INVARIANTS OF JETS AND THE CENTER FOR $\hat{\mathfrak{sl}}_2$

IVAN KARPOV, IVAN LOSEV

Abstract. This is an expository talk for the student learning seminar on the representation theory of affine Kac-Moody algebras at the critical level. We develop the formalism of jet schemes and use it to compute the algebra of invariants for the action of the group $G[[t]]$ on its adjoint representation $\mathfrak{g}[[t]]$. In turn, we use this computation to show that the center of $V_{\kappa_c}(\hat{\mathfrak{sl}}_2)$ is the polynomial algebra freely generated by the Sugawara modes. We then identify the center of $V_{\kappa_c}(\hat{\mathfrak{sl}}_2)$ with the algebra of polynomial functions on the space of projective connections on the disc $D = \operatorname{Spec}(\mathbb{C}[[t]])$ thus getting a coordinate free description of the center. We moslty follow [2].

1. Invariants and the center

1.1. **Introduction.** Throughout the talk, the base field is $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. The corresponding connected algebraic group G acts on $\mathfrak{g}$ (via the adjoint representation), yielding G-actions by graded algebra automorphisms on $\mathbb{C}[\mathfrak{g}](\cong S(\mathfrak{g}))$ and by filtered algebra automorphisms on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ denote a Cartan subalgebra, and W be the corresponding Weyl group. The following is due to Chevalley:

Proposition 1.1.1. (A) *We have a graded algebra isomorphism* $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$.

(B) *The algebras in (A) are isomorphic to the polynomial algebra in* $r := \operatorname{rk}\mathfrak{g}$ *homogeneous generators, to be denoted by* $P_1, \ldots, P_r$.

It is also well-known due to Harish-Chandra (see, e.g., [4, Ch. 23]) that the center $\mathcal{Z}(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ is isomorphic to $\mathbb{C}[\mathfrak{h}]^W$ as a filtered algebra. The Harish-Chandra theorem can be viewed as a finite dimensional counterpart of the main result for the seminar: a description of the center of the completed universal enveloping algebra of $\hat{\mathfrak{g}}$ at the critical level.

We write $\mathcal{O}$ for $\mathbb{C}[[t]]$, $G_{\mathcal{O}}$ for the group of $\mathcal{O}$-points of G and $\mathfrak{g}_{\mathcal{O}}$ for its Lie algebra, $\mathfrak{g}\otimes\mathcal{O}$, compare to [6, Section 3]. The main goal of the first part of the talk is to get an analog of Proposition 1.1.1 for the action of the group $G_{\mathcal{O}}$ on $\mathfrak{g}_{\mathcal{O}}$: we will see that the elements $P_{i,n}$ with $i = 1, \ldots, r$ and $n < 0$ introduced in [6, Section 3.4] are free generators of $\mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}}$. We will use this to show that the Sugawara modes $S_n|0\rangle \in V_{\kappa_c}(\hat{\mathfrak{sl}}_2)$ (with $n \leqslant -2$) generate the center of $V_{\kappa_c}(\hat{\mathfrak{sl}}_2)$.

1.2. **Jet schemes.** In order to compute the algebra $\mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}}$ we will need the formalism of jet schemes (a.k.a. arc spaces).

1.2.1. *Definition via functor of points.* Let CommAlg denote category of commutative associative unital $\mathbb{C}$-algebras, its opposite category is identified with the category of affine schemes over $\operatorname{Spec}(\mathbb{C})$. In particular, an arbitrary scheme X over $\operatorname{Spec}(\mathbb{C})$ gives rise to its *functor of points*

$$\operatorname{Mor}(\operatorname{Spec}(?), X) : \text{CommAlg} \to \text{Sets}$$

sending an algebra R to the *set of R-points of X*. One recovers X uniquely from its functor of points, however, not every functor CommAlg $\to$ Sets is representable (i.e., is a functor of points for a scheme).

Definition 1.2.1. *Let X be a finite type scheme over* $\operatorname{Spec}(\mathbb{C})$. *We define the* jet functor *of X*

$$J_X : \text{CommAlg} \to \text{Sets}$$

by sending R to the set of all morphisms $\operatorname{Spec}(R[[t]]) \to X$ *(of schemes over* $\operatorname{Spec}(\mathbb{C})$*).*

Proposition 1.2.2. *The functor J_X is represented by a scheme to be denoted by JX and called the* jet scheme *(a.k.a.* arc space*) of X.*

We will sketch a proof (and a construction of JX) below in this section.

We also note that for general Yoneda reasons, J is a functor (from the category of finite type schemes to the category of schemes). For a morphism $\varphi : X \to Y$ we write $J\varphi$ for the induced morphism $JX \to JY$.

1.2.2. *Affine case.* We first give a constructive proof of Proposition 1.2.2 in the case when X is affine.

Example 1.2.3. *First, set $X = \mathbb{A}^m = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_m])$. For an arbitrary commutative $\mathbb{C}$-algebra R, the set of $R[[t]]$-points of X is*

$$\mathrm{Hom}_{Alg}(\mathbb{C}[x_1, \ldots, x_m], R[[t]]).$$

Of course, any algebra homomorphism $\phi : \mathbb{C}[x_1, \ldots, x_m] \to R[[t]]$ is uniquely determined from the images $\phi(x_i)$ that are formal power series

$$\phi(x_i) = \sum_{n<0} a_{i,n} t^{-n-1}, a_{i,n} \in R.$$

Thus, the set of R-point of JX is the set $\{a_{i,n} \in R | i = 1, \ldots, m, n < 0\}$ and hence

$$JX = \mathrm{Spec}\,\mathbb{C}[x_{i,n} | i = 1, \ldots, m, n < 0].$$

Example 1.2.4. *Now we consider the case when X is a general finite type affine scheme over $\mathrm{Spec}(\mathbb{C})$, it can be defined as*

$$\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_m]/(F_1, \ldots, F_k)).$$

The same reasoning as in the Example 1.2.3 shows that the set $Mor(\mathrm{Spec}(R), JX)$ can be be identified with the set of $a_i(t) := \phi(x_i) \in R[[t]]$ such that

$$F_j(a_1(t), \ldots, a_n(t)) = 0 \tag{1.2.1}$$

for all $j = 1, \ldots, k$.

To describe this set of formal power series, consider the algebra $\mathcal{R} := \mathbb{C}[x_{i,n}]$ (cf. Example 1.2.3). Define a derivation $T \in \mathrm{Der}_{\mathbb{C}}(\mathcal{R})$ on the free generators by:

$$T : x_{j,n} \mapsto -n x_{j,n-1}.$$

Now, define $F_j^{\#} := F_j(x_{i,-1})$. One can show that the system of equations (1.2.1) is equivalent to $T^\ell F_j^{\#} = 0$ for all possible $\ell \geqslant 0$ and $j = 1, \ldots, k$. So for JX we can take the closed subscheme of $J\mathbb{A}^m$ given by the equations $T^\ell F_j^{\#}$:

$$JX = \mathrm{Spec}(\mathcal{R}/(T^\ell F_j^{\#})).$$

Remark 1.2.5. *We have an algebra homomorphism $\mathbb{C}[X] \to \mathbb{C}[JX]$ sending $F = F(x_1, \ldots, x_m)$ to $F^{\#}$ defined by $F(x_{1,-1}, \ldots, x_{m,-1})$. It yields a scheme morphism $JX \to X$.*

Exercise 1.2.6. *Let X, Y be finite type affine schemes (over $\mathrm{Spec}(\mathbb{C})$). Identify $J(X \times Y)$ with $JX \times JY$. More precisely, let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections. Then $J\pi_1 \times J\pi_2 : J(X \times Y) \xrightarrow{\sim} JX \times JY$.*

1.2.3. *Gluing.* Now we proceed to the case of non-affine finite type schemes Y. We claim that JY can be glued from JX for open affines $X \subset Y$. The key step here is to relate JX and $J(X_f)$ for $f \in \mathbb{C}[X]$, where X_f is the non-vanishing locus for f (known as a principal open subset). We claim that $J(X_f)$ is naturally identified with $(JX)_{f^\sharp}$, where $f^\sharp \in \mathbb{C}[JX]$ is defined in Remark 1.2.5.

Indeed, recall that if $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_m]/(F_1, \ldots, F_k)$, then

$$\mathbb{C}[X_f] = \mathbb{C}[x_1, \ldots, x_m, x]/(F_1, \ldots, F_k, xf - 1).$$

It follows that $\mathbb{C}[J(X_f)] = \mathbb{C}[JX][x_n | n < 0]/(T^\ell(xf-1)^\sharp)$. For $\ell = 0$, the equation $T^\ell(xf-1)^\sharp = 0$ means that $x_{-1} f^\sharp = 1$, i.e., $f^\sharp$ is invertible, and $x_{-1} = (f^\sharp)^{-1}$. The equation $T^\ell(xf-1)^\sharp = 0$ for

$\ell > 0$ then uniquely expresses $x_{-\ell-1}$ as a polynomial in $x_{-1}, \dots, x_{-\ell}, (f^\sharp)^{-1}$ and elements of $\mathbb{C}[JX]$. This gives the required identification $\mathbb{C}[J(X_f)] \cong \mathbb{C}[JX][(f^\sharp)^{-1}]$.

This discussion finishes our sketch of proof of Proposition 1.2.2.

Remark 1.2.7. *Note that we still have a morphism $JY \to Y$. It is affine (of infinite type).*

1.2.4. *nth order jets.* Let X be a finite type scheme over Spec($\mathbb{C}$). It turns out that JX (which is an infinite type scheme) can be presented as the inverse limit of finite type schemes J_nX (*n-th order jet schemes*). By definition, J_nX represents the functor CommAlg $\to$ Sets sending R to the set of morphisms $\operatorname{Spec}(R[t]/(t^{n+1})) \to X$.

For example, for X as in Example 1.2.4, we have

$$J_nX = \operatorname{Spec}(\mathbb{C}[JX]/(x_{i,N} | i = 1, \dots, m, N < -n-1)).$$

As in the case of J, J_n is a functor (in this case, from the category of finite type schemes over Spec($\mathbb{C}$) to itself). The claim that $J = \varprojlim_{n\to\infty} J_n$ is left as an exercise (on the general categorical nonsense).

Exercise 1.2.8. *For X smooth, show that J_1X is the tangent bundle of X.*

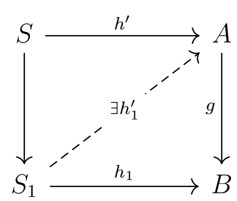
1.2.5. *Smoothness.* The goal of this part is to prove the following statement.

Theorem 1.2.9. *For a smooth morphism $\varphi : X \to Y$, the morphism $J_n\varphi : J_n(X) \to J_n(Y)$ is smooth as well.*[1]

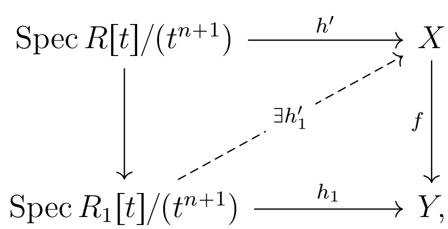
Indeed, let us recall the following criterion of smoothness ([1, Section 1.4]). If R is a commutative $\mathbb{C}$-algebra, then by its *nilpotent extension* we mean a commutative algebra R_1 equipped with an epimorphism $R_1 \twoheadrightarrow R$ whose kernel is a nilpotent ideal.

Proposition 1.2.10. *Suppose that $g : A \to B$ is a morphism of schemes of finite type over $\mathbb{C}$. Then, g is smooth if and only if for any morphism $h : S = \operatorname{Spec}(R) \to B$ which lifts to $h' : S \to A$ the following holds:*

suppose that R_1 is a nilpotent extension of R, that $S_1 = \operatorname{Spec}(R_1)$, and that $h_1 : S_1 \to B$ is any lifting of h. Then h_1 also lifts to $h_1' : S_1 \to A$:



Proof of Theorem 1.2.9. By definition, an R-point of J_nA is an $R[t]/(t^{n+1})$-point of A. Now, we have the diagram



where we need to prove the existence of h_1'. To finish the proof we combine Proposition 1.2.10 with the observation that $R_1[t]/(t^{n+1})$ is a nilpotent extension of $R[t]/(t^{n+1})$. ◻

Remark 1.2.11. *The similar argument proves that, for a surjective smooth morphism f, the morphism J_nf is also surjective (on the level of $\mathbb{C}$-points) for all n.*

Applying Theorem 1.2.9 to $Y = \mathrm{pt}$, we get the following claim.

[1] One can introduce the notion of "formal smoothness". Then, the same statement would be true for the functor J itself (instead of J_n's).

Corollary 1.2.12. *For a smooth variety X, the scheme J_nX is a smooth scheme of finite type.*

The following exercise (based on the generic smoothness) will be used below.

Exercise 1.2.13. *Let $\varphi : X \to Y$ be a dominant morphism to a smooth variety Y. Prove that $J_n\varphi : J_nX \to J_nY$ is dominant.*

1.3. **Jet-theoric Chevalley theorem.** Recall that we write $\mathcal{O}$ for the algebra $\mathbb{C}[[t]]$. For an affine scheme X we will often write $X_{\mathcal{O}}$ for JX.

Let G be an algebraic group. Applying the functoriality of J_n and J to the structure maps of G, we see that J_nG, JG are group schemes over $\mathbb{C}$. In fact, J_nG is an honest algebraic group with Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t]/(t^{n+1})$ – J_nG is the semi-direct product of G with the unipotent group $\exp(t\mathfrak{g}[t]/t^{n+1}\mathfrak{g}[t])$. This description shows, in particular, that $J_{n+1}G \twoheadrightarrow J_nG$ for all n. And JG is the limit $\varprojlim_{n\to\infty} J_nG$, hence a pro-algebraic group.

Applying the functor J to the action morphism $G \times \mathfrak{g} \to \mathfrak{g}$ we get the morphism $J(G \times \mathfrak{g}) \to J\mathfrak{g}$. Under the identification $JG \times J\mathfrak{g} \cong J(G \times \mathfrak{g})$ from Exercise 1.2.6, this gives an action of the pro-algebraic group JG on $J\mathfrak{g}$. We want to compute the algebra of invariant polynomial functions for this action.

The following result is a jet analog of Proposition 1.1.1. Recall that $P_i, i = 1,\ldots,r$, denote free homogeneous generators of the algebra $\mathbb{C}[\mathfrak{g}]^G$. Then we can form the elements $P_{i,n} \in \mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}}$ for all $\ell < 0$ and $i = 1,\ldots,r$, see [6, Section 3.4].

Theorem 1.3.1. *The algebra of invariants $\mathbb{C}[\mathfrak{g}_{\mathcal{O}}]^{G_{\mathcal{O}}}$ is identified with $\mathbb{C}[J(\mathfrak{h}/W)]$, equivalently, is freely generated by the elements $P_{i,\ell}$.*

1.3.1. *Preparation.* We write $\mathfrak{g}/\!/G := \operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G)$. We have the quotient morphism $\pi : \mathfrak{g} \to \mathfrak{g}/\!/G$ induced by the inclusion $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$. It gives rise to $J\pi : J\mathfrak{g} \to J(\mathfrak{g}/\!/G)$. By the Chevalley theorem, $\mathfrak{g}/\!/G$ is an affine space with coordinates $P_1,\ldots,P_r$. The polynomials $P_{i,\ell}$ are nothing else but the coordinates on the infinite dimensional affine space $J(\mathfrak{g}/\!/G)$. So our job is to show that the pullback homomorphism $(J\pi)^*$ identifies $\mathbb{C}[J(\mathfrak{g}/\!/G)]$ with the subalgebra of invariants for $G_{\mathcal{O}} = JG$ in $\mathbb{C}[J\mathfrak{g}]$.

We are going to reduce this to the analogous claim, where J is replaced with J_n: $(J_n\pi)^*$ identifies $\mathbb{C}[J_n(\mathfrak{g}/\!/G)]$ with $\mathbb{C}[J_n\mathfrak{g}]^{J_nG}$. Proving the latter for all n is enough for the following reason. Since $\mathbb{C}[J\mathfrak{g}]$ is the union of its subalgebras $\mathbb{C}[J_n\mathfrak{g}]$, we see that $\mathbb{C}[J\mathfrak{g}]^{JG}$ is the union of its subalgebras $\mathbb{C}[J\mathfrak{g}]^{JG} \cap \mathbb{C}[J_n\mathfrak{g}]$. Our reduction now follows from the next exercise (where one needs to use that $JG \twoheadrightarrow J_nG$ and that the projection $J\mathfrak{g} \to J_n\mathfrak{g}$ is JG-equivariant).

Exercise 1.3.2. $\mathbb{C}[J\mathfrak{g}]^{JG} \cap \mathbb{C}[J_n\mathfrak{g}] = \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$ *as subalgebras in* $\mathbb{C}[J\mathfrak{g}]$.

1.3.2. *1st proof of* $\mathbb{C}[J_n(\mathfrak{g}/\!/J_nG)] \xrightarrow{\sim} \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$. In this proof, different from what is given in [2, Section 3.4] we will use the Kostant slice, a remarkable affine subspace $S \subset \mathfrak{g}$ with the property that the restriction of the quotient morphism $\pi : \mathfrak{g} \to \mathfrak{g}/\!/G := \operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ to S is an isomorphism. For more on Kostant slices see [7]. In particular the claim that $\pi|_S$ is an isomorphism is proved in [7, Section 4].

Let ι denote the inclusion $S \hookrightarrow \mathfrak{g}$. Since $\pi \circ \iota$ is an isomorphism $S \xrightarrow{\sim} \mathfrak{g}/\!/G$, we see that $J_n\pi \circ J_n\iota : J_nS \xrightarrow{\sim} J_n(\mathfrak{g}/\!/G)$. It remains to show that $(J_n\iota)^*$ embeds $\mathbb{C}[J_n\mathfrak{g}]^{J_nG}$ into $\mathbb{C}[J_nS]$.

Let β denote the action map $G \times S \to \mathfrak{g}, (g,s) \mapsto \operatorname{Ad}(g)s$, and ι' denote the embedding $S \hookrightarrow G \times S, s \mapsto (1,s)$. Note that $\iota = \beta \circ \iota'$, hence $J_n\iota = J_n\beta \circ J_n\iota'$. The action of G on $G \times S$ (by left translations on the first factor) gives rise to an action of J_nG on $J_n(G \times S) = J_nG \times J_nS$ (also by left translation on the first factor). So $(J_n\iota')^*$ restricts to an isomorphism $\mathbb{C}[J_n(G\times S)]^{J_nG} \xrightarrow{\sim} \mathbb{C}[J_nS]$. So, the claim that $(J_n\iota^*) : \mathbb{C}[J_n\mathfrak{g}]^{J_nG} \hookrightarrow \mathbb{C}[J_nS]$ is equivalent to $(J_n\beta)^* : \mathbb{C}[J_n\mathfrak{g}]^{J_nG} \hookrightarrow \mathbb{C}[J_n(G\times S)]^{J_nG}$, which will follow from $(J_n\beta)^* : \mathbb{C}[J_n\mathfrak{g}] \hookrightarrow \mathbb{C}[J_n(G\times S)]$. To see the latter injectivity, we remark that $\beta : G\times S \to \mathfrak{g}$ is dominant (Step 1 of the proof of Theorem in [7, Section 4]) and use Exercise 1.2.13. This completes the 1st proof of Theorem 1.3.1.

1.3.3. *2nd proof of* $\mathbb{C}[J_n(\mathfrak{g}/\!/J_nG)] \xrightarrow{\sim} \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$. Now we give a proof that closely follows one in [2]. Consider the open subset of regular elements:

$$\mathfrak{g}^{reg} = \{x \in \mathfrak{g} \mid \dim Z_{\mathfrak{g}}(x) = \operatorname{rk}\mathfrak{g}\},$$

studied in detail in [7, Section 5]. In particular, we have the following claim

(*) The morphism $\pi|_{\mathfrak{g}^{reg}}$ is smooth, and each fiber of $\pi|_{\mathfrak{g}^{reg}} : \mathfrak{g}^{reg} \to \mathfrak{g}/\!/G$ is a single G-orbit (in particular, the morphism is surjective).

Exercise 1.3.3. *For $\mathfrak{g} = \mathfrak{sl}_n$, the subset $\mathfrak{g}^{reg}$ consists precisely of all matrices such that in their Jordan normal form, there is a single block for each eigenvalue.*

Suppose, for a moment, that we know that the direct analog of (*) holds for the action of J_nG on $J_n\mathfrak{g}^{reg}$ and the morphism $J_n(\pi|_{\mathfrak{g}^{reg}}) : J_n\mathfrak{g}^{reg} \to J_n(\mathfrak{g}/\!/G)$. We then can prove that $\mathbb{C}[J_n(\mathfrak{g}/\!/G)] \xrightarrow{\sim} \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$ using the following general result.

Proposition 1.3.4. *Let H be an algebraic group and X, Y be normal algebraic varieties. Suppose H acts on X, and Y is affine. Suppose, further, that $\varphi : X \to Y$ is a surjective H-invariant morphism such that each fiber of ϕ is a single H-orbit. Then $\varphi^* : \mathbb{C}[Y] \xrightarrow{\sim} \mathbb{C}[X]^H$.*

Proof. Clearly, $\varphi^* : \mathbb{C}[Y] \hookrightarrow \mathbb{C}[X]^H$ and we need to prove the surjectivity. Take $f \in \mathbb{C}[X]^H$, and consider the subalgebra of $\mathbb{C}[X]^H$ generated by $\mathbb{C}[Y]$ and f, denote it by A. Then φ factors as $X \to \operatorname{Spec}(A) \to Y$, where both morphisms are dominant. Since each fiber of φ is a single orbit, $\operatorname{Spec}(A) \to Y$ is injective. Any injective dominant morphism is birational, hence f can be viewed as a rational function on Y. It is left as an exercise to show that f has no poles on Y. Since Y is normal, $f \in \operatorname{im}\varphi^*$. This finishes the proof. ☐

We apply this to $X = J_n\mathfrak{g}^{reg}, Y = J_n(\mathfrak{g}/\!/G)$ and $H = J_nG$. Note that $J_n(\mathfrak{g}/\!/G)$ is smooth, hence normal, we use the analog of (*) to deduce $\mathbb{C}[J_n(\mathfrak{g}/\!/J_nG)] \xrightarrow{\sim} \mathbb{C}[J_n(\mathfrak{g}^{reg})]^{J_nG}$. The subvariety $J_n(\mathfrak{g}^{reg}) \subset J_n\mathfrak{g}$ is open and dense. So the restriction homomorphism $\mathbb{C}[J_n\mathfrak{g}] \to \mathbb{C}[J_n(\mathfrak{g}^{reg})]$ is injective. From here we deduce that $\mathbb{C}[J_n(\mathfrak{g}/\!/J_nG)] \xrightarrow{\sim} \mathbb{C}[J_n\mathfrak{g}]^{J_nG}$.

Now, it remains to establish that analog. First, we reformulate the claim.

Exercise 1.3.5. *Let H be an algebraic group acting on a variety X, Y is a variety, and $\varphi : X \to Y$ be an H-invariant morphism. The following claims are equivalent.*

(a) *The morphism φ is smooth and each fiber of φ is a single H-orbit.*
(b) *The morphism $H \times X \to X \times_Y X, (h, x) \mapsto (hx, x)$ is smooth and surjective.*

Apply Exercise 1.3.5 to $H = G, X = \mathfrak{g}^{reg}, Y = \mathfrak{g}/\!/G, \varphi = \pi|_{\mathfrak{g}^{reg}}$ to get that $G \times \mathfrak{g}^{reg} \to \mathfrak{g}^{reg} \times_{\mathfrak{g}/\!/G} \mathfrak{g}^{reg}$ is smooth and surjective. Hence, by Section 1.2.5, $J_n(G \times \mathfrak{g}^{reg}) \to J_n(\mathfrak{g}^{reg} \times_{\mathfrak{g}/\!/G} \mathfrak{g}^{reg})$. One can use the smoothness of $\pi|_{\mathfrak{g}^{reg}}$ and generalize Exercise 1.2.6, to identify $J_n(\mathfrak{g}^{reg} \times_{\mathfrak{g}/\!/G} \mathfrak{g}^{reg})$ with $J_n(\mathfrak{g}^{reg}) \times_{J_n(\mathfrak{g}/\!/G)} J_n(\mathfrak{g}^{reg})$. We get (b) of Exercise 1.3.5 for $H = J_nG, X = J_n(\mathfrak{g}^{reg}), Y = J_n(\mathfrak{g}/\!/G), \varphi = J_n(\pi|_{\mathfrak{g}^{reg}})$, yielding (a), which is what we need to finish the proof.

1.4. **Center of $V_{\kappa_c}(\mathfrak{sl}_2)$.** Suppose $G = \mathrm{SL}_2$. We recall the definition of *Sugawara operators* from [6, Section 1].

$$S_n = \frac{1}{2}\sum_{i=1}^{3}\sum_{j+k=n} :x^i(j)x_i(k): .$$

The elements $S_n|0\rangle$ for $n \leqslant -2$ are central in $V_{\kappa_c}(\mathfrak{sl}_2)$.

Theorem 1.4.1. *The center of the vertex algebra $V_{\kappa_c}(\mathfrak{sl}_2)$ is isomorphic to $\mathbb{C}[S_n|0\rangle]_{n\leqslant -2}$ (as a commutative algebra).*

We leave the proof of this theorem as an exercise. A warm-up is to recall how to prove that the Casimir element is a free generator of the center of the $U(\mathfrak{sl}_2)$ once one knows the description of $\mathbb{C}[\mathfrak{sl}_2]^{\mathrm{SL}_2}$. For details, see [2, Section 3.5].

2. The coordinate-independent description of $V_k(\mathfrak{g})$.

2.1. **The ring O and the field $\mathcal{K}$.** Suppose that X be a smooth curve. Let us define the ring $\mathcal{O}_x$ as the completion of the local ring $\mathcal{O}_{X,x}$ at the maximal ideal $\mathfrak{m}_x$, i.e., $\mathcal{O}_x := \varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_x^n$. Let also $\mathcal{K}_x$ be the field of fractions of $\mathcal{O}_x$. Let $\widehat{\mathfrak{m}}_x$ denote the maximal ideal in $\mathcal{O}_x$.

A choice of an element $t \in \widehat{\mathfrak{m}}_x \backslash \widehat{\mathfrak{m}}_x^2$ is the same as a choice of an isomorphism $\mathcal{O}_x \xrightarrow{\sim} \mathbb{C}[[t]]$.

Exercise 2.1.1. *Prove this statement. A hint: to see that there is an isomorphism use a suitable étale map from an open neighborhood of x to $\mathbb{A}^1$.*

Note that an isomorphism $\mathcal{O}_x \xrightarrow{\sim} \mathbb{C}[[t]]$ induces an isomorphism $\mathcal{K}_x \xrightarrow{\sim} \mathbb{C}((t))$.

2.2. **The algebra $\hat{\mathfrak{g}}_{\kappa,x}$.** We now want to define the Kac-Moody algebra in a coordinate free way, using $\mathcal{K}_x$ instead of $\mathbb{C}((t))$.

Remark 2.2.1. *This is needed, in particular, for globalizing our constructions over the curve X. For more details, an interested reader may consult with the Seminaire Bourbaki talk* [3].

The desired central extension $\hat{\mathfrak{g}}_{\kappa,x}$ comes from the (familiar) short exact sequence

$$0 \to \mathbb{C} \cdot \mathbf{1} \to \hat{\mathfrak{g}}_{\kappa,x} \to \mathfrak{g} \otimes \mathcal{K}_x \to 0,$$

where the cocycle is given by the standard formula $c(A \otimes f, B \otimes g) = -\kappa(A, B)\operatorname{Res}_x(f dg)$.

This definition does not depend on the choice of the local coordinate t on X near x. Any such choice identifies $\hat{\mathfrak{g}}_{\kappa,x}$ with $\hat{\mathfrak{g}}_\kappa$.

In the same fashion, one can redefine the vacuum module.

$$V_\kappa(\mathfrak{g})_x = \operatorname{Ind}_{\mathfrak{g}\otimes\mathcal{O}_x\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa,x}} \mathbb{C}. \tag{2.2.1}$$

Hence, one also has

$$\mathfrak{z}(\widehat{\mathfrak{g}})_x = \mathfrak{z}(V_\kappa(\mathfrak{g})_x) = (V_\kappa(\mathfrak{g})_x)^{\mathfrak{g}\otimes\mathcal{O}_x}.$$

Our goal is to obtain a coordinate-free description of this algebra (note that we do not give a coordinate free description of the vertex algebra $V_\kappa(\hat{\mathfrak{g}})$ itself). A description of $\mathfrak{z}(\widehat{\mathfrak{g}})_x$ that involves a choice of a coordinate is in Theorem 1.4.1.

2.3. **The group of coordinate changes.** We start by studying how the picture from the Section 2.2 interacts with coordinate changes. A coordinate change is understood as an automorphism of $\mathcal{O}_x$. Such automorphisms form a group to be denoted by $\operatorname{Aut}\mathcal{O}$.

Any $\phi \in \operatorname{Aut}\mathcal{O}$ is uniquely determined by its action on the coordinate t. So we can identify $\operatorname{Aut}\mathcal{O}$ with the set of formal power series $\sum_{i=1}^\infty a_i t^i$ with $a_1 \neq 0$. The group operation is the composition: $\phi(t) \circ \psi(t) = \phi(\psi(t))$.

Set $\operatorname{Aut}_+\mathcal{O} := \{\phi \in \operatorname{Aut}\mathcal{O} \mid a_1 = 1\}$. This is a normal subgroup of $\operatorname{Aut}\mathcal{O}$ and, moreover,

$$\operatorname{Aut}\mathcal{O} \simeq \mathbb{C}^\times \ltimes \operatorname{Aut}_+\mathcal{O},$$

where $\mathbb{C}^\times$ is identified with the subgroup of $\operatorname{Aut}\mathcal{O}$ consisting of "loop rotations", i.e., the automorphisms of the form $a : t \mapsto at, a \in \mathbb{C}^\times$.

Remark 2.3.1. *We can also consider, for each $n \geqslant 0$, the group $\operatorname{Aut}(\mathbb{C}[t]/(t^{n+1}))$ together with its decomposition $\operatorname{Aut}(\mathbb{C}[t]/(t^{n+1})) = \mathbb{C}^\times \ltimes \operatorname{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$. Note that $\operatorname{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$ is a unipotent algebraic group, its elements can be thought of as elements $t + a_2 t + \ldots + a_n t^n$ with the group law given by composition followed by truncation, i.e., setting t^{n+1} to 0. So we have an algebraic group epimorphism $\operatorname{Aut}_+(\mathbb{C}[t]/(t^{n+2})) \twoheadrightarrow \operatorname{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$ and the group $\operatorname{Aut}_+\mathcal{O}$ is the inverse limit $\varprojlim_n \operatorname{Aut}_+(\mathbb{C}[t]/(t^{n+1}))$. Therefore it is a pro-unipotent pro-algebraic group.*

The Lie algebras of the algebraic groups of interest are as follows:

$$\mathrm{Lie}(\mathrm{Aut}\,\mathcal{O}) = \mathrm{Der}_0\,\mathcal{O} := t\mathbb{C}[[t]]\partial_t,$$
$$\mathrm{Lie}(\mathrm{Aut}_+\,\mathcal{O}) = \mathrm{Der}_+\,\mathcal{O} := t^2\mathbb{C}[[t]]\partial_t,$$

Note that the Lie subalgebra of $\mathbb{C}^\times \subset \mathrm{Lie}(\mathrm{Aut}\,\mathcal{O})$ is spanned by the Euler vector field $t\partial_t$. We also note that the entire algebra $\mathrm{Der}_0\,\mathcal{O}$ of derivations of $\mathcal{O}$ equals $\mathbb{C}[[t]]\partial_t$, hence is strictly large than the Lie algebra $\mathrm{Lie}(\mathrm{Aut}\,\mathcal{O})$.

Recall the standard notation

$$L_n = -t^{n+1}\frac{\partial}{\partial t}, n \geqslant 0.$$

These elements form a topological basis inside $\mathrm{Der}_0\,\mathbb{C}[[t]]$. Moreover, L_n $(n > 0)$ form a topological basis inside $\mathrm{Der}_+\,\mathbb{C}[[t]]$. In particular, $\mathrm{Der}_0\,\mathcal{O}$ can be embedded into the Virasoro algebra as its positive part.

2.4. **The space $\mathfrak{g}((t))/\mathfrak{g}[[t]]$ vs functions on the 1-forms.** Here, we will be interested in a description of $\mathrm{gr}\,V_k(\mathfrak{g})$. Let us recall the following statement from [6, Section 3.3].

Proposition 2.4.1. *We have the following graded algebra isomorphisms*

$$\mathrm{gr}\,V_k(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{g}((t))/\mathfrak{g}[[t]]) \simeq \mathbb{C}[\mathfrak{g}^*[[t]]dt].$$

We claim that the second isomorphism is coordinate-independent. Let us recall how it is constructed. The algebra $\mathbb{C}[\mathfrak{g}^*[[t]]dt]$ is the algebra of polynomials in the linear functions $r_n : \mathfrak{g}^*[[t]]dt \to \mathbb{C}$ with $n < 0$ given by $\alpha \mapsto \mathrm{Res}_{z=0}(z^n\alpha)$. Let $(\mathfrak{g}^*[[t]]dt)^\vee$ denote the vector space with basis r_n, of course, it is just the space of continuous linear maps $\mathfrak{g}^*[[t]]dt \to \mathbb{C}$ (the continuity is with respect to the t-adic topology on $\mathfrak{g}^*[[t]]dt$ and the discrete topology on $\mathbb{C}$). We then have a linear map

$$\mathfrak{g}((t))/\mathfrak{g}[[t]] \to (\mathfrak{g}^*[[t]]dt)^\vee \tag{2.4.1}$$

given by $f \mapsto r_f$ with $r_f(\alpha) := \mathrm{Res}_{z=0}(f\alpha)$. It is an isomorphism and it is coordinate-independent because taking the residue of a form is classically known to be coordinate-independent.

In what follows in this section we will reprove that (2.4.1) is coordinate-independent from scratch in order to illustrate a general technique that we are going to use in what follows to prove new (and more difficult) results. We note that the group $\mathrm{Aut}\,\mathcal{O}$ acts both on the source and the target of (2.4.1). Our claim that this map is coordinate-independent means just that it is $\mathrm{Aut}(\mathcal{O})$-equivariant. And, since $\mathrm{Aut}(\mathcal{O})$ is connected (as a semi-direct product of a torus and a pro-unipotent group), to show that (2.4.1) is $\mathrm{Aut}(\mathcal{O})$-equivariant, it is enough to show that it is $\mathrm{Der}_0\,\mathcal{O}$-equivariant. In fact, we will see that it is $\mathrm{Der}\,\mathcal{O}$-equivariant. We will do this for an analog of (2.4.1) where $\mathfrak{g}$ is replaced with $\mathbb{C}$.

We have $L_nt^k = -kt^{k+n}$ if $k+m \leqslant -1$ and $L_nt^k = 0$ else. The action of $\mathrm{Der}\,\mathcal{O}$ on $\mathbb{C}[[t]]dt$ is given by

$$L_n(t^{-m-1}dt) = (m-n)t^{n-m-1}dt. \tag{2.4.2}$$

So $\langle L_nr_k, t^{-m-1}dt\rangle = -\langle r_k, L_nt^{-m-1}dt\rangle = (m-n)\langle r_k, t^{n-m-1}dt\rangle = (m-n)\delta_{k+n-m,0}$. We conclude that $L_nr_k = -kr_{k+n}$, which shows that the analog of (2.4.1) (that by definition sends t^k to r_k) is $\mathrm{Der}\,\mathcal{O}$-equivariant.

2.5. $\mathrm{Der}\,\mathcal{O}$**-action on the elements** S_m**.** Section 2.4 suggests that in order to give a coordinate free description of the algebra $\mathbb{C}[S_m|0\rangle | m \leqslant -2]$ we should determine how $\mathrm{Der}\,\mathcal{O}$ acts on the elements S_m (under the natural action of $\mathrm{Der}\,\mathcal{O}$ on $\hat{\mathfrak{g}}$ by derivations).

Choose κ_0 to be the standard trace pairing. With this choice, [6, Corollary 2.17] tells us that if κ is not critical, then

$$L_n.S_m = (n-m)S_{n+m} - \frac{1}{2}(n^3-n)\delta_{n,-m}. \tag{2.5.1}$$

But near $\kappa = \kappa_c$, the element S_m depends continuously on κ, and so (2.5.1) also holds for $\kappa = \kappa_c$.

Remark 2.5.1. *(2.5.1) is true for a suitable normalization of the Sugawara elements for the general simple* $\mathfrak{g}$.

3. Projective connections

3.1. **Definition.** Let us introduce the vector space Ω_D^λ of "λ-forms" on D. Its elements are, by definition, the formal expressions of the form $f(t)(dt)^\lambda$ for $\lambda \in \mathbb{C}$.

The space Ω_D^λ becomes a $\operatorname{Der}\mathcal{O}$-module via the following formula (for $\lambda = 1$ we recover (2.4.2)):

$$\xi(t)\partial_t \cdot f(t)(dt)^\lambda = ((\xi(t)f'(t) + \lambda f(t)\xi'(t)))(dt)^\lambda.$$

Definition 3.1.1. *A projective connection on* $D = \operatorname{Spec}\mathbb{C}[[t]]$ *is a second order differential operator*

$$\rho : \Omega_D^{-1/2} \to \Omega_D^{3/2}$$

of the form $\partial_t^2 - v(t)$. *By definition, this operator sends* $f(t)(dt)^{-1/2}$ *to* $(f''(t) - v(t)f(t))(dt)^{3/2}$.

3.2. **The action of vector fields.** We now would like to write down the action of vector fields on projective connections. The action is as on the linear maps between two $\operatorname{Der}\mathcal{O}$-modules.

First, we compute

$$\xi(t)\partial_t \cdot ((\partial_t^2 - v(t))f(t)(dt)^{-1/2}) = \xi(t)\partial_t((f''(t) - v(t)f(t))(dt)^{3/2}) =$$
$$= (\xi(t)(f'''(t) - v'(t)f(t) - v(t)f'(t)) + \frac{3}{2}((f''(t) - v(t)f(t))\xi'(t)))(dt)^{3/2}.$$

On the other hand,

$$(\partial_t^2 - v(t))(\xi(t)\partial_t \cdot f(t)(dt)^{-1/2}) = (\partial_t^2 - v(t))(\xi(t)f'(t) - \frac{1}{2}f(t)\xi'(t))dt^{-1/2} =$$
$$(\frac{3}{2}\xi'(t)f''(t) + \xi(t)f'''(t) - v(t)\xi(t)f'(t) - \frac{1}{2}f(t)\xi'''(t) + \frac{1}{2}v(t)f(t)\xi'(t))(dt)^{3/2}.$$

Thus, the formula for the action of $\xi(t)\partial_t$ comes by taking the difference between two last quantities:

$$\xi(t)\partial_t : \partial_t^2 - v(t) \mapsto (\xi(t)v'(t) + 2v(t)\xi'(t) - \frac{1}{2}\xi'''(t)). \tag{3.2.1}$$

Remark 3.2.1. *The space of projective connections can be viewed as an affine space with associated vector space* $\mathbb{C}[[t]]$. *We can identify the two by sending* ∂_t^2 *to* 0. *Note that (3.2.1) defines a homomorphism from* $\operatorname{Der}\mathcal{O}$ *to the Lie algebra of the group of affine transformations of* $\mathbb{C}[[t]]$.

Note also that (3.2.1) integrates to an action of $\operatorname{Aut}\mathcal{O}$*: for the coordinate change* $t = \phi(s)$ *(*ϕ *as in 2.3, one has* $\partial_t^2 - v(t) = \partial_s^2 - w(s)$ *for* $w(s) = v(\phi(s))\phi'(s)^2 - \frac{1}{2}\{\phi, s\}$*, where* $\{\phi, s\} := \frac{\phi'''}{\phi'} - \frac{3}{2}(\frac{\phi''}{\phi'})^2$ *is the so-called* Schwarzian derivative.

3.3. **Main theorem.** Now, we can state the main result of this part. We write D_x for $\operatorname{Spec}(\mathcal{O}_x)$, and $\mathcal{P}roj(D_x)$ for the space of projective connections on D_x that we view as an infinite dimensional affine space.

Theorem 3.3.1. *We have a coordinate-independent isomorphism* $\mathfrak{z}(\mathfrak{sl}_2)_x \simeq \mathbb{C}[\mathcal{P}roj(D_x)]$.

Proof. For $k \leqslant -2$, we write p_k for the element of $\mathbb{C}[\mathcal{P}roj(D_x)]$ that sends a projective connection $\partial_t^2 - \sum_{i=0}^{\infty} a_i t^i$ to a_{-k-2}. Note that $\mathbb{C}[\mathcal{P}roj(D_x)] = \mathbb{C}[p_k | k \leqslant -2]$. We claim that the assignment sending $S_k|0\rangle$ to p_k defines a $\operatorname{Der}\mathcal{O}$-equivariant isomorphism $\mathfrak{z}(\mathfrak{sl}_2)_x \simeq \mathbb{C}[\mathcal{P}roj(D_x)]$.

Identify the space $\operatorname{Proj}(D_x)$ with $\mathbb{C}[[t]]$ as in Remark 3.2.1. Thanks to (3.2.1), we have

$$\langle L_n p_k, t^{-m-2}\rangle = \langle p_k, -L_n t^{-m-2}\rangle = \langle p_k, -(m+2)t^{n-m-2} + 2(n+1)t^{n-m-2} - \frac{n^3 - n}{2}t^{n-2}\rangle$$
$$= \delta_{k,m-n}(2n - m) - \delta_{k,-n}\frac{n^3 - n}{2}.$$

We conclude that $L_n p_k = (n-k)p_{k+n} - \frac{n^3-n}{2}\delta_{k,-n}$ (if $k+n > -2$, then the first summand in the right hand side is declared to be zero). This matches (2.5.1) and hence shows that the isomorphism defined by $S_k|0\rangle \mapsto p_k$ is Der $\mathcal{O}$-equivariant, finishing the proof. □

In subsequent talks we will see that Theorem 3.3.1 has a close relative.

Theorem 3.3.2. *We have an algebra isomorphism* $Z(\widetilde{U}_{\kappa_c}(\hat{\mathfrak{sl}}_{2,x})) \simeq \mathbb{C}[\mathcal{P}roj(D_x^\times)]$, *where* $D_x^\times :=$ Spec$(\mathcal{K}_x)$.

References

[1] T. Arakawa, A. Moreau, *Arc spaces and vertex algebras.*
[2] E. Frenkel, *Langlands correspondence for loop groups.*
[3] E. Frenkel, *Vertex algebras and algebraic curves*, Seminaire Bourbaki.
[4] J. Humphreys, *Introduction to Lie Algebras and Representation Theory.*
[5] V. Popov, E. Vinberg, *Invariant theory.*
[6] H. Wan, *Central elements of the completed universal enveloping algebra.* A talk at this seminar.
[7] A note on Kostant slices on the seminar webpage.