1. Recap

Let \((V, |0\rangle, T, Y)\) be a vertex algebra. Recall that to \(A \in V\), one associates the formal sum \(Y(A, z) = \sum_{m \in \mathbb{Z}} A(m) z^{-m-1}\). The following property is a part of the definition of the vertex algebra structure.

The following three properties were proven by Ilya ([DuII]).

1. \(Y(TA, z) = \partial_z Y(A, z)\),
2. \(Y(A, z)B = e^{zT} Y(B, -z)A\),
3. \([A_{(m)}, B_{(k)}] = \sum_{n \geq 0} \binom{m}{n} A_{(n)} B_{(m+k-n)} \Leftrightarrow [A_{(m)}, Y(B, z)] = \sum_{n \geq 0} \binom{m}{n} z^{m-n} Y(A_{(n)} B, z)\).

2. Vertex Poisson algebra structures and \((\text{Der} \mathcal{O}, \text{Aut} \mathcal{O})\)-equivariance

2.1. Commutative vertex algebras. Let us first of all recall (see [DuI]) that a vertex algebra \(V\) is called commutative if

\([Y(A, z), Y(B, w)] = 0\) for all \(A, B \in V\).

Ilya proved that \(V\) is commutative iff for every \(A \in V\), we have \(Y(A, z) \in \text{End}(V)[[z]]\). So, the non-commutativity of \(V\) is “controlled” by the coefficients of

\(Y_-(A, z) := \sum_{m \geq 0} A_m z^{-m-1}\).

Let us also recall that there is an equivalence of categories of commutative vertex algebras and commutative (associative, unital) algebras together with the derivation. This equivalence sends a commutative vertex algebra \((V, |0\rangle, T, Y)\) to \((V, \circ, T)\), where the product \(\circ\) on the vector space \(V\) is defined as follows:

\[A \circ B := A_{(-1)} \cdot B.\]
2.2. Vertex Poisson algebras: motivations and definitions. Recall that both \( \mathfrak{z}(\mathfrak{g}) \), \( W(L\mathfrak{g}) \) are \textit{commutative vertex algebras} and Calder proved that there is an inclusion \( \mathfrak{z}(\mathfrak{g}) \to W(L\mathfrak{g}) \). Our first goal is to introduce a notion of a \textit{vertex Poisson algebra} (this is some additional structure on a commutative vertex algebra), and prove that both \( \mathfrak{z}(\mathfrak{g}) \), \( W(L\mathfrak{g}) \) are vertex Poisson algebras and that the inclusion above is compatible with these structures. We relate the \( \text{Der} \mathcal{O} \)-action to the Poisson vertex algebra structure and use the fact that the isomorphism is Poisson to check that the inclusion above is \( (\text{Der} \mathcal{O}, \text{Aut} \mathcal{O}) \)-equivariant.

2.2.1. Poisson algebras. We start with a motivation: let us recall the notion of a Poisson algebra and how such an object appears naturally via deformations of (commutative) algebras.

Let \( P \) be an associative algebra over \( \mathbb{C} \). Assume that we are given a \textit{deformation} of \( P \) over the ring \( \mathbb{C}[\epsilon]/(\epsilon^k) \) \((k \in \mathbb{Z}_{\geq 2})\). By this, we mean a pair \((P^\epsilon, \iota)\) of a \( \mathbb{C}[\epsilon]/(\epsilon^k) \)-algebra \( P^\epsilon \) which is free as \( \mathbb{C}[\epsilon]/(\epsilon^k) \)-module together with the isomorphism of algebras \( \iota : P^\epsilon / (\epsilon) \to P \).

Assume now that \( P \) is \textit{commutative} and \( k \geq 3 \). Then, we can define an additional structure on \( P \) called the Poisson bracket. For \( a, b \in P \), we define the Poisson bracket \( \{ a, b \} \in P \) as follows:

\[
\{ a, b \} := \frac{\tilde{a} \tilde{b} - \tilde{b} \tilde{a}}{\epsilon} \mod \epsilon \in P,
\]

where \( \tilde{a}, \tilde{b} \in P^\epsilon \) are arbitrary lifts of \( a, b \) (clearly, the definition does not depend on the choice of \( \tilde{a}, \tilde{b} \)).

The following three properties of \( \{ , \} \) are clear from the definitions:

- (i) \( \{ a, b \} = -\{ b, a \} \),
- (ii) \( \{ a, \{ b, c \} \} + \{ c, \{ a, b \} \} + \{ b, \{ c, a \} \} = 0 \),
- (iii) \( \{ a, bc \} = b\{ a, c \} + c\{ a, b \} \).

\textit{Remark 2.1.} Condition \( k \geq 3 \) is needed for the property (ii) to hold.

So, \( \{ , \} \) defines the Lie algebra structure on \( P \) and \( \{ a, - \} \) is a derivation of \( P \). In other words, \( P \) is a Poisson algebra.

2.2.2. Vertex Poisson algebras. Let us now try to guess a candidate for a notion of a “Poisson structure” on a (commutative) vertex algebra \( V \).

Note that it makes sense to talk about a vertex algebra over \( \mathbb{C}[\epsilon]/(\epsilon^k) \), the definition is the same as over \( \mathbb{C} \), operands \( T, Y(-, z) \) must be \( \mathbb{C}[\epsilon]/(\epsilon^k) \)-linear. So it makes sense to talk about a deformation \((V^\epsilon, \iota)\) of \( V \) over \( \mathbb{C}[\epsilon]/(\epsilon^k) \) (recall that \( \iota \) is the identification of vertex algebras \( \iota : V^\epsilon / (\epsilon) \to V \)).

We know that \( Y^\epsilon(-, z) \) is equal to zero modulo \( \epsilon \), so for \( A \in V \) we can define:

\[
Y_-(A, z) := \frac{Y^\epsilon(\tilde{A}, z)}{\epsilon} \mod \epsilon,
\]

where \( \tilde{A} \in V^\epsilon \) is a representative of \( A \). Note that the definition does not depend on the choice of \( \tilde{A} \) since \( Y^\epsilon(-, z) \) is \( \mathbb{C}[\epsilon]/(\epsilon^k) \)-linear.

So, we have equipped commutative vertex algebra \( V \) with an additional structure:

\[
Y_-(A, z) : V \to z^{-1} \text{End}(V)[[z^{-1}]], \quad Y_-(A, z) = \sum_{m \geq 0} A_m z^{-m-1}.
\]

It follows from (1), (2), (3) above that for \( m \geq 0 \), and \( A, B \in V \) we have
• (I) (translation) $Y_-(TA, z) = \partial_z Y_-(A, z)$,
• (II) (skew-symmetry) $Y_-(A, z)B = (e^{zT}Y_-(B, -z)A)_-$,
• (III) (commutator) $[A_{(m)}, Y_-(B, z)] = \sum_{n \geq 0} \binom{m}{n} (z^{m-n}Y_-(A_{(n)}B, z))_-$.

(II) and (III) are analogous to the properties (i) and (ii) in the definition of the Poisson algebra (i.e., the analog of the fact that $\{ , \}$ defines a Lie algebra structure on $P$).

The following exercise should be considered as a vertex algebra counterpart of the property (iii). It claims that the coefficients of $Y_-(A, z)$ are derivations of the commutative product $\odot$ (given by the formula (4)).

**Exercise 2.2.** For every $m \geq 0$ we have

• (IV) $A_{(m)}(B \odot C) = (A_{(m)}B) \odot C + B \odot (A_{(m)}C)$.

**Proof.** Hint: use the definition of $\odot$ (see (4)) to see that (IV) is equivalent to

$$[A_{(m)}, B_{(-1)}] = (A_{(m)}B)_{(-1)}, \quad m \geq 0.$$ 

Rewrite this using some lifts $\tilde{A}, \tilde{B} \in V^c$ of $A, B$ and then use (3). □

**Definition 2.3.** A vertex Poisson algebra is $(V, |0\rangle, T, Y_+, Y_-)$, where $(V, |0\rangle, T, Y_+)$ is a commutative vertex algebra and

$$Y_- : V \to z^{-1} \text{End}(V)[[z^{-1}]]$$

satisfies the conditions (I)–(IV).

### 2.3. Vertex Poisson algebra structure on $\mathfrak{z}(\mathfrak{g})$.

We start with the following example.

**Example 2.4.** If $(V^c, \epsilon)$ is a deformation of some vertex algebra $V$ over $\mathbb{C}[\epsilon]$, then the center $\mathcal{Z}(V)$ carries a natural Poisson vertex algebra structure. Namely, for $A \in \mathcal{Z}(V)$, the operator $Y_-(A, z)$ (given by the equation (5)) is still well-defined and satisfies all the required properties making $\mathcal{Z}(V)$ a Poisson vertex algebra.

Now, let us equip $\mathfrak{z}(\mathfrak{g})$ with a Poisson vertex algebra structure. Fix a $\mathfrak{g}$-invariant scalar product $\kappa_0$ on $\mathfrak{g}$ and consider:

$$\kappa(\epsilon) := \epsilon \kappa_0 + \kappa$$

Consider the family $V_{\kappa(\epsilon)}$, and recall that for every fixed $\epsilon = \epsilon_0$ we have $V_{\kappa(\epsilon_0)} = U(\widehat{\mathfrak{g}}_{\kappa(\epsilon_0)}) \otimes \mathbb{C}[\epsilon] \oplus \mathbb{C}[\epsilon]0$. We can consider $\epsilon$ as a formal variable and define

$$V_{\kappa(\epsilon)} := U(\widehat{\mathfrak{g}}_{\kappa(\epsilon)}) \otimes \mathbb{C}[\epsilon] \oplus \mathbb{C}[\epsilon]1 \text{ is the Lie algebra over } \mathbb{C}[\epsilon] \text{ (with the commutator defined as before but with } \epsilon \text{ now considered as an indeterminate). The same formulas as before define on } V_{\kappa(\epsilon)} \text{ the structure of a vertex algebra over } \mathbb{C}[\epsilon].$$

Reducing modulo $(\epsilon^3)$ we equip $V_{\kappa(\epsilon)}$ with the vertex algebra structure over $\mathbb{C}[\epsilon]/(\epsilon^3)$. This gives us the Poisson vertex algebra structure on the center $\mathcal{Z}(\mathfrak{g})$ of $V_{\kappa(\epsilon)}$ to be denoted by $\mathfrak{z}(\mathfrak{g})_{\kappa_0}$ (note that this structure depends on $\kappa_0$). Since $\kappa_0$ is fixed once and for all, we will sometimes omit it from the notation.

Note that $V_{\kappa(0)} = \pi_0^\mathfrak{z}(\mathfrak{g}) = \pi_0^\mathfrak{h}$ is the Heisenberg vertex algebra so the example above equips $\pi_0 = \pi_{0,0}^\kappa = \pi_0$ with a vertex Poisson algebra structure. This vertex Poisson algebra will be denoted by $\pi_{0,\kappa_0}$ or just by $\pi_0$.

Let us describe this Poisson structure explicitly.
Recall that $\pi^0_{\kappa_0}$ is a Fock module over the Heisenberg Lie algebra $\tilde{h}_{\kappa_0}$ with generators $b_{i,n}$, $i \in \{1, \ldots, \ell\}$, $n \in \mathbb{Z}$ and 1 satisfying the relations:

$$[b_{i,n}, b_{j,m}] = c m \kappa_0(h_i, h_j) \delta_{n,-m} 1.$$ 

Recall also that $\pi_{0,\kappa_0}$ can be deformed to a homomorphism of vertex algebras over $\mathbb{C}[\varepsilon]$:

$$\omega_{\kappa_c} : \kappa \rightarrow W_{0,\kappa_c} = M_{\mathbb{C}[\varepsilon]} \otimes \mathbb{C}[\varepsilon] = M_{\mathbb{C}[\varepsilon]} \otimes \pi_{0,\kappa_0}$$

that can be deformed to a homomorphism of vertex algebras over $\mathbb{C}[[\varepsilon]]$:

$$\omega_{\kappa_c(\varepsilon)} : \kappa(\varepsilon) \rightarrow W_{0,\kappa_c(\varepsilon)} = M_{\mathbb{C}[\varepsilon]} \otimes \pi_{0,\kappa_0}^{\varepsilon \kappa_0}.$$ 

Recall also that the restriction of $\omega_{\kappa_c}$ to $\mathfrak{z}(\mathfrak{g})$ sends it into $\pi_{0,\kappa_0}$ (see [Kl, Lemma 1.2]).

**Remark 2.5.** We see from (6) that the vertex Poisson algebra structure on $\pi_{0,\kappa_0}$ indeed depends on $\kappa_0$.

**2.4. Embedding $\mathfrak{z}(\mathfrak{g})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$ is Poisson.** Recall now that Zeyu constructed a homomorphism of vertex algebras:

$$\omega_{\kappa_c} : \kappa \rightarrow W_{0,\kappa_c} = M_{\mathbb{C}[\varepsilon]} \otimes \mathbb{C}[\varepsilon] = M_{\mathbb{C}[\varepsilon]} \otimes \pi_{0,\kappa_0}$$

that can be deformed to a homomorphism of vertex algebras over $\mathbb{C}[\varepsilon]$:

$$\omega_{\kappa_c(\varepsilon)} : \kappa(\varepsilon) \rightarrow W_{0,\kappa_c(\varepsilon)} = M_{\mathbb{C}[\varepsilon]} \otimes \pi_{0,\kappa_0}^{\varepsilon \kappa_0}.$$ 

Recall also that the restriction of $\omega_{\kappa_c}$ to $\mathfrak{z}(\mathfrak{g})$ sends it into $\pi_{0,\kappa_0}$ (see [Kl, Lemma 1.2]).

**Remark 2.6.** Note that $\pi_0$ is the center of $W_{0,\kappa_c}$.

**Lemma 2.7.** The embedding $\mathfrak{z}(\mathfrak{g})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$ (induced by $\omega_{\kappa_c}$) is a homomorphism of vertex Poisson algebras.

**Proof.** This is a corollary of the following general fact. Let $(V_1^1, \epsilon_1)$, $(V_2^1, \epsilon_2)$ be deformations over $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ of vertex algebras $V_1$, $V_2$ and let $Z(V_1)$, $Z(V_2)$ be their centers. If $\varphi : V_1^1 \rightarrow V_2^1$ is a homomorphism of our vertex algebras over $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ such that $\varphi_0 : V_1 \rightarrow V_2$ restricts to $Z(V_1) \rightarrow Z(V_2)$, then the latter is Poisson. Hint: use the fact that the definition of $Y_{-}(A, z)$ does not depend on the choice of a lift $\tilde{A}$. \qed

**2.5. Vertex Poisson algebra structure on $W(\mathfrak{l} \mathfrak{g})$.** Let us now consider the classical $W$-algebra $W(\mathfrak{l} \mathfrak{g})$. Recall that $W(\mathfrak{l} \mathfrak{g})$ is by the definition the (commutative) vertex subalgebra of $\pi_{0,\kappa_0}^{\varepsilon \kappa_0}$, $\pi_{0,\kappa_0}$ as follows:

$$W(\mathfrak{l} \mathfrak{g}) := \bigcap_{i=1}^{\ell} \ker V_i[1] \subset \pi_{0,\kappa_0}^{\varepsilon \kappa_0},$$

where

$$V_i[1] = \sum_{m \leq 0} V_i[m] D_{V_i,1,m} \cdot D_{V_i,m} \cdot b_j' = a_{ij} \delta_{n,m},$$

and

$$b_j' = b_j \otimes 1 - 1 \otimes b_j.$$
$a_{ij}$ is the Cartan matrix of $Lg$, and

$$
\sum_{n \leq 0} V_i[n] z^{-n} = \exp \left( - \sum_{m > 0} \frac{b_i^m - m}{m} z^m \right).
$$

Let $\kappa^\vee$ be the invariant product on $\mathfrak{h}^*$ corresponding to $\kappa_0$ (in other words, if we consider $\kappa_0$ as the identification $\kappa_0 : \mathfrak{h} \sim \mathfrak{h}^*$, then $\kappa^\vee_0 : \mathfrak{h}^* \sim \mathfrak{h}$ is nothing else but $\kappa_0^{-1}$). We have $\kappa_0 = \kappa^\vee_0$ in Calder’s notations.

**Lemma 2.8.** $W(Lg)$ is a vertex Poisson subalgebra of $\pi^\vee_{0,\kappa^\vee_0}$ (to be denoted $W(Lg)_{\kappa^\vee_0}$).

**Proof.** Recall that $V_i[1]$ is the limit as $\epsilon \to 0$ of $\frac{1}{\epsilon} \cdot \left( \frac{2}{n!} \right) V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$, where $V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$ is the residue of the vertex operator

$$
V_{\kappa^\vee_i}^{-\kappa^\vee_0} (z) = T_{-\kappa_i} \exp \left( \sum_{n \geq 0} \frac{\alpha_i \otimes t^n}{n} z^{-n} \right) \exp \left( \sum_{n > 0} \frac{\alpha_i \otimes t^n}{n} z^{-n} \right).
$$

Recall also that by [MF, Section 8.1.2], $\ker V_{\kappa^\vee_i}^{-\kappa^\vee_0} (z)$ is the vertex subalgebra of $\pi^\vee_{0,\kappa^\vee_0}$.

Now, we claim that $\frac{1}{\epsilon} V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$ defines a flat deformation of $\ker V_i[1]$. Note that it is enough to prove this claim for $g = sl_2$ (use that operator $V_{\kappa^\vee_i}^{-\kappa^\vee_0}$ is equal to identity on the component corresponding to $\alpha_i^\perp \subset \mathfrak{h}$, in other words, if we identify $V_0(\mathfrak{h})$ with $V_0(\alpha_i^\perp) \otimes V_0(\text{Span}_{\mathbb{C}}(\alpha_i))$, then $V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$ will become $\text{Id}$ tensor the corresponding operator for $sl_2$). Note also that $\ker V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$ has (graded) dimension at least $\text{grdim} \pi^\vee_0 - \text{grdim} \pi^\vee_{-\kappa^\vee_0}$. It is an exercise to see that the difference above is equal to the graded dimension of $\mathfrak{g}(sl_2)$. So, it remains to check that the graded dimension of $\ker V_i[1]$ is not greater than the one of $\mathfrak{g}(sl_2)$. This (and actually the equality) will follow from the results of the second talk (namely, from the identification of $\ker V_i[1]$ with functions on $\text{Op}_{\text{PGL}_2}(D)$).

As soon as we know that the deformation $\ker V_{\kappa^\vee_i}^{-\kappa^\vee_0} [1]$ is flat, it immediately follows from the construction in Section 2.2.2 that $\ker V_i[1]$ is Poisson.

Let us also explain in more detail why the kernel of $V_{\kappa^\vee_i}^{-\kappa^\vee_0} (z)$ is a vertex subalgebra. Recall that in [Kl, Section 2.1], a notion of a module over a vertex algebra was introduced. Recall also (see [MF, Section 8.1.2]) that

$$
V_{\kappa^\vee_i}^{-\kappa^\vee_0} (z) = Y_{\pi^\vee_{-\kappa^\vee_0}, \pi^\vee_{-\alpha_i}} \{ -\alpha_i \}, z \in \text{Hom}(\pi^\vee_{0,\kappa^\vee_0}, \pi^\vee_{-\alpha_i})[|z|^{\pm 1}].
$$

It follows from [Fr, Equation (7.2-3)] that if $M$ is a module over a vertex algebra $V$, then for every $A \in V$ and $B \in M$, we have (we use the same notaion for both $Y(A, z)$ and $Y_M(A, z)$):

$$
\int Y_{V,M}(B, z) dz, Y(A, w) = Y_{V,M} \left( \int Y_{V,M}(B, z) dz \cdot A, w \right).
$$

Set $S := \int Y_{V,M}(B, z) dz$. Equation above implies that for $A \in \ker S$, we have

$$
S \cdot Y(A, w) = Y_M(A, w) \cdot S.
$$

This means that for $C \in \ker S$, we have

$$
S(Y(A, w)C) = (Y_M(A, w) \cdot S)C = 0
$$
so \( Y(A, w) \) preserves the kernel of \( S \). This is the main property that one has to check to show that something is a vertex subalgebra. It is an exercise to finish the argument and check that \( \ker S \) is indeed a vertex subalgebra of \( V \).

2.6. **The embedding** \( \mathfrak{z}(\mathfrak{g})_{\kappa_0} \hookrightarrow W(L\mathfrak{g})_{\kappa_0} \) **is Poisson.** Let us now recall that by [MF, Section 8.1.3] in Calder’s notes we have an isomorphism of commutative vertex algebras

\[
\pi_0(\mathfrak{g}) \xrightarrow{\sim} \pi_0(L\mathfrak{g})
\]

inducing the embedding \( \mathfrak{z}(\mathfrak{g}) \hookrightarrow W(L\mathfrak{g}) \). All of the vertex algebras above are equipped with vertex Poisson algebra structures (depending on a choice of \( \kappa_0 \)) and we already know that the embeddings \( \mathfrak{z}(\mathfrak{g})_{\kappa_0} \hookrightarrow \pi_0(\mathfrak{g}), W(L\mathfrak{g})_{\kappa_0} \hookrightarrow \pi_0(\mathfrak{g})_{\kappa_0} \) are Poisson. So, to see that the embedding \( \mathfrak{z}(\mathfrak{g})_{\kappa_0} \hookrightarrow W(L\mathfrak{g})_{\kappa_0} \) is Poisson, it remains to prove the following lemma.

**Lemma 2.9.** The isomorphism \( \pi_0(\mathfrak{g}),\kappa_0 \xrightarrow{\sim} \pi_0(\mathfrak{g}),\kappa_0 \) is Poisson.

**Proof.** This isomorphism is the specialization to \( \epsilon = 0 \) of the family of isomorphisms of vertex algebras over \( \mathbb{C}[\epsilon] \):

\[
\pi_0(\epsilon \mathfrak{g}),\kappa_0 \xrightarrow{\epsilon} \pi_0(\mathfrak{g}),\kappa_0 / \epsilon
\]

given by

\[
b_{i,n} \mapsto -b'_{i,n},
\]

where \( b'_{i,n} = \epsilon \frac{2}{\kappa_0(h_i, h_i)} b_{i,n} \). The claim now follows from the definitions. \( \square \)

So, we finally obtain the following stronger version of the theorem proved by Calder.

**Theorem 2.10.** There is a commutative diagram of vertex Poisson algebras:

\[
\begin{array}{ccc}
\pi_0(\mathfrak{g}),\kappa_0 & \xrightarrow{\sim} & \pi_0(\mathfrak{g}),\kappa_0 \\
\mathfrak{z}(\mathfrak{g})_{\kappa_0} & \hookrightarrow & W(L\mathfrak{g})_{\kappa_0}
\end{array}
\]

2.7. **Equivariance w.r.t.** \((\text{Der} \mathcal{O}, \text{Aut} \mathcal{O})\). Recall that \( \mathfrak{z}(\mathfrak{g}) \) carries a natural action of \( \text{Der} \mathcal{O} \) (coming from the natural action on \( V_{\kappa_0}(\mathfrak{g}) \)), the action of \( \text{Aut} \mathcal{O} \) is obtained by the exponentiation of the action of \( \text{Der}_0 \mathcal{O} \) (recall that \( \text{Der}_0 \mathcal{O} = \mathbb{C}\llbracket t \rrbracket \partial t \rrbracket \)).

We claim that the action of \( \text{Der} \mathcal{O} \) on \( \mathfrak{z}(\mathfrak{g}) \) is controlled by the vertex Poisson algebra structure on \( \mathfrak{z}(\mathfrak{g})_{\kappa_0} \). Indeed, recall (see [Wa, Sections 5,6]) that the \( \text{Der} \mathcal{O} \)-action on the deformation \( V_{\kappa(\epsilon)}(\mathfrak{g}) \) is generated by the Fourier coefficients \( L_n^\epsilon, n \geq -1 \) of the vertex operator

\[
Y(S_{\kappa(\epsilon)}, z) = \sum_{n \in \mathbb{Z}} L_n^\epsilon z^{-n-2},
\]

where \( S_{\kappa(\epsilon)} \) is the conformal vector:

\[
S_{\kappa(\epsilon)} = \frac{\kappa_0}{\kappa(\epsilon) - \kappa_c} S_1 = \epsilon^{-1} S_1
\]

and

\[
S_1 = \frac{1}{2} \sum_a J_{a,(-1)}^a J_{a,(-1)}(0),
\]
$J_{a}, J_{\theta} \in \mathfrak{g}$ is the dual basis w.r.t. $\kappa_{0}$.

The action of $\text{Der} \mathcal{O}$ on $\mathfrak{g}(\mathfrak{g})$ can be obtained as a limit of the action above, namely limits $L_{n}^{0}$ of the operators $L_{n}^{0}$ ($n \geq -1$) generate the action of $\text{Der} \mathcal{O}$ on $\mathfrak{g}(\mathfrak{g})$ (c.f. [Wa, Section 6]). Note that these limits are indeed well-defined and equal to the coefficients of the series

$$Y_{-}(S_{1}, z) = \sum_{n \geq -1} L_{n} z^{-n-2}.$$  

Recall now that we have an embedding of Poisson algebras $\omega_{\kappa}: \mathfrak{g}(\mathfrak{g})_{\kappa} \hookrightarrow W(L_{\mathfrak{g}})_{\kappa_{0}}$. It follows that the Fourier coefficients of the vertex operator $Y_{-}(\omega_{\kappa}(S_{1}), z)$ equip $W(L_{\mathfrak{g}})_{\kappa_{0}}$ with the action of $\text{Der}(\mathcal{O})$. The embedding above is $\text{Der}(\mathcal{O})$-equivariant by the definition.

Recall that we have an embedding of vertex Poisson algebras $W(L_{\mathfrak{g}})_{\kappa_{0}} \rightarrow \pi_{0,\kappa_{0}}^{\vee}$. We then obtain the action of $\text{Der}(\mathcal{O})$ on the whole $\pi_{0,\kappa_{0}}^{\vee}$ (via the Fourier coefficients of the vertex operator $Y_{-}$ corresponding to $\omega_{\kappa_{0}}(S_{1}) \in \pi_{0,\kappa_{0}}^{\vee}$). Let us describe this action explicitly.

**Lemma 2.11.** The action of $L_{n} = -t^{n+1} \partial_{t} \in \text{Der} \mathcal{O}$, $n \geq -1$ on $\pi_{0,\kappa_{0}}^{\vee}$ is given by the derivations of the algebra structure which are uniquely determined by:

$$L_{n} \cdot b'_{t,n} = -m b'_{t,n+m}, \quad n < -m,$$

$$L_{n} \cdot b'_{t,-n} = -n(n+1), \quad n > 0,$$

$$L_{n} \cdot b'_{t,n} = 0, \quad n > -m.$$

**Proof.** The claim follows from [Wa, Section 7] (see also [Fr, Equation 6.2-13]) where the action of $L_{n}$ on $\pi_{0}$ is described together with the fact that the identification $\pi_{0} \sim \pi_{0}^{\vee}$ is $\text{Der} \mathcal{O}$-equivariant and sends $b_{t,n}$ to $-b'_{t,n}$. \qed

### 3. Miura opers

The goal of this section is to construct an $\text{Aut} \mathcal{O}$-equivariant isomorphism between $W(L_{\mathfrak{g}})$ and the algebra $\text{Fun Op}_{L_{\mathfrak{g}}}(D)$ of functions on the space of $L_{\mathfrak{g}}$-opers on the disc. This will allow us to compute the character of $W(L_{\mathfrak{g}})$ (using that we know the character of $\text{Fun Op}_{L_{\mathfrak{g}}}(D)$) and to conclude that the embedding $\mathfrak{g}(\mathfrak{g}) \hookrightarrow W(L_{\mathfrak{g}})$ is actually an isomorphism. Composing isomorphisms $\mathfrak{g}(\mathfrak{g}) \sim W(L_{\mathfrak{g}}) \sim \text{Fun Op}_{L_{\mathfrak{g}}}(D)$ we will finally obtain the desired $\text{Aut} \mathcal{O}$-equivariant identification

$$\mathfrak{g}(\mathfrak{g}) \sim \text{Fun Op}_{L_{\mathfrak{g}}}(D).$$

We start with constructing the isomorphism $W(L_{\mathfrak{g}}) \simeq \text{Fun Op}_{L_{\mathfrak{g}}}(D)$. Let us explain how this will be done.

Recall that $W(L_{\mathfrak{g}})$ is an $\text{Aut} \mathcal{O}$-equivariant subspace of $\pi_{0}^{\vee}$ equal to the intersection of kernels of certain operators $V_{i}[1]$. We will identify $\pi_{0}^{\vee}$ with the vector space of functions on the space $\text{MOp}_{L_{\mathfrak{g}}}(D)_{\text{gen}}$ of so-called generic Miura $L_{\mathfrak{g}}$-opers on the disc $D$. This will be done by identifying $\text{MOp}_{L_{\mathfrak{g}}}(D)_{\text{gen}}$ with the space $\text{Conn}(\Omega_{D}^{\mathfrak{g}})$ of connections in the $H$-bundle $\Omega_{D}^{\mathfrak{g}}$ (introduced in [Wa, Section 7]).

There is a natural $\text{Aut} \mathcal{O}$-equivariant (surjective) morphism from generic Miura opers to opers that induces a morphism

$$\mu: \text{Conn}(\Omega_{D}^{\mathfrak{g}}) \sim \text{MOp}_{L_{\mathfrak{g}}}(D)_{\text{gen}} \rightarrow \text{Op}_{L_{\mathfrak{g}}}(D)$$
called the *Miura transformation*. We will show that the image of \( \mu^* \) is precisely the intersection of kernels of \( V_i[1] \)'s. So, we will obtain the desired identification \( W(Lg) \simeq \text{Fun Op}_{G}(D) \).

### 3.1. Miura opers.

Let \( X \) be a smooth curve or \( D \) or \( D^\times \). Recall that on all principal bundles that we consider group acts from the *right*. This is nothing but our convention. Recall also that \( G \) is an adjoint simple group.

**Definition 3.1.** A Miura \( G \)-oper on \( X \) is a quadruple \((F, \nabla, F_B, F_B')\), where \((F, \nabla, F_B)\) is a \( G \)-oper on \( X \) and \( F_B \) is a \( B \)-reduction in \( F \) which is preserved by \( \nabla \).

Note that a \( B \)-reduction \( F_B \subset F \) in a \( G \)-bundle \( F \) on a space \( X \) is the same as a section of the map \( F/B \to X \).

We will say that two \( B \)-reductions \( F_B, F'_B \subset F \) are in generic relative position if the image of the section corresponding to \( F'_B \) lies in \( F_B \times B_{\mathbb{G}/B} \subset F_B \times B_{\mathbb{G}/B} \), where \( B_{\mathbb{G}/B} \subset B_{\mathbb{G}/B} \) is the open Bruhat cell (\( w_0 \) is the longest element in the Weyl group of \( G \)).

A Miura \( G \)-oper on \( X \) is called generic if \( F_B, F'_B \) are in generic relative position.

The space of generic Miura \( G \)-opers on \( X \) will be denoted by \( \text{MOp}_{G}(X)_{\text{gen}} \).

**Example 3.2.** For example, for \( g = \mathfrak{sl}_n \), Miura \( G \)-oper is given by the following connection (in the trivial bundle):

\[
\partial_t + \begin{pmatrix}
* & 0 & \ldots & 0 \\
1 & * & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & \ldots & 1 & *
\end{pmatrix}.
\]

Let us now describe the space \( \text{MOp}_{G}(X)_{\text{gen}} \).

Consider the line bundle \( \Omega_X^1 \) and let \( \Omega_X \subset \Omega_X^1 \) be the complement to its zero section; \( \Omega_X \) a \( \mathbb{C}^\times \)-bundle on \( X \). Let \( \hat{\rho}: \mathbb{C}^\times \to H \) be the cocharacter of \( H \) corresponding to \( \rho := \frac{1}{2} \sum_{i} \alpha_i^\vee \) (\( \hat{\rho} \) indeed determines the cocharacter of \( H \) since \( G \) is adjoint). The cocharacter \( \hat{\rho} \) defines the action of \( \mathbb{C}^\times \) on \( H \). Set

\[
\Omega^\hat{\rho} := \Omega_X \times \mathbb{C}^\times H.
\]

Let \( \text{Conn}(\Omega^\hat{\rho}_X) \) be the space of connections on the \( T \)-bundle \( \Omega^\hat{\rho}_X \). The rest of this section will be devoted to the proof of the following proposition.

**Proposition 3.3.** There exists a natural \( \text{Aut \ X-equivariant isomorphism} \)

\[
\text{MOp}_{G}(X)_{\text{gen}} \simeq \text{Conn}(\Omega^\hat{\rho}_X).
\]

First of all, recall that for any \( B \)-bundle \( P_B \), we can consider the corresponding \( H \)-bundle \( P_H := P_B/N \), where \( N \subset B \) is the unipotent radical. We start with two lemmas.

The following statement was discussed in [Wa, Section 7].

**Lemma 3.4.** Let \((F, \nabla, F_B)\) be a \( G \)-oper, then there exists a canonical isomorphism of \( H \)-bundles:

\[
F_H \simeq \Omega^\hat{\rho}_X.
\]
Set $B_- := w_0^{-1}Bw_0$. Note that $B \cap B_- = H$. For a $B$-bundle $\mathcal{F}_B$, we will denote by $\mathcal{F}_{B_-}$ the $B_-$-bundle $\mathcal{F}_{B_0}^B \subset \mathcal{F}$ (we apply $w_0 \in G$ to $\mathcal{F}_B^B$ using the right action of $G$ on $\mathcal{F}$, the resulting space is a $w_0^{-1}Bw_0 = B_-$-torsor). Similarly, for an $H$-bundle $\mathcal{F}_H^H \subset \mathcal{F}$, we denote by $\mathcal{F}_H^Hw_0$ the corresponding $w_0^{-1}Hw_0 = H$-bundle.

**Lemma 3.5.** If $\mathcal{F}_B, \mathcal{F}_B^\prime \subset \mathcal{F}$ are in generic relative position, then $\mathcal{F}_B \cap \mathcal{F}_B^\prime$ defines $H$-reductions in both $\mathcal{F}_B$ and $\mathcal{F}_B^\prime$. We then obtain the identifications

$$\mathcal{F}_B \cap \mathcal{F}_B^\prime \cong \mathcal{F}_H, \mathcal{F}_B \cap \mathcal{F}_B^\prime \cong \mathcal{F}_H^\prime w_0,$$

where $\mathcal{F}_H := \mathcal{F}_B/N, \mathcal{F}_H^\prime := \mathcal{F}_B^\prime/N$.

So, in particular, $\mathcal{F}_H \cong \mathcal{F}_H^\prime w_0$ as $H$-bundles.

**Proof.** We just need to check that for every $x \in X$, $(\mathcal{F}_B|_x) \cap (\mathcal{F}_B^\prime|_x) \subset \mathcal{F}_B|_x$ is a principal homogeneous $H$-space. We fix a trivialization $\mathcal{F}_B|_x \simeq B$, it induces the identification $\mathcal{F}|_x = \mathcal{F}_B|_x \times B G \simeq G$. So, we have identified $\mathcal{F}_B|_x \subset \mathcal{F}|_x$ with $B \subset G$. Then $\mathcal{F}_B^\prime|_x \subset G$ identifies with $bw_0Bw_0 = w_0B_-$ for some $b \in B$ (since $\mathcal{F}_B$ and $\mathcal{F}_B^\prime$ are in generic relative position). So, the intersection $(\mathcal{F}_B|_x) \cap (\mathcal{F}_B^\prime|_x)$ gets identified with $B \cap bB_-= b(B \cap B_-) = bH$ which is clearly a (right) $H$-torsor. □

We are now ready to prove Proposition 3.3.

**Proof.** Note that Lemmas 3.4, 3.5 imply that if $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B^\prime)$ is a generic Miura oper, then we have canonical identifications

$$\mathcal{F} \simeq \Omega_X^0 \times^H G, \mathcal{F}_B \simeq \Omega_X^0 \times^H B, \mathcal{F}_B^\prime \simeq (\Omega_X^0 \times^H B_-)w_0.$$

So, to obtain the identification from Proposition 3.3 we just need to construct a connection $\nabla \in \mathcal{F}$ starting with a connection $\nabla$ in $\Omega_X^0$ and vice versa.

Let us construct a map from the LHS to the RHS. The connection $\nabla$ preserves the $B$-bundle $\mathcal{F}_B$ so induces a connection $\nabla$ on the $H$-bundle $\mathcal{F}_H$ and so on $\Omega^0 \simeq \mathcal{F}_H \simeq \mathcal{F}_H^\prime w_0$ (see Lemma 3.5). This is the connection on $\Omega^0$ that we need. This gives rise to a map $f : \text{MOp}_G(D)_{\text{gen}} \to \text{Conn}(\Omega_X^0), \nabla \mapsto \nabla$.

Let us construct a map in the opposite direction (it was sketched in [Wa, Section 7]). We start with a connection $\nabla$ on $\Omega_X^0$. The connection $\nabla$ induces a connection on $\mathcal{F} = \Omega_X^0 \times^H G$ to be denoted by the same symbol.

Observe now that the space $\text{Conn}(\mathcal{F})$ of connections on $\mathcal{F}$ is the affine space over the vector space $\Gamma(X, \mathfrak{g}_X \otimes_{\mathcal{O}_X} \Omega_X^1)$, where

$$\mathfrak{g}_X = \mathcal{F} \times^G \mathfrak{g} = \Omega_X^0 \times^H \mathfrak{g} = \Omega_X^0 \times^H (\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) = (\Omega_X^0 \times^H \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (\Omega_X^1)^{\otimes (\langle \beta, \alpha \rangle + 1)}.$$

So, we can identify $\text{Conn}(\mathcal{F})$ with the space

$$\text{Conn}(\Omega_X^0) \times \Gamma(X, \bigoplus_{\alpha \in \Delta} (\Omega^1)^{\otimes (\langle \beta, \alpha \rangle + 1)}).$$

For every negative simple root $-\alpha_i$, we see that the term $(\Omega^1)^{\otimes (\langle \beta, -\alpha \rangle + 1)}$ is just the structure sheaf $\mathcal{O}_X$, so the choice of a generator $f_i \in \mathfrak{g}_{-\alpha_i}$ defines the element $p_{-1} := \sum_i f_i \in \Gamma(X, \bigoplus_{\alpha \in \Delta} (\Omega^1)^{\otimes (\langle \beta, \alpha \rangle + 1)})$. Now $\nabla := \nabla + p_{-1}$. It follows from the definitions that the maps $\nabla \mapsto \nabla$ and $\nabla \mapsto \nabla$ are inverse to each other. □
References

[Bo1] E. Bogdanova, Seminar notes, Part I
[Bo2] E. Bogdanova, Seminar notes, Part II
[Wa] Zeyu Wang, Seminar notes, 2024.