VERTEX POISSON ALGEBRAS AND MIURA OPERS I

VASILY KRYLOV

1. Recap

Let $(V, |0\rangle, T, Y)$ be a vertex algebra. Recall that to $A \in V$, one associates the formal sum $Y(A,z) = \sum_{m\in\mathbb{Z}} A_{(m)} z^{-m-1}$. The following property is a part of the definition of the vertex algebra structure.

The following three properties were proven by Ilya ([DuII]).

$$Y(TA, z) = \partial_z Y(A,z), \tag{1}$$

$$Y(A,z)B = e^{zT} Y(B,-z)A \tag{2}$$

(3)

$$[A_{(m)}, B_{(k)}] = \sum_{n\geqslant 0} \binom{m}{n} (A_{(n)}B)_{(m+k-n)} \Leftrightarrow [A_{(m)}, Y(B,z)] = \sum_{n\geqslant 0} \binom{m}{n} z^{m-n} Y(A_{(n)}B, z).$$

2. Vertex Poisson algebra structures and $(\operatorname{Der}\mathcal{O}, \operatorname{Aut}\mathcal{O})$-equivariance

2.1. **Commutative vertex algebras.** Let us first of all recall (see [DuI]) that a vertex algebra V is called commutative if

$$[Y(A,z), Y(B,w)] = 0 \text{ for all } A, B \in V.$$

Ilya proved that V is commutative iff for every $A \in V$, we have $Y(A,z) \in \operatorname{End}(V)[[z]]$. So, the non-commutativity of V is "controlled" by the coefficients of

$$Y_-(A,z) := \sum_{m\geqslant 0} A_m z^{-m-1}.$$

Let us also recall that there is an equivalence of categories of commutative vertex algebras and commutative (associative, unital) algebras together with the derivation. This equivalence sends a commutative vertex algebra $(V, |0\rangle, T, Y)$ to $(V, \circ, T)$, where the product $\circ$ on the vector space V is defined as follows:

$$A \circ B := A_{(-1)} \cdot B. \tag{4}$$

2.2. **Vertex Poisson algebras: motivations and definitions.** Recall that both $\mathfrak{z}(\widehat{\mathfrak{g}})$, $W(^L\mathfrak{g})$ are *commutative vertex algebras* and Calder proved that there is an inclusion $\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow W(^L\mathfrak{g})$. Our first goal is to introduce a notion of a *vertex Poisson algebra* (this is some additional structure on a commutative vertex algebra), and prove that both $\mathfrak{z}(\widehat{\mathfrak{g}})$, $W(^L\mathfrak{g})$ are vertex Poisson algebras and that the inclusion above is compatible with these structures. We relate the $\operatorname{Der}\mathcal{O}$-action to the Poisson vertex algebra structure and use the fact that the isomorphism is Poisson to check that the inclusion above is $(\operatorname{Der}\mathcal{O}, \operatorname{Aut}\mathcal{O})$-equivariant.

2.2.1. *Poisson algebras.* We start with a motivation: let us recall the notion of a Poisson algebra and how such an object appears naturally via deformations of (commutative) algebras.

Let P be an associative algebra over $\mathbb{C}$. Assume that we are given a *deformation* of P over the ring $\mathbb{C}[\epsilon]/(\epsilon^k)$ ($k \in \mathbb{Z}_{\geqslant 2}$). By this, we mean a pair (P^ϵ, ι) of a $\mathbb{C}[\epsilon]/(\epsilon^k)$-algebra P^ϵ which is free as $\mathbb{C}[\epsilon]/(\epsilon^k)$-module together with the isomorphism of algebras $\iota\colon P^\epsilon/(\epsilon) \xrightarrow{\sim} P$.

Assume now that P is *commutative* and $k \geqslant 3$. Then, we can define an additional structure on P called the Poisson bracket. For $a, b \in P$, we define the Poisson bracket $\{a, b\} \in P$ as follows:

$$\{a, b\} := \frac{\tilde{a}\tilde{b} - \tilde{b}\tilde{a}}{\epsilon} \bmod \epsilon \in P,$$

where $\tilde{a}, \tilde{b} \in P^\epsilon$ are arbitrary lifts of a, b (clearly, the definition does not depend on the choice of $\tilde{a}, \tilde{b}$).

The following three properties of $\{\,,\,\}$ are clear from the definitions:

- (i) $\{a, b\} = -\{b, a\}$,
- (ii) $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$,
- (iii) $\{a, bc\} = b\{a, c\} + c\{a, b\}$.

Remark 2.1. Condition $k \geqslant 3$ is needed for the property (ii) to hold.

So, $\{\,,\,\}$ defines the Lie algebra structure on P and $\{a, -\}$ is a derivation of P. In other words, P is a Poisson algebra.

2.2.2. *Vertex Poisson algebras.* Let us now try to guess a candidate for a notion of a "Poisson structure" on a (commutative) vertex algebra V.

Note that it makes sense to talk about a vertex algebra over $\mathbb{C}[\epsilon]/(\epsilon^k)$, the definition is the same as over $\mathbb{C}$, operaions T, $Y(-, z)$ must be $\mathbb{C}[\epsilon]/(\epsilon^k)$-linear. So it makes sense to talk about a deformation (V^ϵ, ι) of V over $\mathbb{C}[\epsilon]/(\epsilon^k)$ (recall that ι is the identification of vertex algebras $\iota\colon V^\epsilon/(\epsilon) \xrightarrow{\sim} V$).

We know that $Y^\epsilon_-(-, z)$ is equal to zero modulo ϵ, so for $A \in V$ we can define:

$$Y_-(A, z) := \frac{Y^\epsilon_-(\tilde{A}, z)}{\epsilon} \bmod \epsilon, \tag{5}$$

where $\tilde{A} \in V^\epsilon$ is a representative of A. Note that the definition does not depend on the choice of $\tilde{A}$ since $Y^\epsilon(-, z)$ is $\mathbb{C}[\epsilon]/(\epsilon^k)$-linear.

So, we have equipped commutative vertex algebra V with an additional structure:

$$Y_-(-, z)\colon V \to z^{-1}\operatorname{End}(V)[[z^{-1}]],\ Y_-(A, z) = \sum_{m \geqslant 0} A_{(m)} z^{-m-1}.$$

It follows from (1), (2), (3) above that for $m \geqslant 0$, and $A, B \in V$ we have

- (I) (translation) $Y_-(TA,z) = \partial_z Y_-(A,z)$,
- (II) (skew-symmetry) $Y_-(A,z)B = (e^{zT}Y_-(B,-z)A)_-$,
- (III) (commutator) $[A_{(m)}, Y_-(B,z)] = \sum_{n\geqslant 0}\binom{m}{n}(z^{m-n}Y_-(A_{(n)}B,z))_-$,

(II) and (III) are analogous to the properties (i) and (ii) in the definition of the Poisson algebra (i.e., the analog of the fact that $\{\,,\,\}$ defines a Lie algebra structure on P).

The following exercise should be considered as a vertex algebra counterpart of the property (iii). It claims that the coefficients of $Y_-(A,z)$ are *derivations* of the commutative product $\circ$ (given by the formula (4)).

Exercise 2.2. For every $m \geqslant 0$ we have

- (IV) $A_{(m)}(B\circ C) = (A_{(m)}B)\circ C + B\circ(A_{(m)}C)$.

Proof. Hint: use the definition of $\circ$ (see (4)) to see that (IV) is equivalent to

$$[A_{(m)}, B_{(-1)}] = (A_{(m)}B)_{(-1)},\ m \geqslant 0.$$

Rewrite this using some lifts $\tilde{A},\tilde{B} \in V^\epsilon$ of A,B and then use (3). □

Definition 2.3. A vertex Poisson algebra is $(V,|0\rangle,T,Y_+,Y_-)$, where $(V,|0\rangle,T,Y_+)$ is a commutative vertex algebra and

$$Y_-: V \to z^{-1}\,\mathrm{End}(V)[[z^{-1}]]$$

satisfies the conditions (I)–(IV).

2.3. **Vertex Poisson algebra structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$.** We start with the following example.

Example 2.4. If (V^ϵ,ι) is a deformation of some vertex algebra V over $\mathbb{C}[\epsilon]$, then the center $\mathcal{Z}(V)$ carries a natural Poisson vertex algebra structure. Namely, for $A \in \mathcal{Z}(V)$, the operator $Y_-(A,z)$ (given by the equation (5)) is still well-defined and satisfies all the required properties making $\mathcal{Z}(V)$ a Poisson vertex algebra.

Now, let us equip $\mathfrak{z}(\widehat{\mathfrak{g}})$ with a Poisson vertex algebra structure.

Fix a $\mathfrak{g}$-invariant scalar product κ_0 on $\mathfrak{g}$ and consider:

$$\kappa(\epsilon) := \epsilon\kappa_0 + \kappa_c$$

Consider the family $V_{\kappa(\epsilon)}$, and recall that for every fixed $\epsilon = \epsilon_0$ we have $V_{\kappa(\epsilon_0)} = U(\widehat{\mathfrak{g}}_{\kappa(\epsilon_0)}) \otimes_{U(\mathfrak{g}[[t]])\oplus\mathbb{C}1} \mathbb{C}|0\rangle$. We can consider ϵ as a formal variable and define

$$V_{\kappa(\epsilon)} := U(\widehat{\mathfrak{g}}_{\kappa(\epsilon)}) \otimes_{U(\mathfrak{g}[[t]])\otimes\mathbb{C}[\epsilon]\oplus\mathbb{C}[\epsilon]\mathbf{1}} \mathbb{C}[\epsilon]|0\rangle,$$

where $\widehat{\mathfrak{g}}_{\kappa(\epsilon)} = \mathfrak{g}((t))\otimes\mathbb{C}[\epsilon]\oplus\mathbb{C}[\epsilon]\mathbf{1}$ is the Lie algebra over $\mathbb{C}[\epsilon]$ (with the commutator defined as before but with ϵ now considered as an indeterminante). The same formulas as before define on $V_{\kappa(\epsilon)}$ the structure of a vertex algebra over $\mathbb{C}[\epsilon]$. Reducing modulo (ϵ^3) we equip $V_{\kappa(\epsilon)}$ with the vertex algebra structure over $\mathbb{C}[\epsilon]/(\epsilon^3)$. This gives us the Poisson vertex algebra structure on the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V_{\kappa_c}(\mathfrak{g})$ to be denoted by $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$ (note that this structure *depends* on κ_0). Since κ_0 is fixed once and for all, we will sometimes omit it from the notation.

Note that $V_\kappa(\mathfrak{h}) = \pi_0^\kappa(\mathfrak{g}) = \pi_0^\kappa$ is the Heisenberg vertex algebra so the example above equips $\pi_0 = \pi_0^{\epsilon\kappa_0}|_{\epsilon=0}$ with a vertex Poisson algebra structure. This vertex Poisson algebra will be denoted by π_{0,κ_0} or just by π_0.

Let us describe this Poisson structure explicitly.

Recall that $\pi_0^{\epsilon\kappa_0}$ is a Fock module over the Heisenberg Lie algebra $\widehat{\mathfrak{h}}_{\epsilon\kappa_0}$ with generators $b_{i,n}$, $i \in 1,\dots,\ell$, $n \in \mathbb{Z}$ and $\mathbf{1}$ satisfying the relations:

$$[b_{i,n}, b_{j,m}] = \epsilon n \kappa_0(h_i, h_j)\delta_{n,-m}\mathbf{1}.$$

Recall also that π_{0,κ_0} can be identified with the space of monomials in $b_{i,n}$, $i = 1,\dots,\ell$, $n < 0$ (via the action on the vacuum $|0\rangle$). It follows from the definitions that for $n < 0$ and $i = 1,\dots,\ell$ we have:

$$Y_-(b_{i,-1}|0\rangle, z) = \{b_i(z), -\} := \sum_{n\geqslant 0}\{b_{i,n}, -\}z^{-n-1}, \tag{6}$$

where $\{\,,\,\}$ is the Poisson bracket defined by

$$\{b_{i,n}, b_{j,m}\} = n\kappa_0(h_i, h_j)\delta_{n,-m}.$$

In other words,

$$Y_-(b_{i,-1}|0\rangle, z) = \sum_{n\geqslant 0}\Big(\sum_{j=1}^{\ell} n\kappa_0(h_i, h_j)\frac{\partial}{\partial b_{j,-n}}\Big)z^{-n-1}.$$

Remark 2.5. We see from (6) that the vertex Poisson algebra structure on π_{0,κ_0} indeed depends on κ_0.

2.4. **Embedding $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$ is Poisson.** Recall now that Zeyu constructed a homomorphism of vertex algebras:

$$\omega_{\kappa_c}\colon V_{\kappa_c}(\mathfrak{g}) \to W_{0,\kappa_c} = M_{\mathfrak{g}} \otimes V_0(\mathfrak{h}) = M_{\mathfrak{g}} \otimes \pi_0$$

that can be deformed to a homomorphism of vertex algebras over $\mathbb{C}[\epsilon]$:

$$\omega_{\kappa(\epsilon)}\colon V_{\kappa(\epsilon)} \to W_{0,\kappa(\epsilon)} = M_{\mathfrak{g}} \otimes \pi_0^{\epsilon\kappa_0}.$$

Recall also that the restriction of ω_{κ_c} to $\mathfrak{z}(\widehat{\mathfrak{g}})$ sends it into π_0 (see [Kl, Lemma 1.2]).

Remark 2.6. Note that π_0 is the center of W_{0,κ_c}.

Lemma 2.7. *The embedding $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$ (induced by ω_{κ_c}) is a homomorphism of vertex Poisson algebras.*

Proof. This is a corollary of the following general fact. Let (V_1^ϵ, ι_1), (V_2^ϵ, ι_2) be deformations over $\mathbb{C}[\epsilon]/(\epsilon^3)$ of vertex algebras V_1, V_2 and let $\mathcal{Z}(V_1)$, $\mathcal{Z}(V_2)$ be their centers. If $\varphi_\epsilon\colon V_1^\epsilon \to V_2^\epsilon$ is a homomorphism of our vertex algebras over $\mathbb{C}[\epsilon]/(\epsilon^3)$ such that $\varphi_0\colon V_1 \to V_2$ restricts to $\mathcal{Z}(V_1) \to \mathcal{Z}(V_2)$, then the latter is Poisson. Hint: use the fact that the definition of $Y_-(A,z)$ does not depend on the choice of a lift $\tilde{A}$. □

2.5. **Vertex Poisson algebra structure on $W({}^L\mathfrak{g})$.** Let us now consider the classical W-algebra $W({}^L\mathfrak{g})$. Recall that $W({}^L\mathfrak{g})$ is by the definition the (commutative) vertex subalgebra of $\pi_0^\vee = \pi_0({}^L\mathfrak{g})$ defined as follows:

$$W({}^L\mathfrak{g}) := \bigcap_{i=1}^{\ell} \ker V_i[1] \subset \pi_0^\vee,$$

where

$$V_i[1] = \sum_{m\leqslant 0} V_i[m]D_{b'_{i,m-1}},\ D_{b'_{i,m}} \cdot b'_{j,n} = a_{ij}\delta_{n,m},$$

a_{ij} is the Cartan matrix of ${}^L\mathfrak{g}$, and

$$\sum_{n\leqslant 0} V_i[n]z^{-n} = \exp\Big(-\sum_{m>0}\frac{b'_{i,-m}}{m}z^m\Big).$$

Let $\kappa_0^\vee$ be the invariant product on $\mathfrak{h}^*$ corresponding to κ_0 (in other words, if we consider κ_0 as the identification $\kappa_0\colon \mathfrak{h}\xrightarrow{\sim}\mathfrak{h}^*$, then $\kappa_0^\vee\colon \mathfrak{h}^*\xrightarrow{\sim}\mathfrak{h}$ is nothing else but κ_0^{-1}). We have $\nu_0=\kappa_0^\vee$ in Calder's notations.

Lemma 2.8. *$W({}^L\mathfrak{g})$ is a vertex Poisson subalgebra of $\pi^\vee_{0,\kappa_0^\vee}$ (to be denoted $W({}^L\mathfrak{g})_{\kappa_0^\vee}$).*

Proof. Recall that $V_i[1]$ is the limit as $\epsilon\to 0$ of $\frac{1}{\epsilon}\cdot(\frac{2}{\kappa_0^\vee(h_i,h_i)}V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1])$, where $V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1]$ is the residue of the vertex operator

$$V^{\epsilon\kappa_0^\vee}_{-\alpha_i}(z) = T_{-\alpha_i}\exp\Big(\sum_{n<0}\frac{\alpha_i\otimes t^n}{n}z^{-n}\Big)\exp\Big(\sum_{n>0}\frac{\alpha_i\otimes t^n}{n}z^{-n}\Big).$$

Recall also that by [MF, Section 8.1.2], $\ker V^{\epsilon\kappa_0^\vee}_{-\alpha_i}(z)$ is the vertex subalgebra of $\pi_0^{\epsilon\kappa_0^\vee}$.

Now, we claim that $\ker \frac{1}{\epsilon}V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1]$ defines a *flat* deformation of $\ker V_i[1]$. Note that it is enough to prove this claim for $\mathfrak{g}=\mathfrak{sl}_2$ (use that operator $V^{\epsilon\kappa_0^\vee}_{-\alpha_i}$ is equal to identity on the component corresponding to $\alpha_i^\perp\subset\mathfrak{h}$, in other words, if we identify $V_0(\mathfrak{h})$ with $V_0(\alpha_i^\perp)\otimes V_0(\mathrm{Span}_{\mathbb{C}}(\alpha_i))$, then $V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1]$ will become Id tensor the corresponding operator for $\mathfrak{sl}_2$). Note also that $\ker V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1]$ has (graded) dimension at least $\mathrm{grdim}\,\pi_0^\vee - \mathrm{grdim}\,\pi^\vee_{-\alpha_i}$. It is an exercise to see that the difference above is equal to the graded dimension of $\mathfrak{z}(\widehat{\mathfrak{sl}}_2)$. So, it remains to check that the graded dimension of $\ker V_i[1]$ is not *greater* than the one of $\mathfrak{z}(\widehat{\mathfrak{sl}}_2)$. This (and actually the equality) will follow from the results of the second talk (namely, from the identification of $\ker V_i[1]$ with functions on $\mathrm{Op}_{\mathrm{PGL}_2}(D)$).

As soon as we know that the deformation $\ker V^{\epsilon\kappa_0^\vee}_{-\alpha_i}[1]$ is flat, it immediately follows from the construction in Section 2.2.2 that $\ker V_i[1]$ is Poisson.

Let us also explain in more detail why the kernel of $V^{\epsilon\kappa_0^\vee}_{-\alpha_i}(z)$ is a vertex subalgebra. Recall that in [Kl, Section 2.1], a notion of a module over a vertex algebra was introduced. Recall also (see [MF, Section 8.1.2]) that

$$V^{\epsilon\kappa_0^\vee}_{-\alpha_i}(z) = Y_{\pi_0^{\epsilon\kappa_0^\vee},\pi_{-\alpha_i}^{\epsilon\kappa_0^\vee}}(|-\alpha_i\rangle,z)\in \mathrm{Hom}(\pi_0^{\epsilon\kappa_0^\vee},\pi_{-\alpha_i}^{\epsilon\kappa_0^\vee})[[z^{\pm 1}]].$$

It follows from [Fr, Equation (7.2-3)] that if M is a module over a vertex algebra V, then for every $A\in V$ and $B\in M$, we have (we use the same notaion for both $Y(A,z)$ and $Y_M(A,z)$):

$$\Big[\int Y_{V,M}(B,z)dz, Y(A,w)\Big] = Y_{V,M}\Big(\int Y_{V,M}(B,z)dz\cdot A, w\Big). \tag{7}$$

Set $S:=\int Y_{V,M}(B,z)dz$. Equation above implies that for $A\in\ker S$, we have

$$S\cdot Y(A,w) = Y_M(A,w)\cdot S.$$

This means that for $C\in\ker S$, we have

$$S(Y(A,w)C) = (Y_M(A,w)\cdot S)C = 0$$

so $Y(A, w)$ preserves the kernel of S. This is the main property that one has to check to show that something is a vertex subalgebra. It is an exercise to finish the argument and check that $\ker S$ is indeed a vertex subalgebra of V.

$\square$

2.6. **The embedding $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow W({}^L\mathfrak{g})_{\kappa_0^\vee}$ is Poisson.** Let us now recall that by [MF, Section 8.1.3] in Calder's notes we have an isomorphism of commutative vertex algebras

$$\pi_0(\mathfrak{g}) \xrightarrow{\sim} \pi_0^\vee({}^L\mathfrak{g})$$

inducing the embedding $\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow W({}^L\mathfrak{g})$. All of the vertex algebras above are equipped with vertex Poisson algebra structures (depending on a choice of κ_0) and we already know that the embeddings $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$, $W({}^L\mathfrak{g})_{\kappa_0^\vee} \hookrightarrow \pi^\vee_{0,\kappa_0^\vee}$ are Poisson. So, to see that the embedding $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow W({}^L\mathfrak{g})_{\kappa_0^\vee}$ is Poisson, it remains to prove the following lemma.

Lemma 2.9. *The isomorphism $\pi_{0,\kappa_0} \xrightarrow{\sim} \pi^\vee_{0,\kappa_0^\vee}$ is Poisson.*

Proof. This isomorphism is the specialization to $\epsilon = 0$ of the family of isomorphisms of vertex algebras over $\mathbb{C}[\epsilon]$:

$$\pi_0^{\epsilon\kappa_0} \xrightarrow{\sim} \pi_0^{\kappa_0^\vee/\epsilon}$$

given by

$$b_{i,n} \mapsto -\mathbf{b}'_{i,n},$$

where $\mathbf{b}'_{i,n} = \epsilon\frac{2}{\kappa_0^\vee(h_i,h_i)}\mathbf{b}_{i,n}$. The claim now follows from the definitions. $\square$

So, we finally obtain the following stronger version of the theorem proved by Calder.

Theorem 2.10. *There is a commutative diagram of vertex Poisson algebras:*

$$\begin{tikzcd} \pi_{0,\kappa_0} \arrow[r, "\simeq"] & \pi^\vee_{0,\kappa_0^\vee} \\ \mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \arrow[u] \arrow[r] & W({}^L\mathfrak{g})_{\kappa_0^\vee} \arrow[u] \end{tikzcd}$$

2.7. **Equivariance w.r.t.** $(\operatorname{Der}\mathcal{O}, \operatorname{Aut}\mathcal{O})$**.** Recall that $\mathfrak{z}(\widehat{\mathfrak{g}})$ carries a natural action of $\operatorname{Der}\mathcal{O}$ (coming from the natural action on $V_{\kappa_c}(\mathfrak{g})$), the action of $\operatorname{Aut}\mathcal{O}$ is obtained by the exponentiation of the action of $\operatorname{Der}_0\mathcal{O}$ (recall that $\operatorname{Der}_0\mathcal{O} = \mathbb{C}[[t]]\partial_t$).

We claim that the action of $\operatorname{Der}\mathcal{O}$ on $\mathfrak{z}(\widehat{\mathfrak{g}})$ is controlled by the vertex Poisson algebra structure on $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$. Indeed, recall (see [Wa, Sections 5,6]) that the $\operatorname{Der}\mathcal{O}$-action on the deformation $V_{\kappa(\epsilon)}(\mathfrak{g})$ is generated by the Fourier coefficients L_n^ϵ, $n \geqslant -1$ of the vertex operator

$$Y(S_{\kappa(\epsilon)}, z) = \sum_{n\in\mathbb{Z}} L_n^\epsilon z^{-n-2},$$

where $S_{\kappa(\epsilon)}$ is the conformal vector:

$$S_{\kappa(\epsilon)} = \frac{\kappa_0}{\kappa(\epsilon) - \kappa_c} S_1 = \epsilon^{-1} S_1$$

and

$$S_1 = \frac{1}{2}\sum_a J^a_{(-1)} J_{a,(-1)}|0\rangle,$$

$J^a, J_a \in \mathfrak{g}$ is the dual basis w.r.t. κ_0.

The action of $\operatorname{Der}\mathcal{O}$ on $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be obtained as a limit of the action above, namely limits L_n^0 of the operators L_n^ϵ $(n \geqslant -1)$ generate the action of $\operatorname{Der}\mathcal{O}$ on $\mathfrak{z}(\widehat{\mathfrak{g}})$ (c.f. [Wa, Section 6]). Note that these limits are indeed well-defined and equal to the coefficients of the series

$$Y_-(S_1, z) = \sum_{n \geqslant -1} L_n z^{-n-2}.$$

Recall now that we have an embedding of Poisson algebras $\omega_{\kappa_c} \colon \mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0^\vee} \hookrightarrow W({}^L\mathfrak{g})_{\kappa_0^\vee}$. It follows that the Fourier coefficients of the vertex operator $Y_-(\omega_{\kappa_c}(S_1), z)$ equip $W({}^L\mathfrak{g})_{\kappa_0^\vee}$ with the action of $\operatorname{Der}(\mathcal{O})$. The embedding above is $\operatorname{Der}(\mathcal{O})$-equivariant *by the definition*.

Recall that we have an embedding of vertex Poisson algebras $W({}^L\mathfrak{g})_{\kappa_0^\vee} \hookrightarrow \pi^\vee_{0,\kappa_0^\vee}$. We then obtain the action of $\operatorname{Der}(\mathcal{O})$ on the whole $\pi^\vee_{0,\kappa_0^\vee}$ (via the Fourier coefficients of the vertex operator Y_- corresponding to $\omega_{\kappa_c}(S_1) \in \pi^\vee_{0,\kappa_0^\vee}$). Let us describe this action explicitly.

Lemma 2.11. *The action of $L_n = -t^{n+1}\partial_t \in \operatorname{Der}\mathcal{O}$, $n \geqslant -1$ on $\pi^\vee_{0,\kappa_0^\vee}$ is given by the derivations of the algebra structure which are uniquely determined by:*

$$L_n \cdot \mathbf{b}'_{i,m} = -m\mathbf{b}'_{i,n+m}, \;\; n < -m,$$
$$L_n \cdot \mathbf{b}'_{i,-n} = -n(n+1), \;\; n > 0,$$
$$L_n \cdot \mathbf{b}'_{i,m} = 0, \;\; n > -m.$$

Proof. The claim follows from [Wa, Section 7] (see also [Fr, Equation 6.2-13]) where the action of L_n on π_0 is described together with the fact that the identification $\pi_0 \xrightarrow{\sim} \pi_0^\vee$ is $\operatorname{Der}\mathcal{O}$-equivariant and sends $b_{i,n}$ to $-\mathbf{b}'_{i,n}$. □

3. Miura opers

The goal of this section is to construct an $\operatorname{Aut}\mathcal{O}$-equivariant isomorphism between $W({}^L\mathfrak{g})$ and the algebra $\operatorname{Fun}\operatorname{Op}_{{}^LG}(D)$ of functions on the space of ${}^LG$-opers on the disc. This will allow us to compute the character of $W({}^L\mathfrak{g})$ (using that we know the character of $\operatorname{Fun}\operatorname{Op}_{{}^LG}(D)$) and to conclude that the embedding $\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow W({}^L\mathfrak{g})$ is actually an isomorphism. Composing isomorphisms $\mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} W({}^L\mathfrak{g}) \xrightarrow{\sim} \operatorname{Fun}\operatorname{Op}_{{}^LG}(D)$ we will finally obtain the desired $\operatorname{Aut}\mathcal{O}$-equivariant identification

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \operatorname{Fun}\operatorname{Op}_{{}^LG}(D).$$

We start with constructing the isomorphism $W({}^L\mathfrak{g}) \simeq \operatorname{Fun}\operatorname{Op}_{{}^LG}(D)$. Let us explain how this will be done.

Recall that $W({}^L\mathfrak{g})$ is an $\operatorname{Aut}\mathcal{O}$-equivariant subspace of $\pi_0^\vee$ equal to the intersection of kernels of certain operators $V_i[1]$. We will identify $\pi_0^\vee$ with the vector space of functions on the space $\operatorname{MOp}_{{}^LG}(D)_{\mathrm{gen}}$ of so-called *generic* Miura ${}^LG$-opers on the disc D. This will be done by identifying $\operatorname{MOp}_{{}^LG}(D)_{\mathrm{gen}}$ with the space $\operatorname{Conn}(\Omega_D^{\check\rho})$ of connections in the H-bundle $\Omega_D^{\check\rho}$ (introduced in [Wa, Section 7]).

There is a natural $\operatorname{Aut}\mathcal{O}$-equivariant (surjective) morphism from generic Miura opers to opers that induces a morphism

$$\mu \colon \operatorname{Conn}(\Omega_D^{\check\rho}) \xrightarrow{\sim} \operatorname{MOp}_{{}^LG}(D)_{\mathrm{gen}} \to \operatorname{Op}_{{}^LG}(D)$$

called the *Miura transformation*. We will show that the image of μ^* is precisely the intersection of kernels of $V_i[1]$'s. So, we will obtain the desired identification $W({}^L\mathfrak{g}) \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(D)$.

3.1. **Miura opers.** Let X be a smooth curve or D or $D^\times$. Recall that on all principal bundles that we consider group acts from the *right*. This is nothing but our convention. Recall also that G is an adjoint simple group.

Definition 3.1. A *Miura* G-oper on X is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$, where $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a G-oper on X and $\mathcal{F}'_B$ is a B-reduction in $\mathcal{F}$ which is preserved by ∇.

Note that a B-reduction $\mathcal{F}_B \subset \mathcal{F}$ in a G-bundle $\mathcal{F}$ on a space X is the same as a section of the map $\mathcal{F}/B \to X$.

We will say that two B-reductions $\mathcal{F}_B, \mathcal{F}'_B \subset \mathcal{F}$ are *in generic relative position* if the image of the section corresponding to $\mathcal{F}'_B$

$$s' \colon X \to \mathcal{F}/B \simeq \mathcal{F} \times^G G/B \simeq \mathcal{F}_B \times^B G/B$$

lies in

$$\mathcal{F}_B \times^B (Bw_0B) \subset \mathcal{F}_B \times^B G/B,$$

where $Bw_0B \subset G/B$ is the open Bruhat cell (w_0 is the longest element in the Weyl group of G).

A Miura G-oper on X is called *generic* if $\mathcal{F}_B, \mathcal{F}'_B$ are in generic relative position. The space of generic Miura G-opers on X will be denoted by $\mathrm{MOp}_G(X)_{\mathrm{gen}}$.

Example 3.2. For example, for $\mathfrak{g} = \mathfrak{sl}_n$, Miura G-oper is given by the following connection (in the trivial bundle):

$$\partial_t + \begin{pmatrix} * & 0 & \dots & 0 \\ 1 & * & \dots & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & \dots & 1 & * \end{pmatrix}.$$

Let us now describe the space $\mathrm{MOp}_G(X)_{\mathrm{gen}}$.

Consider the line bundle Ω^1_X and let $\Omega^*_X \subset \Omega^1_X$ be the complement to its zero section; Ω^*_X a $\mathbb{C}^\times$-bundle on X. Let $\check{\rho} \colon \mathbb{C}^\times \to H$ be the cocharacter of H corresponding to $\check{\rho} := \frac{1}{2}\sum_i \alpha_i^\vee$ ($\check{\rho}$ indeed determines the cocharacter of H since G is adjoint). The cocharacter $\check{\rho}$ defines the action of $\mathbb{C}^\times$ on H. Set

$$\Omega^{\check{\rho}} := \Omega^*_X \times^{\mathbb{C}^\times} H.$$

Let $\mathrm{Conn}(\Omega^{\check{\rho}}_X)$ be the space of connections on the T-bundle $\Omega^{\check{\rho}}_X$. The rest of this section will be devoted to the proof of the following proposition.

Proposition 3.3. *There exists a natural* $\operatorname{Aut} X$*-equivariant isomorphism*

$$\mathrm{MOp}_G(X)_{\mathrm{gen}} \simeq \mathrm{Conn}(\Omega^{\check{\rho}}_X).$$

First of all, recall that for any B-bundle $\mathcal{P}_B$, we can consider the corresponding H-bundle $\mathcal{P}_H := \mathcal{P}_B/N$, where $N \subset B$ is the unipotent radical. We start with two lemmas.

The following statement was discussed in [Wa, Section 7].

Lemma 3.4. *Let* $(\mathcal{F}, \nabla, \mathcal{F}_B)$ *be a* G*-oper, then there exists a canonical isomorphism of* H*-bundles:*

$$\mathcal{F}_H \simeq \Omega^{\check{\rho}}_X.$$

Set $B_- := w_0^{-1}Bw_0$. Note that $B \cap B_- = H$. For a B-bundle $\mathcal{F}'_B$, we will denote by $\mathcal{F}'_{B_-}$ the B_--bundle $\mathcal{F}'_B w_0 \subset \mathcal{F}$ (we apply $w_0 \in G$ to $\mathcal{F}'_B$ using the right action of G on $\mathcal{F}$, the resulting space is a $w_0^{-1}Bw_0 = B_-$-torsor). Similarly, for an H-bundle $\mathcal{F}'_H \subset \mathcal{F}$, we denote by $\mathcal{F}'_H w_0$ the corresponding $w_0^{-1}Hw_0 = H$-bundle.

Lemma 3.5. *If $\mathcal{F}_B, \mathcal{F}'_B \subset \mathcal{F}$ are in generic relative position, then $\mathcal{F}_B \cap \mathcal{F}'_{B_-}$ defines H-reductions in both $\mathcal{F}_B$ and $\mathcal{F}'_{B_-}$. We then obtain the identifications*

$$\mathcal{F}_B \cap \mathcal{F}'_{B_-} \xrightarrow{\sim} \mathcal{F}_H, \ \mathcal{F}_B \cap \mathcal{F}'_{B_-} \xrightarrow{\sim} \mathcal{F}'_H w_0,$$

where $\mathcal{F}_H := \mathcal{F}_B/N$, $\mathcal{F}'_H := \mathcal{F}'_B/N$.

So, in particular, $\mathcal{F}_H \simeq \mathcal{F}'_H w_0$ as H-bundles.

Proof. We just need to check that for every $x \in X$, $(\mathcal{F}_B|_x) \cap (\mathcal{F}'_{B_-}|_x) \subset \mathcal{F}_B|_x$ is a principal homogeneous H-space. We fix a trivialization $\mathcal{F}_B|_x \simeq B$, it induces the identification $\mathcal{F}|_x = \mathcal{F}_B|_x \times^B G \simeq G$. So, we have identified $\mathcal{F}_B|_x \subset \mathcal{F}|_x$ with $B \subset G$. Then $\mathcal{F}'_{B_-}|_x \subset G$ identifies with $bw_0B \cdot w_0 = bB_-$ for some $b \in B$ (since $\mathcal{F}_B$ and $\mathcal{F}'_B$ are in generic realive position). So, the intersection $(\mathcal{F}_B|_x) \cap (\mathcal{F}'_{B_-}|_x)$ gets identified with $B \cap bB_- = b(B \cap B_-) = bH$ which is clearly a (right) H-torsor. $\square$

We are now ready to prove Proposition 3.3.

Proof. Note that Lemmas 3.4, 3.5 imply that if $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ is a generic Miura oper, then we have *canonical* identifications

$$\mathcal{F} \simeq \Omega_X^{\check{\rho}} \times^H G, \ \mathcal{F}_B \simeq \Omega_X^{\check{\rho}} \times^H B, \ \mathcal{F}'_B \simeq (\Omega_X^{\check{\rho}} \times^H B_-)w_0.$$

So, to obtain the identification from Proposition 3.3 we just need to construct a connection ∇ in $\mathcal{F}$ starting with a connection $\widehat{\nabla}$ in $\Omega_X^{\check{\rho}}$ and vice versa.

Let us construct a map from the LHS to the RHS. The connection ∇ preserves the B-bundle $\mathcal{F}'_B$ so induces a connection $\overline{\nabla}$ on the H-bundle $\mathcal{F}'_H$ and so on $\Omega^{\check{\rho}} \simeq \mathcal{F}_H \simeq \mathcal{F}'_H w_0$ (see Lemma 3.5). This is the connection on $\Omega^{\check{\rho}}$ that we need. This gives rise to a map $f\colon \operatorname{MOp}_G(D)_{\mathrm{gen}} \to \operatorname{Conn}(\Omega_D^{\check{\rho}})$, $\nabla \mapsto \widehat{\nabla}$.

Let us construct a map in the opposite direction (it was sketched in [Wa, Section 7]). We start with a connection $\widehat{\nabla}$ on $\Omega_X^{\check{\rho}}$. The connection $\widehat{\nabla}$ induces a connection on $\mathcal{F} = \Omega_X^{\check{\rho}} \times^H G$ to be denoted by the same symbol.

Observe now that the space $\operatorname{Conn}(\mathcal{F})$ of connections on $\mathcal{F}$ is the affince space over the vector space $\Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes_{\mathcal{O}_X} \Omega_X^1)$, where

$$\mathfrak{g}_{\mathcal{F}} = \mathcal{F} \times^G \mathfrak{g} = \Omega_X^{\check{\rho}} \times^H \mathfrak{g} = \Omega_X^{\check{\rho}} \times^H (\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) = (\Omega_X^{\check{\rho}} \times^H \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (\Omega_X^1)^{\otimes \langle \check{\rho}, \alpha \rangle}.$$

So, we can identify $\operatorname{Conn}(\mathcal{F})$ with the space

$$\operatorname{Conn}(\Omega_X^{\check{\rho}}) \times \Gamma(X, \bigoplus_{\alpha \in \Delta} (\Omega^1)^{\otimes (\langle \check{\rho}, \alpha \rangle + 1)})$$

For every negative simple root $-\alpha_i$, we see that the term $(\Omega^1)^{\otimes (\langle \check{\rho}, -\alpha_i \rangle + 1)}$ is just the structure sheaf $\mathcal{O}_X$, so the choice of a generator $f_i \in \mathfrak{g}_{-\alpha_i}$ defines the element $p_{-1} := \sum_i f_i \in \Gamma(X, \bigoplus_{\alpha \in \Delta} (\Omega^1)^{\otimes (\langle \check{\rho}, \alpha \rangle + 1)})$. Now $\nabla := \widehat{\nabla} + p_{-1}$. It follows from the definitions that the maps $\nabla \mapsto \widehat{\nabla}$ and $\widehat{\nabla} \mapsto \nabla$ are inverse to each other. $\square$

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