VERTEX POISSON ALGEBRAS AND MIURA OPERS II

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From now on, we assume that $X = D$. Let me recall that last time we proved the following proposition.

Proposition 0.1. *There exists a natural* $\operatorname{Der}\mathcal{O}$*-equivariant isomorphism*

$$\mathrm{MOp}_G(D)_{\mathrm{gen}} \simeq \mathrm{Conn}(\Omega_D^{\check{\rho}}). \tag{1}$$

Composing the identification (1) with the natural map

$$\mathrm{MOp}_G(D) \to \mathrm{Op}_G(D),\ (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B) \mapsto (\mathcal{F}, \nabla, \mathcal{F}_B)$$

we obtain the morphism

$$\mu\colon\ \mathrm{Conn}(\Omega_X^{\check{\rho}}) \to \mathrm{Op}_G(X)$$

called the *Miura transformation*. We will see that μ is dominant so induces the embedding

$$\mu^*\colon\ \operatorname{Fun}\mathrm{Op}_G(X) \hookrightarrow \operatorname{Fun}\mathrm{Conn}(\Omega_X^{\check{\rho}}).$$

We will also recall the identification $\operatorname{Fun}\mathrm{Conn}(\Omega_X^{\check{\rho}}) \simeq \pi_0$ constructed by Zeyu in [Wa, Section 7] and show that the image of μ^* coincides with the intersection of kernels of the operators $V_i[1]$, $i = 1, \ldots, \ell$. This will imply the desired $\operatorname{Der}\mathcal{O}$-equivariant identification

$$W(\mathfrak{g}) \simeq \operatorname{Fun}\mathrm{Op}_G(D).$$

We will then compute the character of $W(\mathfrak{g})$ (by computing the one for $\operatorname{Fun}\mathrm{Op}_G(D)$) and conclude that the embedding

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow W({}^L\mathfrak{g})$$

is actually an isomorphism. In other words, we finish the proof of the main theorem.

Theorem 0.2. *There exists an* $\operatorname{Aut}\mathcal{O}$*-equivariant isomorphism:*

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} W({}^L\mathfrak{g}).$$

0.1. **Miura transformation: explicit realization for $X = D$.** We fix a coordinate t on the disc, then it follows from [Bo1, Corollary 3.18] that we have an identification:

$$\mathrm{Op}_G(D) \simeq \{\partial_t + S[[t]]\},$$

where $S = p_{-1} + \mathfrak{z}_{\mathfrak{g}}(p_1) \subset \mathfrak{g}$ is the Konstant slice. In other words, every element of $\mathrm{Op}_G(D)$ has a canonical representative

$$\partial_t + p_{-1} + \sum_{i=1}^{\ell} v_i(t)c_i,$$

where $\{c_i\}$ is some fixed basis in $V = \mathfrak{z}_{\mathfrak{g}}(p_1)$, consisiting of eigenvectors of $2\check{\rho} = [p_1, p_{-1}]$ and

$$v_i(t) = \sum_{n<0} v_{i,n}t^{-n-1}.$$

Thus,

$$\operatorname{Fun}\operatorname{Op}_G(D)=\mathbb{C}[v_{i,n}]_{i=1,\dots,\ell;\, n<0}.$$

Now, by the proof of Proposition 0.1, each generic Miura oper can be represented by a connection operator

$$\partial_t+p_{-1}+u(t),\ u(t)\in\mathfrak{h}[[t]].$$

Set $u_i(t):=\alpha_i(u(t))$ and write

$$u_i(t)=\sum_{n<0}u_{i,n}t^{-n-1}.$$

We see that

$$\operatorname{Fun}\operatorname{MOp}_G(D)_{\mathrm{gen}}=\operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check\rho})=\mathbb{C}[u_{i,n}]_{i=1,\dots,\ell;\, n<0}.$$

In these terms, the identification $\operatorname{MOp}_G(D)_{\mathrm{gen}}\simeq\operatorname{Conn}(\Omega_D^{\check\rho})$ is given by $\partial_t+p_{-1}+u(t)\mapsto\partial_t+u(t)$ and the morphism $\mu\colon \operatorname{Conn}(\Omega_D^{\check\rho})\to\operatorname{Op}_G(D)$ sends $\partial_t+u(t)$ to the class of $\partial_t+p_{-1}+u(t)$ in

$$\operatorname{Op}_G(D)=\{\partial_t+p_{-1}+\mathfrak{b}[[t]]\}/N[[t]].$$

Example 0.3. Let us compute the Miura transformation μ for $\mathfrak{g}=\mathfrak{sl}_2$. We start with a generic Miura oper

$$\partial_t+\begin{pmatrix}\frac{1}{2}u(t) & 0\\ 1 & -\frac{1}{2}u(t)\end{pmatrix} \tag{2}$$

and now consider it as an element of the quotient $\{\partial_t+p_{-1}+\mathfrak{b}[[t]]\}/N[[t]]$.

To compute μ, we need to find an element $g(t)\in N[[t]]$ such that under the gauge action of $g(t)$, (2) goes to an element of the form $\partial_t+\begin{pmatrix}0 & v(t)\\ 1 & 0\end{pmatrix}$.

It is an exercise to check that

$$\begin{pmatrix}1 & -\frac{1}{2}u(t)\\ 0 & 1\end{pmatrix}\cdot\left(\partial_t+\begin{pmatrix}\frac{1}{2}u(t) & 0\\ 1 & -\frac{1}{2}u(t)\end{pmatrix}\right)=\partial_t+\begin{pmatrix}0 & \frac{1}{4}u(t)^2+\frac{1}{2}\partial_t u(t)\\ 1 & 0\end{pmatrix}$$

so we conclude that

$$\mu(u(t))=\frac{1}{4}u(t)^2+\frac{1}{2}\partial_t u(t).$$

Let us finish this section by recalling how $\operatorname{Der}(\mathcal{O})$ acts on $\operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check\rho})$ (see [Wa, Section 7]).

Lemma 0.4. *The action of* $\operatorname{Der}(\mathcal{O})$ *on* $\operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check\rho})$ *is given by:*

$$L_n\cdot u_{i,m}=-mu_{i,n+m},\ n<-m,$$
$$L_n\cdot u_{i,-n}=-n(n+1),\ n>0,$$
$$L_n\cdot u_{i,m}=0,\ n>-m.$$

Comparing this lemma with [Kr, Lemma 2.11] we see that we have a $\operatorname{Der}\mathcal{O}$-equivariant isomorphisms

$$\operatorname{Fun}\operatorname{MOp}_G(D)_{\mathrm{gen}}\simeq\operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check\rho})\simeq\pi_0(\mathfrak{g}).$$

In particular, the action of L_0 induces the grading on both of our algebras and the isomorphism is compatible with this action. Note that

$$\deg u_{i,m}=-m.$$

0.2. **The image of the pullback under Miura map vs screening operators.** Our goal in this section will be to prove that the Miura transformation μ is *surjective* and identifies the image of $\mu^*\colon \operatorname{Fun}\operatorname{Op}_G(D) \hookrightarrow \operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check\rho})$ with $W(\mathfrak{g}) \subset \pi_0(\mathfrak{g})$.

We start with the following lemma that we will later use to give another description of $\operatorname{MOp}_G(D)_{\mathrm{gen}}$.

Lemma 0.5. *Let ∇ be a connection in a G-bundle $\mathcal{F}$ on a disc D. For every right B-torsor $\mathcal{F}'_{B,0} \subset \mathcal{F}|_0$ there exist the unique B-reduction $\mathcal{F}'_B \subset \mathcal{F}$ preserved by ∇ and equal to $\mathcal{F}'_{B,0}$ at 0.*

Proof. Standard. If we use a small disc instead of a formal one, we parallel-transport $\mathcal{F}'_{B,0}$ to the other points. By parallel-transport we mean the following.

Recall (see [Bo1]) that a connection on $\mathcal{F}$ is a G-equivariant section

$$\omega\colon \mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{C}} \mathfrak{g}^* \to \Omega^1_{\mathcal{F}}$$

of the natural (surjective) map $\Omega^1_{\mathcal{F}} \twoheadrightarrow \mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{C}} \mathfrak{g}^*$. We can compose ω with the natural embedding $\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{C}} (\mathfrak{g}/\mathfrak{b})^* \hookrightarrow \mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{C}} \mathfrak{g}^*$ and obtain a B-equivariant map:

$$\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{C}} (\mathfrak{g}/\mathfrak{b})^* \to \Omega^1_{\mathcal{F}}$$

which (from B-invariance) descends to $\mathcal{F}/B$ with the image lying in $\Omega^1_{\mathcal{F}/B}$:

$$\omega_{/B}\colon \mathcal{F} \times^B (\mathfrak{g}/\mathfrak{b})^* \to \Omega^1_{\mathcal{F}/B}.$$

Map $\omega_{/B}$ is a section of the (surjective) map $\varphi\colon \Omega^1_{\mathcal{F}/B} \twoheadrightarrow \mathcal{F} \times^B (\mathfrak{g}/\mathfrak{b})^*$ appearing in the exact sequence:

$$0 \to f_B^*\Omega^1_D \to \Omega^1_{\mathcal{F}/B} \twoheadrightarrow \mathcal{F} \times^B (\mathfrak{g}/\mathfrak{b})^* \to 0,$$

where $f_B\colon \mathcal{F}/B \twoheadrightarrow D$ is the natural map. Passing to the dual sequence

$$0 \to \mathcal{F} \times^B (\mathfrak{g}/\mathfrak{b}) \to \mathcal{T}_{\mathcal{F}/B} \to f_B^*\mathcal{T}_D \to 0 \tag{3}$$

we see that $\omega^*_{/B}$ defines the splitting of every fiber of $\mathcal{T}_{\mathcal{F}/B}$ into the direct sum of "horizontal" and "vertical" directions.

A section $s\colon D \hookrightarrow \mathcal{F}/B$ is preserved by the connection ∇ iff

$$s^*\omega_{/B} \circ (ds)^* = 0. \tag{4}$$

Here, we consider the exact sequence

$$0 \to s^*(\mathcal{F} \times^B (\mathfrak{g}/\mathfrak{b})) \to s^*\mathcal{T}_{\mathcal{F}/B} \to \mathcal{T}_D \to 0 \tag{5}$$

on D (obtained as a pull back of (3) via s^*) and denote by $(ds)^*\colon \mathcal{T}_D \to s^*\mathcal{T}_{\mathcal{F}/B}$ the morphism dual to $ds\colon s^*\Omega^1_{\mathcal{F}/B} \to \Omega^1_D$.

Now, using $\omega^*_{/B}$, we can parallel-transport $\mathcal{F}'_{B,0} \in (\mathcal{F}/B)|_0$. ∎

Let $\mathcal{F}_{\mathrm{univ}} \to \operatorname{Op}_G(D)$ be the *universal* G-bundle on $\operatorname{Op}_G(D)$ whose fiber at $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is $\mathcal{F}_0$. Let $\mathcal{F}_{B,\mathrm{univ}}$ be the universal B bundle whose fiber at $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is $\mathcal{F}_{B,0}$ (note that $\mathcal{F}_{B,\mathrm{univ}}$ is a B-reduction of $\mathcal{F}_{\mathrm{univ}}$).

Corollary 0.6. *We have natural isomorphisms*

$$\mathcal{F}_{\mathrm{univ}} \times^G (G/B) \xrightarrow{\sim} \operatorname{MOp}_G(D),\ \mathcal{F}_{B,\mathrm{univ}} \times^B (Bw_0B)/B \xrightarrow{\sim} \operatorname{MOp}_G(D)_{\mathrm{gen}}.$$

In particular, the Miura morphism μ is the trivial principal N-bundle (we choose the trivialization of $\Omega_D^{\check\rho}$ that induces the trivialization of $\mathcal{F}_B = \Omega_D^{\check\rho} \times^H B$, which, in turn, induces the trivialization of $\mathcal{F}_{B,\mathrm{univ}}$).

Proof. The first identification directly follows from Lemma 0.5: note that a point of $\mathcal{F}_{\mathrm{univ}} \times^G (G/B)$ is nothing else but an oper $(\mathcal{F}, \nabla, \mathcal{F}_B)$ together with a choice of a B-torsor $\mathcal{F}'_{B,0} \subset \mathcal{F}|_0$. The second identification follows from the first identification. □

Warning 0.7. *Note that there is no similar description of* $\mathrm{MOp}_G(X)$ *for an arbitrary curve* X*. For example, morphism* $\mathrm{MOp}_G(X) \to \mathrm{Op}_G(X)$ *may fail to be surjective.*

So, we conclude that:

$$\mathrm{Fun}\,\mathrm{Op}_G(D) = (\mathrm{Fun}\,\mathrm{MOp}_G(D)_{\mathrm{gen}})^N = (\mathrm{Fun}\,\mathrm{MOp}_G(D)_{\mathrm{gen}})^{\mathfrak{n}}.$$

Recall now that $\mathrm{Fun}\,\mathrm{MOp}_G(D)_{\mathrm{gen}} \simeq \mathrm{Fun}\,\mathrm{Conn}(\Omega_D^{\check{\rho}})$ so by *transport de structure* we obtain an action of N and $\mathfrak{n}$ on $\mathrm{Fun}\,\mathrm{Conn}(\Omega_D^{\check{\rho}}) = \mathbb{C}[u_{i,n}]$. Our next goal will be to understand this action.

Choose $e_j \in \mathfrak{n}$, $j = 1, \ldots, \ell$ such that $(e_j, [e_j, f_j], f_j)$ is an $\mathfrak{sl}_2$-triple. Note that $\mathfrak{n}$ is generated by the elements e_j (as a Lie algebra), so

$$\mathrm{Fun}\,\mathrm{Op}_G(D) = \bigcap_{i=1}^{\ell} \mathbb{C}[u_{i,n}]^{e_j}.$$

In the remaining part of this section, we will explicitly compute the action of e_j and will see that it acts via the operator $V_j[1]$. As a corollary, we will obtain the desired identification $\mathrm{Fun}\,\mathrm{Op}_G(D) \simeq W(\mathfrak{g})$.

Let us now describe the action of $N \curvearrowright \mathrm{Conn}\,\Omega_D^{\check{\rho}}$. We fix a trivialization φ of $\Omega_D^{\check{\rho}}$. Recall the space $\widetilde{\mathrm{Op}}_G(D) = \{\partial_t + p_{-1} + \mathfrak{b}[[t]]\}$ introduced in [Bo1, Definition 3.9]. This space is a moduli space of $(\mathcal{F}, \nabla, \mathcal{F}_B, \psi)$, where $(\mathcal{F}, \nabla, \mathcal{F}_B) \in \mathrm{Op}_G(D)$ and ψ is a trivialization of $\mathcal{F}_B$ that becomes equal to φ on $\Omega_D^{\check{\rho}} = \mathcal{F}_B/N$.

Now, we have an embedding

$$\mathrm{MOp}_G(D) \hookrightarrow \widetilde{\mathrm{Op}}_G(D)$$

sending $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ to $(\mathcal{F}, \nabla, \mathcal{F}_B, \psi)$, where ψ is induced by the identification $\mathcal{F}_B \simeq \Omega_D^{\check{\rho}} \times^H B$ (comming from the H-reduction in $\mathcal{F}_B$ induced by $\mathcal{F}'_B$, see [Kr, Lemma 3.4]) and the trivialization φ. After the identification $\mathrm{Conn}\,\Omega_D^{\check{\rho}} \simeq \mathrm{MOp}_G(D)$, the embedding above is given by

$$\partial_t + u(t) \mapsto \partial_t + p_{-1} + u(t).$$

Now, the action of N on $\mathrm{MOp}_G(D)_{\mathrm{gen}}$ sends $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ to $(\mathcal{F}, \nabla, \mathcal{F}_B, {}^n\mathcal{F}'_B)$, so their images in $\widetilde{\mathrm{Op}}_G(D)$ are $(\mathcal{F}, \nabla, \mathcal{F}_B, \psi)$, $(\mathcal{F}, \nabla, \mathcal{F}_B, {}^n\psi)$, where ${}^n\psi$ is some trivialization of $\mathcal{F}_B$ depending on n. Recall now that $N[[t]]$ acts on $\widetilde{\mathrm{Op}}_G(D)$ by changing the trivialization ψ (after the identification $\widetilde{\mathrm{Op}}_G(D) = \{\partial_t + p_{-1} + \mathfrak{b}[[t]]\}$, this action becomes the action via gauge transformations). It follows from the definitions that the points $(\mathcal{F}, \nabla, \mathcal{F}_B, \psi)$, $(\mathcal{F}, \nabla, \mathcal{F}_B, {}^n\psi)$ in $\widetilde{\mathrm{Op}}_G(D)$ differ by the action of $g(t) \in N[[t]]$ such that $g(0) = n$.

Now, $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ corresponds to some $\partial_t + u(t) \in \mathrm{Conn}(\Omega_D^{\check{\rho}})$. Reformulating, we have a map

$$a_{u(t)} = a\colon N \to N[[t]],\ n \mapsto g(t)$$

and the action of n on $\partial_t + u(t)$ is equal to $\partial_t + \tilde{u}(t)$, where $\tilde{u}(t) \in \mathfrak{h}[[t]]$ is such that

$$\partial_t + g(p_{-1} + u(t))g^{-1} - g^{-1}(\partial_t g) = \partial_t + p_{-1} + \tilde{u}(t) \in \partial_t + p_{-1} + \mathfrak{h}[[t]].$$

Remark 0.8. *Note that $g(t)$ is determined uniquely by the containment*

$$\partial_t + g(p_{-1} + u(t))g^{-1} - g^{-1}(\partial_t g) \in \partial_t + p_{-1} + \mathfrak{h}[[t]]$$

together with the condition $g(0) = n$. This is an exercise: use that the action of $N[[t]]$ on $\partial_t + p_{-1} + \mathfrak{b}[[t]]$ is free together with the fact that

$$N\backslash \operatorname{Conn}\Omega^{\check{\rho}} = \operatorname{Op}_G(D) = N[[t]]\backslash(\partial_t + p_{-1} + \mathfrak{b}[[t]]).$$

Let us also emphasize that the map a depends *on the point $u(t)$ and is* not *a homomorphism in general.*

We want to compute the infinitesimal action of $e_i \in \mathfrak{g}_{\alpha_i} \subset \mathfrak{n}$. Recall that $\mathfrak{g}_{\alpha_i}$ integrates to the subgroup $N_i \subset N$ (isomorphic to $\mathbb{G}_a$). We claim that the map a restricts to the map $N_i \to N_i[[t]]$. To see that, it is enough (by Remark 0.8) to prove the existence of the map $a_i \colon N_i \to N_i[[t]]$ such that for $n \in N_i$ and $g := a_i(n)$, we have

$$\partial_t + g(p_{-1} + u(t))g^{-1} - g^{-1}(\partial_t g) \in \partial_t + p_{-1} + \mathfrak{h}[[t]] \text{ and } g(0) = n. \tag{6}$$

Recall that we have the decomposition $\mathfrak{h}[[t]] = \mathfrak{h}_i[[t]] \oplus \mathfrak{h}_i^{\perp}[[t]]$, where $\mathfrak{h}_i \subset \mathfrak{h}$ is the Cartan subalgebra for $\mathfrak{sl}_2 = \langle e_i, [e_i, f_i], f_i\rangle$. Decompose $u(t) = {}^i u(t) + {}^i u^{\perp}(t)$. We can also write $p_{-1} = f_i + p'$, where $p' := \sum_{j\neq i} f_j$. We see that for $g \in N_i[[t]]$

$$\partial_t + g(p_{-1} + u(t))g^{-1} - g^{-1}(\partial_t g) = \partial_t + g(f_i + {}^i u(t))g^{-1} - g^{-1}(\partial_t g) + p' + {}^i u^{\perp}(t).$$

In other words, $g \in N_i[[t]]$ satisfies the conditions (6) iff

$$g(f_i + {}^i u)g^{-1} - g^{-1}(\partial_t g) \in f_i + \mathfrak{h}_i[[t]],\ g(0) = n. \tag{7}$$

This proves the existence of a_i as above (since we already know that it exists for $\mathfrak{sl}_2$) and moreover reduces the computation of a_i to the case of $\mathfrak{g} = \mathfrak{sl}_2$.

Now, we fix a point $\begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 0 & -\frac{1}{2}u(t)\end{pmatrix} \in \mathfrak{h}[[t]]$ and we want to find ψ_i in the following form:

$$a_i \colon \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix} \mapsto \begin{pmatrix} 1 & zx_z(t) \\ 0 & 1\end{pmatrix},\ x_z(0) = 1,\ z \in \mathbb{C}.$$

The condition (7) is equivalent to

$$x_z'(t) = -u(t)x_z(t) - zx_z(t)^2,\ x_z(0) = 1. \tag{8}$$

Clearly, this equation has a unique solution (note also that this solution clearly *depends* on $u(t)$).

Recall now that our goal was to compute the infinitesimal action of e_i. In other words, we need to compute $x(t) := x_0(t)$ in (8). By (8), $x(t)$ is the solution of the equation

$$x'(t) = -u(t)x(t),\ x(0) = 1. \tag{9}$$

It is an exercise to check that the unique solution is

$$x(t) = \sum_{n\leqslant 0} x_n t^{-n} = \exp\Big(-\sum_{m>0} \frac{u_{-m}}{m} t^m\Big). \tag{10}$$

So, we have computed $x(t)$ and conclude that e acts on $\operatorname{Fun}\operatorname{Conn}\Omega_D^{\check{\rho}} = \mathbb{C}[u_n]$ as follows: function

$$\begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 0 & -\frac{1}{2}u(t)\end{pmatrix} \mapsto u(t)$$

maps to the difference of diagonal entries of

$$\Big[\begin{pmatrix}0 & x(t)\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}u(t) & 0\\ 1 & -\frac{1}{2}u(t)\end{pmatrix}\Big] - \begin{pmatrix}0 & x'(t)\\ 0 & 0\end{pmatrix} = \begin{pmatrix}x(t) & 0\\ 0 & -x(t)\end{pmatrix}.$$

So,

$$e\cdot u(t) = 2x(t). \tag{11}$$

Remark 0.9. Recall that in the formula (11) we are dealing with $\mathfrak{g}=\mathfrak{sl}_2$, from the discussion above it follows that for arbitrary $\mathfrak{g}$ the action of e_i on $u_j(t)$ is given by $x_i(t)\langle\alpha_j,\check{\alpha}_i\rangle = a_{ij}x_i(t)$.

In other words, e acts by the derivation

$$2\sum_{n\leqslant 0} x_n\frac{\partial}{\partial u_{n-1}},$$

where x_n are the coefficients of (10).

Remark 0.10. Again, for arbitrary $\mathfrak{g}$, we see that e_i acts by the derivation

$$\sum_{j=1}^{\ell} a_{ij}\sum_{n\leqslant 0} x_{i,n}\frac{\partial}{\partial u_{j,n-1}}.$$

Recall now that $W(\mathfrak{sl}_2)\subset \pi_0(\mathfrak{sl}_2)$ is the kernel of the operator

$$V[1] = \sum_{m\leqslant 0} V[m]D_{b_{m-1}},$$

where

$$D_{b_m}b_n = 2\delta_{n,m}$$

and

$$\sum_{n\leqslant 0} V[n]z^{-n} = \exp\Big(-\sum_{m>0}\frac{b_{-m}}{m}z^m\Big).$$

We see that after the identifications $u_{-m}\mapsto b_{-m}$, operators $e\cdot -$, $V[1]$ become the same. It follows that for arbitrary simple $\mathfrak{g}$ and adjoint G we have

$$\operatorname{Fun}\operatorname{Op}_G(D)\simeq W(\mathfrak{g}).$$

0.3. **Character of $W(\mathfrak{g})\simeq \operatorname{Fun}\operatorname{Op}_G(D)$ and the main theorem.** We are now ready to compute the character of $W(\mathfrak{g})$ and then finish the proof of Theorem 0.2.

Recall that the algebra $\operatorname{Fun}\operatorname{Op}_G(D)$ is graded via $L_0=-t\partial_t$. Derivation $-t\partial_t$ is the Lie derivative at $\epsilon=1$ of the family of automorphisms of $\mathcal{O}$ given by $t\mapsto \epsilon^{-1}t$. Recall now that

$$\operatorname{Op}_G(D)\simeq\{\partial_t+S[[t]]\}$$

and it follows from [Bo2, Section 1] that the action of the automorphism $t\mapsto\epsilon^{-1}t$ is given by:

$$\partial_t+p_{-1}+v(t)\mapsto \partial_t+p_{-1}+\tilde{v}_\epsilon(t),$$

where

$$\tilde{v}_\epsilon(t)=\epsilon\check{\rho}(\epsilon)v(\epsilon t)\check{\rho}(\epsilon)^{-1}.$$

We see that the infinitesimal action is given by

$$L_0\cdot v(t) = v(t)+[\check{\rho},v(t)]+(t\partial_t)v(t). \tag{12}$$

Recall now that

$$V = \mathfrak{z}_{\mathfrak{g}}(p_1) = \bigoplus_{i=1}^{\ell} V_{d_i},$$

where d_i are exponents (degrees of the free generators of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ *minus one*). The action of $\check{\rho}$ on V_{d_i} is via the multiplication by d_i. We conclude that the character of $\operatorname{Fun}\operatorname{Op}_G(D)$ (w.r.t. L_0) is equal to

$$\operatorname{ch}_{L_0} \operatorname{Fun}\operatorname{Op}_G(D) = \prod_{i=1}^{\ell} \prod_{n_i \geqslant d_i+1} \frac{1}{1-q^{n_i}}.$$

This is nothing else but the character of $\mathfrak{z}(\widehat{\mathfrak{g}})$ (see [MF, Section 8.1.2]). We obtain the following corollary.

Corollary 0.11. *We have*

$$\operatorname{ch}_{L_0} W(\mathfrak{g}) = \operatorname{ch}_{L_0} \operatorname{Fun}\operatorname{Op}_G(D) = \operatorname{ch}_{L_0} \mathfrak{z}(\widehat{\mathfrak{g}})$$

so the embedding $\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow W({}^L\mathfrak{g})$ *is an isomorphism.*

We have finally proved the main theorem.

Theorem 0.12. *There is a commutative diagram (of vertex Poisson algebras) preserving the* $(\operatorname{Der}\mathcal{O}, \operatorname{Aut}\mathcal{O})$*-actions:*

$$\begin{array}{ccccc}
\pi_0(\mathfrak{g}) & \xrightarrow{\simeq} & \pi_0^{\vee}({}^L\mathfrak{g}) & \xrightarrow{\simeq} & \operatorname{Fun}\operatorname{Conn}(\Omega_D^{\check{\rho}}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{z}(\widehat{\mathfrak{g}}) & \xrightarrow{\simeq} & W({}^L\mathfrak{g}) & \xrightarrow{\simeq} & \operatorname{Fun}\operatorname{Op}_{{}^L G}(D)
\end{array}$$

vertical arrows are embeddings, and the upper arrows are given by $b_{i,n} \mapsto -\mathbf{b}'_{i,n} \mapsto -u_{i,n}$.

Remark 0.13. Feigin and Frenkel proved that for generic values of κ_0, there exists an isomorphism $W_{\kappa_0}(\mathfrak{g}) \simeq W_{\kappa_0^{\vee}}({}^L\mathfrak{g})$ between the corresponding W-algebras. Theorem 0.12 should be considered as a limit at the critical level of this identification.

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