials $P_{i,\ell}$ are nothing else but the coordinates on the infinite dimensional affine space $J(\mathfrak{g}/G)$. So our job is to show that the pullback homomorphism $(J\pi)^*$ identifies $\mathbb{C}[J(\mathfrak{g}/G)]$ with the subalgebra of invariants for $G_O = JG$ in $\mathbb{C}[\mathfrak{g}].$

We are going to reduce this to the analogous claim, where $J$ is replaced with $J_n$: $(J_n\pi)^*$ identifies $\mathbb{C}[J_n(\mathfrak{g}/G)]$ with $\mathbb{C}[J_n\mathfrak{g}]^{J_nG}$. Proving the latter for all $n$ is enough for the following reason. Since $\mathbb{C}[\mathfrak{g}]$ is the union of its subalgebras $\mathbb{C}[J_n\mathfrak{g}]$, we see that $\mathbb{C}[\mathfrak{g}]^{J_G}$ is the union of its subalgebras $\mathbb{C}[\mathfrak{g}]^{J_G} \cap \mathbb{C}[J_n\mathfrak{g}]$. Our reduction now follows from the next exercise (where one needs to use that $JG \to J_nG$ and that the projection $J\mathfrak{g} \to J_n\mathfrak{g}$ is $JG$-equivariant).

Exercise 1.3.2. $\mathbb{C}[\mathfrak{g}]^{J_G} \cap \mathbb{C}[J_n\mathfrak{g}] = \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$ as subalgebras in $\mathbb{C}[\mathfrak{g}]$.

1.3.2. 1st proof of $\mathbb{C}[J_n(\mathfrak{g}/J_nG)] \cong \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$. In this proof, different from what is given in [2, Section 3.4] we will use the Kostant slice, a remarkable affine subspace $S \subseteq \mathfrak{g}$ with the property that the restriction of the quotient morphism $\pi : \mathfrak{g} \to \mathfrak{g}/G := \text{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ to $S$ is an isomorphism. For more on Kostant slices see [7]. In particular the claim that $\pi|_S$ is an isomorphism is proved in [7, Section 4].

Let $i$ denote the inclusion $S \subseteq \mathfrak{g}$. Since $i \circ \iota$ is an isomorphism $S \cong \mathfrak{g}/G$, we see that $J_n\pi \circ J_n\iota : J_nS \cong J_n(\mathfrak{g}/G)$. It remains to show that $(J_n\iota)^* \iota$ embeds $\mathbb{C}[J_n\mathfrak{g}]^{J_nG}$ into $\mathbb{C}[J_n\mathfrak{g}]$. Let $\beta$ denote the action map $G \times S \to \mathfrak{g}$, $(g, s) \mapsto \text{Ad}(g)s$, and $\iota'$ denote the embedding $S \subseteq G \times S$, $s \mapsto (1, s)$. Note that $\iota = \beta \circ \iota'$, hence $J_n\iota = J_n\beta \circ J_n\iota'$. The action of $G$ on $G \times S$ (by left translations on the first factor) gives rise to an action of $J_nG$ on $J_n(G \times S) = J_nG \times J_nS$ (also by left translation on the first factor). So $(J_n\iota'^*)$ restricts to an isomorphism $\mathbb{C}[J_n\mathfrak{g}]^{J_nG} \cong \mathbb{C}[J_nS]$. So, the claim that $(J_n\iota'^*)^* \mathbb{C}[J_n\mathfrak{g}]^{J_nG} \subseteq \mathbb{C}[J_n\mathfrak{g}]$ is equivalent to $(J_n\beta'^*)^* \mathbb{C}[J_n\mathfrak{g}]^{J_nG} \subseteq \mathbb{C}[J_n(G \times S)]^{J_nG}$, which will follow from $(J_n\beta'^*)^* \mathbb{C}[J_n\mathfrak{g}] \subseteq \mathbb{C}[J_n(G \times S)]$. To see the latter injectivity, we remark that $\beta : G \times S \to \mathfrak{g}$ is dominant (Step 1 of the proof of Theorem in [7, Section 4]) and use Exercise 1.2.13. This completes the 1st proof of Theorem 1.3.1.
1.3.3. 2nd proof of $\mathbb{C}[J_n(\mathfrak{g}/J_n G)] \sim \mathbb{C}[J_n \mathfrak{g}]^{J_n G}$. Now we give a proof that closely follows one in [2]. Consider the open subset of regular elements:

$$\mathfrak{g}^{\text{reg}} = \{ x \in \mathfrak{g} \mid \dim Z_{\mathfrak{g}}(x) = \text{rk} \mathfrak{g} \},$$

studied in detail in [7, Section 5]. In particular, we have the following claim

(*) The morphism $\pi|_{\mathfrak{g}^{\text{reg}}}$ is smooth, and each fiber of $\pi|_{\mathfrak{g}^{\text{reg}}} : \mathfrak{g}^{\text{reg}} \to \mathfrak{g}/G$ is a single $G$-orbit (in particular, the morphism is surjective).

**Exercise 1.3.3.** For $\mathfrak{g} = \mathfrak{s}l_2$, the subset $\mathfrak{g}^{\text{reg}}$ consists precisely of all matrices such that in their Jordan normal form, there is a single block for each eigenvalue.

Suppose, for a moment, that we know that the direct analog of (*) holds for the action of $J_n G$ on $J_n \mathfrak{g}^{\text{reg}}$ and the morphism $J_n(\pi|_{\mathfrak{g}^{\text{reg}}}) : J_n \mathfrak{g}^{\text{reg}} \to J_n(\mathfrak{g}/G)$. We then can prove that $\mathbb{C}[J_n(\mathfrak{g}/G)] \sim \mathbb{C}[J_n \mathfrak{g}]^{J_n G}$ using the following general result.

**Proposition 1.3.4.** Let $H$ be an algebraic group and $X,Y$ be normal algebraic varieties. Suppose $H$ acts on $X$, and $Y$ is affine. Suppose, further, that $\varphi : X \to Y$ is a surjective $H$-invariant morphism such that each fiber of $\varphi$ is a single $H$-orbit. Then $\varphi^* : \mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]^H$.

**Proof.** Clearly, $\varphi^* : \mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]^H$ and we need to prove the surjectivity. Take $f \in \mathbb{C}[X]^H$, and consider the subalgebra of $\mathbb{C}[X]^H$ generated by $\mathbb{C}[Y]$ and $f$, denote it by $A$. Then $\varphi$ factors as $X \to \text{Spec}(A) \to Y$, where both morphisms are dominant. Since each fiber of $\varphi$ is a single orbit, $	ext{Spec}(A) \to Y$ is injective. Any injective dominant morphism is birational, hence $f$ can be viewed as a rational function on $Y$. It is left as an exercise to show that $f$ has no poles on $Y$. Since $Y$ is normal, $f \in \text{im} \varphi^*$. This finishes the proof. \qed

We apply this to $X = J_n \mathfrak{g}^{\text{reg}}, Y = J_n(\mathfrak{g}/G)$ and $H = J_n G$. Note that $J_n(\mathfrak{g}/G)$ is smooth, hence normal, we use the analog of (*) to deduce $\mathbb{C}[J_n(\mathfrak{g}/J_n G)] \sim \mathbb{C}[J_n \mathfrak{g}^{\text{reg}}]^{J_n G}$. The subvariety $J_n(\mathfrak{g}^{\text{reg}}) \subseteq J_n \mathfrak{g}$ is open and dense. So the restriction homomorphism $\mathbb{C}[J_n \mathfrak{g}] \to \mathbb{C}[J_n \mathfrak{g}^{\text{reg}}]$ is injective. From here we deduce that $\mathbb{C}[J_n(\mathfrak{g}/J_n G)] \sim \mathbb{C}[J_n \mathfrak{g}]^{J_n G}$.

Now, it remains to establish that analog. First, we reformulate the claim.

**Exercise 1.3.5.** Let $H$ be an algebraic group acting on a variety $X$, $Y$ is a variety, and $\varphi : X \to Y$ be an $H$-invariant morphism. The following claims are equivalent.

(a) The morphism $\varphi$ is smooth and each fiber of $\varphi$ is a single $H$-orbit.

(b) The morphism $H \times X \to X \times_Y X, (h,x) \mapsto (hx,x)$ is smooth and surjective.

Apply Exercise 1.3.5 to $H = G, X = \mathfrak{g}^{\text{reg}}, Y = \mathfrak{g}/G, \varphi = \pi|_{\mathfrak{g}^{\text{reg}}}$ to get that $G \times \mathfrak{g}^{\text{reg}} \to \mathfrak{g}^{\text{reg}} \times \mathfrak{g}/G \mathfrak{g}^{\text{reg}}$ is smooth and surjective. Hence, by Section 1.2.5, $J_n(G \times \mathfrak{g}^{\text{reg}}) \to J_n(\mathfrak{g}^{\text{reg}} \times \mathfrak{g}/G \mathfrak{g}^{\text{reg}})$. One can use the smoothness of $\pi|_{\mathfrak{g}^{\text{reg}}}$ and generalize Exercise 1.2.6, to identify $J_n(\mathfrak{g}^{\text{reg}} \times \mathfrak{g}/G \mathfrak{g}^{\text{reg}})$ with $J_n(\mathfrak{g}^{\text{reg}}) \times J_n(\mathfrak{g}/G)$. We get (b) of Exercise 1.3.5 for $H = J_n G, X = J_n(\mathfrak{g}^{\text{reg}}), Y = J_n(\mathfrak{g}/G), \varphi = J_n(\pi|_{\mathfrak{g}^{\text{reg}}})$, yielding (a), which is what we need to finish the proof.

1.4. Center of $V_{\kappa}(\mathfrak{s}l_2)$. Suppose $G = \text{SL}_2$. We recall the definition of *Sugawara operators* from [6, Section 1].

$$S_n = \frac{1}{2} \sum_{i=1}^{3} \sum_{j+k=n} x^i(j)x_i(k) : .$$

The elements $S_n|0\rangle$ for $n \leq -2$ are central in $V_{\kappa}(\mathfrak{s}l_2)$.

**Theorem 1.4.1.** The center of the vertex algebra $V_{\kappa}(\mathfrak{s}l_2)$ is isomorphic to $\mathbb{C}[S_n|0\rangle]_{n \leq -2}$ (as a commutative algebra).

We leave the proof of this theorem as an exercise. A warm-up is to recall how to prove that the Casimir element is a free generator of the center of the $U(\mathfrak{s}l_2)$ once one knows the description of $\mathbb{C}[\mathfrak{s}l_2]^{\text{SL}_2}$. For details, see [2, Section 3.5].
2. The coordinate-independent description of $V_\kappa(\mathfrak{g})$.

2.1. The ring $O$ and the field $K$. Suppose that $X$ be a smooth curve. Let us define the ring $O_x$ as the completion of the local ring $O_{X,x}$ at the maximal ideal $\mathfrak{m}_x$, i.e., $O_x := \varprojlim O_{X,x}/\mathfrak{m}_x^n$. Let also $K_x$ be the field of fractions of $O_x$. Let $\hat{\mathfrak{m}}_x$ denote the maximal ideal in $O_x$.

A choice of an element $t \in \hat{\mathfrak{m}}_x \setminus \hat{\mathfrak{m}}_x^2$ is the same as a choice of an isomorphism $O_x \xrightarrow{\sim} \mathbb{C}[[t]]$.

**Exercise 2.1.1.** Prove this statement. A hint: to see that there is an isomorphism use a suitable étale map from an open neighborhood of $x$ to $\mathbb{A}^1$.

Note that an isomorphism $O_x \xrightarrow{\sim} \mathbb{C}[[t]]$ induces an isomorphism $K_x \xrightarrow{\sim} \mathbb{C}(t)$.

2.2. The algebra $\hat{\mathfrak{g}}_{\kappa,x}$. We now want to define the Kac-Moody algebra in a coordinate free way, using $K_x$ instead of $\mathbb{C}(t)$.

**Remark 2.2.1.** This is needed, in particular, for globalizing our constructions over the curve $X$. For more details, an interested reader may consult with the Seminaire Bourbaki talk [3].

The desired central extension $\hat{\mathfrak{g}}_{\kappa,x}$ comes from the (familiar) short exact sequence

$$0 \to \mathbb{C} \cdot 1 \to \hat{\mathfrak{g}}_{\kappa,x} \to \mathfrak{g} \otimes K_x \to 0,$$

where the cocycle is given by the standard formula $c(A \otimes f, B \otimes g) = -\kappa(A, B) \text{Res}_x(fdg)$. This definition does not depend on the choice of the local coordinate $t$ on $X$ near $x$. Any such choice identifies $\hat{\mathfrak{g}}_{\kappa,x}$ with $\hat{\mathfrak{g}}_{\kappa}$.

In the same fashion, one can redefine the vacuum module.

$$(2.2.1) \quad V_\kappa(\mathfrak{g})_x = \text{Ind}_{\hat{\mathfrak{g}}_{\kappa,x} \otimes \mathbb{C}1}^{\mathfrak{g} \otimes K_x} \mathbb{C}.$$ 

Hence, one also has

$$\mathfrak{z}(\mathfrak{g})_x = \mathfrak{z}(V_\kappa(\mathfrak{g})_x) = (V_\kappa(\mathfrak{g})_x)^{\mathfrak{g} \otimes O_x}.$$ 

Our goal is to obtain a coordinate-free description of this algebra (note that we do not give a coordinate free description of the vertex algebra $V_\kappa(\hat{\mathfrak{g}})$ itself). A description of $\mathfrak{z}(\hat{\mathfrak{g}})_x$ that involves a choice of a coordinate is in Theorem 1.4.1.

2.3. The group of coordinate changes. We start by studying how the picture from the Section 2.2 interacts with coordinate changes. A coordinate change is understood as an automorphism of $O_x$. Such automorphisms form a group to be denoted by $\text{Aut } O$.

Any $\phi \in \text{Aut } O$ is uniquely determined by its action on the coordinate $t$. So we can identify $\text{Aut } O$ with the set of formal power series $\sum_{i=1}^{\infty} a_i t^i$ with $a_1 \neq 0$. The group operation is the composition: $\phi(t) \circ \psi(t) = \phi(\psi(t))$.

Set $\text{Aut}_+ O := \{ \phi \in \text{Aut } O \mid a_1 = 1 \}$. This is a normal subgroup of $\text{Aut } O$ and, moreover,

$$\text{Aut } O \simeq \mathbb{C}^\times \ltimes \text{Aut}_+ O,$$

where $\mathbb{C}^\times$ is identified with the subgroup of $\text{Aut } O$ consisting of “loop rotations”, i.e., the automorphisms of the form $a : t \mapsto at, a \in \mathbb{C}^\times$.

**Remark 2.3.1.** We can also consider, for each $n \geq 0$, the group $\text{Aut}(\mathbb{C}[t]/(t^{n+1}))$ together with its decomposition $\text{Aut}(\mathbb{C}[t]/(t^{n+1})) = \mathbb{C}^\times \ltimes \text{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$. Note that $\text{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$ is a unipotent algebraic group, its elements can be thought of as elements $t + a_2 t^2 + \ldots + a_n t^n$ with the group law given by composition followed by truncation, i.e., setting $t^{n+1}$ to 0. So we have an algebraic group epimorphism $\text{Aut}_+(\mathbb{C}[t]/(t^{n+2})) \to \text{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$ and the group $\text{Aut}_+ O$ is the inverse limit $\varprojlim_n \text{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$. Therefore it is a pro-unipotent pro-algebraic group.
The Lie algebras of the algebraic groups of interest are as follows:

\[
\text{Lie}(\text{Aut } \mathcal{O}) = \text{Der}_0 \mathcal{O} := tC[[t]] \partial_t, \\
\text{Lie}(\text{Aut}_+ \mathcal{O}) = \text{Der}_+ \mathcal{O} := t^2 C[[t]] \partial_t,
\]

Note that the Lie subalgebra of \( C^\times \subset \text{Lie}(\text{Aut } \mathcal{O}) \) is spanned by the Euler vector field \( t \partial_t \). We also note that the entire algebra \( \text{Der}_0 \mathcal{O} \) of derivations of \( \mathcal{O} \) equals \( C[[t]] \partial_t \), hence is strictly larger than the Lie algebra \( \text{Lie}(\text{Aut } \mathcal{O}) \).

Recall the standard notation

\[
L_n = -t^{n+1} \frac{\partial}{\partial t}, n \geq 0.
\]

These elements form a topological basis inside \( \text{Der}_0 C[[t]] \). Moreover, \( L_n \) \((n > 0)\) form a topological basis inside \( \text{Der}_+ C[[t]] \). In particular, \( \text{Der}_0 \mathcal{O} \) can be embedded into the Virasoro algebra as its positive part.

2.4. The space \( \mathfrak{g}(t)/\mathfrak{g}[[t]] \) vs functions on the 1-forms. Here, we will be interested in a description of \( \text{gr} V_k(\mathfrak{g}) \). Let us recall the following statement from [6, Section 3.3].

**Proposition 2.4.1.** We have the following graded algebra isomorphisms

\[
\text{gr} V_k(\mathfrak{g}) \simeq \text{Sym}(\mathfrak{g}(t)/\mathfrak{g}[[t]]) \simeq C[\mathfrak{g}^*[[t]]]dt.
\]

We claim that the second isomorphism is coordinate-independent. Let us recall how it is constructed. The algebra \( C[\mathfrak{g}^*[[t]]]dt \) is the algebra of polynomials in the linear functions \( r_n : \mathfrak{g}^*[[t]]dt \to C \) with \( n < 0 \) given by \( \alpha \mapsto \text{Res}_{z=0} (z^n \alpha) \). Let \( (\mathfrak{g}^*[[t]]dt)^\vee \) denote the vector space with basis \( r_n \), of course, it is just the space of continuous linear maps \( \mathfrak{g}^*[[t]]dt \to C \) (the continuity is with respect to the \( t \)-adic topology on \( \mathfrak{g}^*[[t]]dt \) and the discrete topology on \( C \)). We then have a linear map

\[
(2.4.1) \quad \mathfrak{g}(t)/\mathfrak{g}[[t]] \to (\mathfrak{g}^*[[t]]dt)^\vee
\]

given by \( f \mapsto r_f \) with \( r_f(\alpha) := \text{Res}_{z=0} (f \alpha) \). It is an isomorphism and it is coordinate-independent because taking the residue of a form is classically known to be coordinate-independent.

In what follows in this section we will reprove that (2.4.1) is coordinate-independent from scratch in order to illustrate a general technique that we are going to use in what follows to prove new (and more difficult) results. We note that the group \( \text{Aut } \mathcal{O} \) acts both on the source and the target of (2.4.1). Our claim that this map is coordinate-independent means just that it is \( \text{Aut}(\mathcal{O}) \)-equivariant. And, since \( \text{Aut}(\mathcal{O}) \) is connected (as a semi-direct product of a torus and a pro-unipotent group), to show that (2.4.1) is \( \text{Aut}(\mathcal{O}) \)-equivariant, it is enough to show that it is \( \text{Der}_0 \mathcal{O} \)-equivariant. In fact, we will see that it is \( \text{Der } \mathcal{O} \)-equivariant. We will do this for an analog of (2.4.1) where \( \mathfrak{g} \) is replaced with \( C \).

We have \( L_n t^k = -kt^{k+n} \) if \( k + m \leq -1 \) and \( L_n t^k = 0 \) else. The action of \( \text{Der } \mathcal{O} \) on \( C[[t]]dt \) is given by

\[
(2.5.2) \quad L_n (t^{-m-1}dt) = (m-n)t^{n-m-1}dt.
\]

So \( \langle L_n r_k, t^{-m-1}dt \rangle = \langle r_k, L_n t^{-m-1}dt \rangle = (m-n) \langle r_k, t^{n-m-1}dt \rangle = (m-n) \delta_{k+n-m,0} \). We conclude that \( L_n r_k = -k r_{k+m} \), which shows that the analog of (2.4.1) (that by definition sends \( t^k \) to \( r_k \)) is \( \text{Der } \mathcal{O} \)-equivariant.

2.5. \( \text{Der } \mathcal{O} \)-action on the elements \( S_m \). Section 2.4 suggests that in order to give a coordinate free description of the algebra \( C[S_m|0]|m \leq -2 \) we should determine how \( \text{Der } \mathcal{O} \) acts on the elements \( S_m \) (under the natural action of \( \text{Der } \mathcal{O} \) on \( \mathfrak{g} \) by derivations).

Choose \( \kappa_0 \) to be the standard trace pairing. With this choice, [6, Corollary 2.17] tells us that if \( \kappa \) is not critical, then

\[
(2.5.1) \quad L_n S_m = (n-m) S_{n+m} - \frac{1}{2} (n^3 - n) \delta_{n,-m}.
\]

But near \( \kappa = \kappa_c \), the element \( S_m \) depends continuously on \( \kappa \), and so (2.5.1) also holds for \( \kappa = \kappa_c \).
Remark 2.5.1. (2.5.1) is true for a suitable normalization of the Sugawara elements for the general simple \( \mathfrak{g} \).

3. Projective connections

3.1. Definition. Let us introduce the vector space \( \Omega_D^\lambda \) of \( \lambda \)-forms on \( D \). Its elements are, by definition, the formal expressions of the form \( f(t)(dt)^\lambda \) for \( \lambda \in \mathbb{C} \).

The space \( \Omega_D^\lambda \) becomes a \( \text{Der} \mathcal{O} \)-module via the following formula (for \( \lambda = 1 \) we recover (2.4.2)):

\[
\xi(t) \partial_t \cdot f(t)(dt)^\lambda = ((\xi(t)f'(t) + \lambda f(t)\xi'(t)))(dt)^\lambda.
\]

Definition 3.1.1. A projective connection on \( D = \text{Spec} \mathbb{C}[[t]] \) is a second order differential operator

\[
\rho : \Omega_D^{-1/2} \to \Omega_D^{3/2}
\]

of the form \( \partial_t^2 - v(t) \). By definition, this operator sends \( f(t)(dt)^{-1/2} \) to \((f''(t) - v(t)f(t))(dt)^{3/2} \).

3.2. The action of vector fields. We now would like to write down the action of vector fields on projective connections. The action is as on the linear maps between two \( \text{Der} \mathcal{O} \)-modules.

First, we compute

\[
\xi(t) \partial_t \cdot ((\partial_t^2 - v(t))f(t)(dt)^{-1/2}) = \xi(t) \partial_t((f''(t) - v(t)f(t))(dt)^{3/2}) =
\]

\[
= (\xi(t)(f''(t) - v'(t)f(t) - v(t)f'(t)) + \frac{3}{2}((f''(t) - v(t)f(t))\xi'(t)))(dt)^{3/2}.
\]

On the other hand,

\[
(\partial_t^2 - v(t))(\xi(t) \partial_t \cdot f(t)(dt)^{-1/2}) = (\partial_t^2 - v(t))(\xi(t)f'(t) - \frac{1}{2}f(t)\xi'(t))dt^{-1/2} =
\]

\[
(\frac{3}{2}\xi'(t)f''(t) + \xi(t)f'''(t) - v(t)\xi(t)f'(t) - \frac{1}{2}f(t)\xi''(t) + \frac{1}{2}v(t)f(t)\xi'(t))(dt)^{3/2}.
\]

Thus, the formula for the action of \( \xi(t) \partial_t \) comes by taking the difference between two last quantities:

\[
(3.2.1) \quad \xi(t) \partial_t : \partial_t^2 - v(t) \mapsto (\xi(t)v'(t) + 2v(t)\xi'(t) - \frac{1}{2}\xi''(t)).
\]

Remark 3.2.1. The space of projective connections can be viewed as an affine space with associated vector space \( \mathbb{C}[[t]] \). We can identify the two by sending \( \partial_t^2 \) to 0. Note that (3.2.1) defines a homomorphism from \( \text{Der} \mathcal{O} \) to the Lie algebra of the group of affine transformations of \( \mathbb{C}[[t]] \).

Note also that (3.2.1) integrates to an action of \( \text{Aut} \mathcal{O} \): for the coordinate change \( t = \phi(s) \) (\( \phi \) as in 2.3, one has \( \partial_t^2 - v(t) = \partial_s^2 - w'(s) \) for \( w(s) = v(\phi(s))\phi'(s)^2 - \frac{1}{2}\{\phi, s\} \), where \( \{\phi, s\} := \frac{\phi''}{\phi} - \frac{3}{2}(\frac{\phi''}{\phi})^2 \) is the so-called Schwarzian derivative.

3.3. Main theorem. Now, we can state the main result of this part. We write \( D_x \) for \( \text{Spec}(\mathcal{O}_x) \), and \( \mathcal{P} \text{roj}(D_x) \) for the space of projective connections on \( D_x \) that we view as an infinite dimensional affine space.

Theorem 3.3.1. We have a coordinate-independent isomorphism \( \mathfrak{sl}_2 \) \( \simeq \mathbb{C}[\mathcal{P} \text{roj}(D_x)] \).

Proof. For \( k \leq -2 \), we write \( p_k \) for the element of \( \mathbb{C}[\mathcal{P} \text{roj}(D_x)] \) that sends a projective connection \( \partial_t^2 - \sum_{i=0}^{-k} a_i t^i \) to \( a_{-k-2} \). Note that \( \mathbb{C}[\mathcal{P} \text{roj}(D_x)] = \mathbb{C}[p_k[k \leq -2]] \). We claim that the assignment sending \( S_k[0] \) to \( p_k \) defines a \( \text{Der} \mathcal{O} \)-equivariant isomorphism \( \mathfrak{sl}_2 \) \( \simeq \mathbb{C}[\mathcal{P} \text{roj}(D_x)] \).

Identify the space \( \mathcal{P} \text{roj}(D_x) \) with \( \mathbb{C}[[t]] \) as in Remark 3.2.1. Thanks to (3.2.1), we have

\[
\langle L_n p_k, t^{-m-2} \rangle = \langle p_k, -L_n t^{-m-2} \rangle = \langle p_k, -(m + 2)t^{n-m-2} + 2(n + 1)t^{n-m-2} - \frac{n^3}{2} - \frac{n}{2} \rangle
\]

\[
= \delta_{k,m-n}(2n - m) - \delta_{k,m-n} \frac{n^3 - n}{2}.
\]
We conclude that $L_n p_k = (n-k)p_{k+n} - \frac{n^2-n}{2} \delta_{k,-n}$ (if $k+n > -2$, then the first summand in the right hand side is declared to be zero). This matches (2.5.1) and hence shows that the isomorphism defined by $S_k|0 \mapsto p_k$ is Der $\mathcal{O}$-equivariant, finishing the proof. □

In subsequent talks we will see that Theorem 3.3.1 has a close relative.

**Theorem 3.3.2.** We have an algebra isomorphism $Z(\bar{U}_{\kappa_c}(\hat{sl}_2, x)) \cong \mathbb{C}[\text{Proj}(D^x_\mathbf{x})]$, where $D^x_\mathbf{x} := \text{Spec}(\mathcal{K}_\mathbf{x})$.

**References**