SCHUR-WEYL DUALITY FOR QUANTUM GROUPS

YI SUN

Abstract. These are notes for a talk in the MIT-Northeastern Fall 2014 Graduate seminar on Hecke algebras and affine Hecke algebras. We formulate and sketch the proofs of Schur-Weyl duality for the pairs $(U_q(\mathfrak{sl}_n), H_q(m))$, $(Y(\mathfrak{sl}_n), \Lambda_m)$, and $(U_q(\widehat{\mathfrak{sl}}_n), \mathcal{H}_q(m))$. We follow mainly [Ara99, Jim86, Dri86, CP96], drawing also on the presentation of [BGHP93, Mol07].

Contents

1. Introduction

Let $V = \mathbb{C}^n$ be the fundamental representation of $\mathfrak{sl}_n$. The vector space $V^{\otimes m}$ may be viewed as a $U(\mathfrak{sl}_n)$ and S_m-representation, and the two representations commute. Classical Schur-Weyl duality gives a finer understanding of this representation. We first state the classifications of representations of S_m and $\mathfrak{sl}_n$.

Theorem 1.1. The finite dimensional irreducible representations of S_m are parametrized by partitions $\lambda \vdash m$. For each such λ, the corresponding representation S_λ is called a Specht module.

Theorem 1.2. The finite dimensional irreducible representations of $\mathfrak{sl}_n$ are parametrized by signatures λ with $\ell(\lambda) \leq n$ and $\sum_i \lambda_i = 0$. For any partition λ with $\ell(\lambda) \leq n$, there is a unique shift λ' of λ so that $\sum_i \lambda'_i = 0$. We denote the irreducible with this highest weight by L_λ.

The key fact underlying classical Schur-Weyl duality is the following decomposition of a tensor power of the fundamental representation.

Theorem 1.3. View $V^{\otimes m}$ as a representation of S_m and $U(\mathfrak{sl}_n)$. We have the following:

(a) the images of $\mathbb{C}[S_m]$ and $U(\mathfrak{sl}_n)$ in $\mathrm{End}(W)$ are commutants of each other, and

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(b) as a $\mathbb{C}[S_m] \otimes U(\mathfrak{sl}_n)$-module, we have the decomposition

$$V^{\otimes m} = \bigoplus_{\substack{\lambda \vdash m \\ \ell(\lambda) \leq n}} S_\lambda \boxtimes L_\lambda.$$

We now reframe this result as a relation between categories of representations; this reformulation will be the one which generalizes to the affinized setting. Say that a representation of $U(\mathfrak{sl}_n)$ is of weight m if each of its irreducible components occurs in $V^{\otimes m}$. In general, the weight of a representation is not well-defined; however, for small weight, we have the following characterization from the Pieri rule.

Lemma 1.4. The irreducible L_λ is of weight $m \leq n-1$ if and only if $\lambda = \sum_i c_i \omega_i$ with $\sum_i ic_i = m$.

Given a S_m-representation W, define the $U(\mathfrak{sl}_n)$-representation $\mathsf{FS}(W)$ by

$$\mathsf{FS}(W) = \mathrm{Hom}_{S_m}(W, V^{\otimes m}),$$

where the $U(\mathfrak{sl}_n)$-action is inherited from the action on $V^{\otimes m}$. Evidently, FS is a functor $\mathrm{Rep}(S_m) \to \mathrm{Rep}(U(\mathfrak{sl}_n))$, and we may rephrase Theorem 1.3 as follows.

Theorem 1.5. For $n > m$, the functor FS is an equivalence of categories between $\mathrm{Rep}(S_m)$ and the subcategory of $\mathrm{Rep}(U(\mathfrak{sl}_n))$ consisting of weight m representations.

In this talk, we discuss generalizations of this duality to the quantum group setting. In each case, $U(\mathfrak{sl}_n)$ will be replaced with a quantization ($U_q(\mathfrak{sl}_n)$, $Y_\hbar(\mathfrak{sl}_n)$, or $U_q(\widehat{\mathfrak{sl}}_n)$), and $\mathbb{C}[S_m]$ will be replaced by a Hecke algebra ($H_q(m)$, Λ_m, or $\mathcal{H}_q(m)$).

2. Finite-type quantum groups and Hecke algebras

2.1. **Definition of the objects.** Our first generalization of Schur-Weyl duality will be to the finite type quantum setting. In this case, $U_q(\mathfrak{sl}_n)$ will replace $U(\mathfrak{sl}_n)$, and the Hecke algebra $H_q(m)$ of type A_{m-1} will replace S_m. We begin by defining these objects.

Definition 2.1. Let $\mathfrak{g}$ be a simple Kac-Moody Lie algebra of simply laced type with Cartan matrix $A = (a_{ij})$. The Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ is the Hopf algebra given as follows. As an algebra, it is generated by $x_i^\pm$ and q^{h_i} for $i = 1, \ldots, n-1$ so that $\{q^{h_i}\}$ are invertible and commute, and we have have the relations

$$q^{h_i} x_j^\pm q^{-h_i} = q^{\pm a_{ij}} e_j, \qquad [x_i^+, x_j^-] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \qquad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix} (x_i^\pm)^r x_j^\pm (x_i^\pm)^{1-a_{ij}-r} = 0.$$

The coalgebra structure is given by the coproduct

$$\Delta(x_i^+) = x_i^+ \otimes q^{h_i} + 1 \otimes x_i^+, \qquad \Delta(x_i^-) = x_i^- \otimes 1 + q^{-h_i} \otimes x_i^-, \qquad \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i},$$

and counit $\varepsilon(x_i^\pm) = 0$ and $\varepsilon(q^{h_i}) = 1$, and the antipode is given by

$$S(x_i^+) = -x_i^+ q^{-h_i}, \qquad S(x_i^-) = -q^{h_i} x_i^-, \qquad S(q^{h_i}) = q^{-h_i}.$$

Definition 2.2. The Hecke algebra $H_q(m)$ of type A_{m-1} is the associative algebra given by

$$H_q(m) = \Big\langle T_1, \ldots, T_{m-1} \mid (T_i - q^{-1})(T_i + q) = 0, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, [T_i, T_j] = 0 \text{ for } |i - j| \neq 1 \Big\rangle.$$

2.2. ***R*-matrices and the Yang-Baxter equation.** To obtain $H_q(m)$-representations from $U_q(\mathfrak{sl}_n)$-representations, we use the construction of R-matrices.

Proposition 2.3. There exists a unique *universal R-matrix* $\mathcal{R} \in U_q(\mathfrak{sl}_n) \hat{\otimes} U_q(\mathfrak{sl}_n)$ such that:

(a) $\mathcal{R} \in q^{\sum_i x_i \otimes x_i}(1 + (U_q(\mathfrak{n}_+) \hat{\otimes} U_q(\mathfrak{n}_-))_{>0})$ for $\{x_i\}$ an orthonormal basis of , and
(b) $\mathcal{R}\Delta(x) = \Delta^{21}(x)\mathcal{R}$, and
(c) $(\Delta \otimes 1)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}$ and $(1 \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}$.

We say that such an $\mathcal{R}$ defines a *pseudotriangular structure* on $U_q(\mathfrak{sl}_n)$. Let $P(x \otimes y) = y \otimes x$ denote the flip map, and let $\widehat{\mathcal{R}} = P \circ \mathcal{R}$. From Proposition 2.3, we may derive several additional properties of $\mathcal{R}$ and $\widehat{\mathcal{R}}$.

Corollary 2.4. The universal R-matrix of $U_q(\mathfrak{sl}_n)$:

(a) satisfies the Yang-Baxter equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12};$$

(b) gives an isomorphism $\widehat{\mathcal{R}} : W \otimes V \to V \otimes W$ for any $V, W \in \mathrm{Rep}(U_q(\mathfrak{sl}_n))$;

(c) satisfies a different version of the Yang-Baxter equation

$$\widehat{\mathcal{R}}^{23}\widehat{\mathcal{R}}^{12}\widehat{\mathcal{R}}^{23} = \widehat{\mathcal{R}}^{12}\widehat{\mathcal{R}}^{23}\widehat{\mathcal{R}}^{12};$$

(d) when evaluated in the tensor square $V^{\otimes 2}$ of the fundamental representation of $U_q(\mathfrak{sl}_n)$ is given by

$$\mathcal{R}|_{V\otimes V} = q\sum_i E_{ii} \otimes E_{ii} + \sum_{i\neq j} E_{ii} \otimes E_{jj} + (q - q^{-1})\sum_{i>j} E_{ij} \otimes E_{ji}. \tag{1}$$

2.3. **From the Yang-Baxter equation to the Hecke relation.** We wish to use Corollary 2.4 to define a $H_q(m)$-action on $V^{\otimes m}$. Define the map $\sigma^m : H_q(m) \to \mathrm{End}(V^{\otimes m})$ by

$$\sigma^m : T_i \mapsto \widehat{\mathcal{R}}^{i,i+1}.$$

Lemma 2.5. The map σ^m defines a representation of $H_q(m)$ on $V^{\otimes m}$.

Proof. The braid relation follows from Corollary 2.4(c) and the commutativity of non-adjacent reflections from the definition of σ^m. The Hecke relation follows from a direct check on the eigenvalues of the triangular matrix $\mathcal{R}|_{V\otimes V}$ from (1). □

2.4. **Obtaining Schur-Weyl duality.** We have analogues of Theorems 1.3 and 1.5 for $V^{\otimes m}$.

Theorem 2.6. If q is not a root of unity, we have:

(a) the images of $U_q(\mathfrak{sl}_n)$ and $H_q(m)$ in $\mathrm{End}(V^{\otimes m})$ are commutants of each other;

(b) as a $H_q(m) \otimes U_q(\mathfrak{sl}_n)$-module, we have the decomposition

$$V^{\otimes m} = \bigoplus_{\substack{\lambda\vdash m \\ \ell(\lambda)\leq n}} S_\lambda \boxtimes L_\lambda,$$

where S_λ and L_λ are quantum deformations of the classical representations of S_m and $U(\mathfrak{sl}_n)$.

Proof. We explain a proof for $n > m$, though the result holds in general. For (a), we use a dimension count from the non-quantum case. By the definition of σ^m in terms of R-matrices, each algebra lies inside the commutant of the other. We now claim that $\sigma^m(H_q(m))$ spans $\mathrm{End}_{U_q(\mathfrak{sl}_n)}(V^{\otimes m})$. If q is not a root of unity, the decomposition of $V^{\otimes m}$ into $U_q(\mathfrak{sl}_n)$-isotypic components is the same as in the classical case, meaning that its commutant has the same dimension as in the classical case. Similarly, $H_q(m)$ is isomorphic to $\mathbb{C}[S_m]$; because σ^m is faithful, this means that $\sigma^m(H_q(m))$ has the same dimension as the classical case, and thus $\sigma^m(H_q(m))$ is the entire commutant of $U_q(\mathfrak{sl}_n)$. Finally, because $U_q(\mathfrak{sl}_n)$ is semisimple and $V^{\otimes m}$ is finite-dimensional, $U_q(\mathfrak{sl}_n)$ is isomorphic to its double commutant, which is the commutant of $H_q(m)$. For (b), $V^{\otimes m}$ decomposes into such a sum by (a), so it suffices to identify the multiplicity space of L_λ with S_λ; this holds because it does under the specialization $q \to 1$. □

Corollary 2.7. For $n > m$, the functor $\mathsf{FS}_q : \mathrm{Rep}(H_q(m)) \to \mathrm{Rep}(U_q(\mathfrak{sl}_n))$ defined by

$$\mathsf{FS}_q(W) = \mathrm{Hom}_{H_q(m)}(W, V^{\otimes m})$$

with $U_q(\mathfrak{sl}_n)$-module structure induced from $V^{\otimes m}$ is an equivalence of categories between $\mathrm{Rep}(H_q(m))$ and the subcategory of weight m representations of $U_q(\mathfrak{sl}_n)$.

Proof. From semisimplicity and the explicit decomposition of $V^{\otimes m}$ provided by Theorem 2.6(b). □

3. Yangians and degenerate affine Hecke algebras

3.1. **Yang-Baxter equation with spectral parameter and Yangian.** We extend the results of the previous section to the analogue of $U_q(\mathfrak{sl}_n)$ given by the solution to the Yang-Baxter equation with spectral parameter. This object is known as the Yangian $Y(\mathfrak{sl}_n)$, and it will be Schur-Weyl dual to the degenerate affine Hecke algebra Λ_m. We first introduce the Yang-Baxter equation with spectral parameter

$$R^{12}(u-v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u-v). \tag{2}$$

We may check that (2) has a solution in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ given by

$$R(u) = 1 - \frac{P}{u}.$$

This solution allows us to define the Yangian $Y(\mathfrak{gl}_n)$ via the RTT formalism.

Definition 3.1. The Yangian $Y(\mathfrak{gl}_n)$ is the Hopf algebra with generators $t_{ij}^{(k)}$ and defining relation

$$R^{12}(u-v)t^1(u)t^2(v) = t^2(v)t^1(u)R^{12}(u-v), \tag{3}$$

where $t(u) = \sum_{i,j} t_{ij}(u) \otimes E_{ij} \in Y(\mathfrak{sl}_n) \otimes \mathrm{End}(\mathbb{C}^n)$, $t_{ij}(u) = \delta_{ij}u^{-1} + \sum_{k\geq 1} t_{ij}^{(k)} u^{-k-1} \in Y(\mathfrak{sl}_n)[[u^{-1}]]$, the superscripts denote action in a tensor coordinate, and the relation should be interpreted in $Y(\mathfrak{sl}_n)((v^{-1}))[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$. The coalgebra structure is given by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u)$$

and the antipode by $S(t(u)) = t(u)^{-1}$.

Remark. There is an embedding of Hopf algebras $U(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)$ given by $t_{ij} \mapsto t_{ij}^{(0)}$.

Remark. Relation (3) is equivalent to the relations

$$[t_{ij}^{(r)}, t_{kl}^{(s-1)}] - [t_{ij}^{(r-1)}, t_{kl}^{(s)}] = t_{kj}^{(r-1)}t_{il}^{(s-1)} - t_{kj}^{(s-1)}t_{il}^{(r-1)} \tag{4}$$

for $1 \leq i, j, k, l \leq n$ and $r, s \geq 1$ (where $t_{ij}^{-1} = \delta_{ij}$). For $r = 0$ and $i = j = a$, this implies that

$$[t_{aa}^{(0)}, t_{kl}^{(s-1)}] = \delta_{ka}t_{al}^{(s-1)} - \delta_{al}t_{ka}^{(s-1)}, \tag{5}$$

meaning that $t_{ij}^{(k)}$ and $t_{ij}^{(0)}$ map between the same $U(\mathfrak{gl}_n)$-weight spaces.

Remark. For any a, the map $\mathrm{ev}_a : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ given by

$$\mathrm{ev}_a : t_{ij}(u) \mapsto 1 + \frac{E_{ij}}{u-a}$$

is an algebra homomorphism but not a Hopf algebra homomorphism. Pulling back $U(\mathfrak{gl}_n)$-representations through this map gives the *evaluation representations* of $Y(\mathfrak{gl}_n)$.

3.2. **The Yangian of $\mathfrak{sl}_n$.** For any formal power series $f(u) = 1 + f_1u^{-1} + f_2u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]$, the map

$$t(u) \mapsto f(u)t(u)$$

defines an automorphism μ_f of $Y(\mathfrak{gl}_n)$. One can check that the elements of $Y(\mathfrak{gl}_n)$ fixed under μ_f form a Hopf subalgebra.

Definition 3.2. The Yangian $Y(\mathfrak{sl}_n)$ of $\mathfrak{sl}_n$ is $Y(\mathfrak{sl}_n) = \{x \in Y(\mathfrak{gl}_n) \mid \mu_f(x) = x\}$.

We may realize $Y(\mathfrak{sl}_n)$ as a quotient of $Y(\mathfrak{gl}_n)$. Define the quantum determinant of $Y(\mathfrak{gl}_n)$ by

$$\mathrm{qdet}\, t(u) = \sum_{\sigma\in S_n} (-1)^\sigma t_{\sigma(1),1}(u)t_{\sigma(2),2}(u-1)\cdots t_{\sigma(n),n}(u-n+1) \tag{6}$$

Proposition 3.3. We have the following:

(a) the coefficients of $\mathrm{qdet}\, t(u)$ generate $Z(Y(\mathfrak{gl}_n))$;
(b) $Y(\mathfrak{gl}_n)$ admits the tensor decomposition $Z(Y(\mathfrak{gl}_n)) \otimes Y(\mathfrak{sl}_n)$;
(c) $Y(\mathfrak{sl}_n) = Y(\mathfrak{gl}_n)/(\mathrm{qdet}\, t(u) - 1)$.

Observe that any representation of $Y(\mathfrak{gl}_n)$ pulls back to a representation of $Y(\mathfrak{sl}_n)$ under the embedding $Y(\mathfrak{sl}_n) \to Y(\mathfrak{gl}_n)$. Further, the image of $U(\mathfrak{sl}_n)$ under the previous embedding $U(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)$ lies in $Y(\mathfrak{sl}_n)$, so we may consider any $Y(\mathfrak{sl}_n)$-representation as a $U(\mathfrak{sl}_n)$-representation. We say that a representation of $Y(\mathfrak{sl}_n)$ is of weight m if it is of weight m as a representation of $U(\mathfrak{sl}_n)$.

3.3. **Degenerate affine Hecke algebra.** The Yangian will be Schur-Weyl dual to the degenerate affine Hecke algebra Λ_m, which may be viewed as a $q \to 1$ limit of the affine Hecke algebra.

Definition 3.4. The degenerate affine Hecke algebra Λ_m is the associative algebra given by

$$\Lambda_m = \Big\langle s_1, \ldots, s_{m-1}, x_1, \ldots, x_m \,|s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, [x_i, x_j] = 0,$$
$$s_i x_i - x_{i+1} s_i = 1, [s_i, s_j] = [s_i, x_j] = 0 \text{ if } |i-j| \neq 1 \Big\rangle.$$

Remark. We have the following facts about Λ_m:

- s_i and x_i generate copies of $\mathbb{C}[S_m]$ and $\mathbb{C}[x_1, \ldots, x_m]$ inside Λ_m;
- the center of Λ_m is $\mathbb{C}[x_1, \ldots, x_m]^{S_m}$;
- the elements $y_i = x_i - \sum_{j<i} s_{ij}$ in Λ_m give an alternate presentation via

$$\Lambda = \Big\langle s_1, \ldots, s_{m-1}, y_1, \ldots, y_m \mid s y_i = y_{s(i)} s, [y_i, y_j] = (y_i - y_j) s_{ij} \Big\rangle.$$

3.4. **The Drinfeld functor.** We now upgrade FS to a functor between $\mathrm{Rep}(\Lambda_m)$ and $\mathrm{Rep}(Y(\mathfrak{sl}_n))$. For a Λ_m-representation W, define the linear map $\rho_W : Y(\mathfrak{gl}_n) \to \mathrm{End}(\mathsf{FS}(W))$ by

$$\rho_W : t(u) \mapsto T^{1,\star}(u - x_1) T^{2,\star}(u - x_2) \cdots T^{m,\star}(u - x_m),$$

where

$$T(u - x_l) = 1 + \frac{1}{u - x_l} \sum_{ab} E_{ab} \otimes E_{ab} \in \mathrm{End}(W \otimes V \otimes V)$$

should be thought of as the image of the evaluation map $\mathrm{ev}_a : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ given by $t_{ij}(u) \mapsto 1 + \frac{E_{ij}}{u-a}$ at "$a = x_l$".

Proposition 3.5. The map ρ_W gives a representation of $Y(\mathfrak{gl}_n)$ on $\mathsf{FS}(W)$.

Proof. Define $S = \sum_{ab} E_{ab} \otimes E_{ab}$. We first check the image of ρ_W lies in $\mathrm{Hom}_{S_m}(W, V^{\otimes m})$. For any $f : W \to V^{\otimes m}$, we must check that $\rho_W(f)(s_i w) = P^{i,i+1} \rho_W(f)(w)$. Because all coefficients of $\prod_l (u - x_l)$ are central in Λ_m, it suffices to check this for

$$\widetilde{\rho}_W : t(u) \mapsto \prod_l (u - x_l) \rho_W(t(u)) = \prod_l (u - x_l + S^{l,\star}).$$

Notice that $(u - x_j + S^{j,\star})$ commutes with the action of s_i and $P^{i,i+1}$ unless $j = i, i+1$, so it suffices to check that

$$(u - x_i + S^{i,\star})(u - x_{i+1} + S^{i+1,\star}) f(s_i w) = P^{i,i+1} (u - x_i + S^{i,\star})(u - x_{i+1} + S^{i+1,\star}) f(w).$$

We compute the first term as

$$(u - x_i + S^{i,\star})(u - x_{i+1} + S^{i+1,\star}) f(s_i w)$$
$$= (u + S^{i,\star})(u + S^{i+1,\star}) f(s_i w) - (u + S^{i+1,\star}) f(x_i s_i w) - (u + S^{i,\star}) f(x_{i+1} s_i w) + f(x_i x_{i+1} s_i w).$$

Now notice that

$$(u + S^{i,\star})(u + S^{i+1,\star}) f(s_i w) = (u + S^{i,\star})(u + S^{i+1,\star}) P^{i,i+1} f(w)$$
$$= P^{i,i+1} (u + S^{i,\star})(u + S^{i+1,\star}) f(w) + P^{i,i+1} [S^{i+1,\star}, S^{i,\star}] f(w)$$

and

$$-(u + S^{i+1,\star}) f(x_i s_i w) = -(u + S^{i+1,\star}) f((s_i x_{i+1} + 1) w)$$
$$= -P^{i,i+1} (u + S^{i,\star}) f(x_{i+1} w) - (u + S^{i+1,\star}) f(w)$$

and

$$-(u+S^{i,\star})f(x_{i+1}s_iw) = -(u+S^{i,\star})f((s_ix_i-1)w)$$
$$= -P^{i,i+1}(u+S^{i+1,\star})f(x_iw)+(u+S^{i,\star})f(w)$$

and

$$f(x_ix_{i+1}s_iw) = P^{i,i+1}f(x_ix_{i+1}w).$$

Putting these together, we find that

$$(u-x_i+S^{i,\star})(u-x_{i+1}+S^{i+1,\star})f(s_iw) = P^{i,i+1}(u-x_i+S^{i,\star})(u-x_{i+1}+S^{i+1,\star})f(w)$$
$$+\left(P^{i,i+1}[S^{i+1,\star},S^{i,\star}]+S^{i,\star}-S^{i+1,\star}\right)f(w).$$

We may check in coordinates that $[S^{i,\star},S^{i+1,\star}] = [P^{i,i+1},S^{i,\star}]$ so that

$$P^{i,i+1}[S^{i+1,\star},S^{i,\star}] = P^{i,i+1}S^{i,\star}P^{i,i+1}-S^{i,\star} = S^{i+1,\star}-S^{i,\star},$$

which yields the desired. To check that ρ_W is a valid $Y(\mathfrak{gl}_n)$-representation, we note that the x_l form a commutative subalgebra of Λ_m, hence the same proof that ev_a is a valid map of algebras shows that ρ_W is a representation, since the action of the x_i commutes with the action of $U(\mathfrak{gl}_n)$. □

Lemma 3.6. We may reformulate the action of $Y(\mathfrak{gl}_n)$ on $\mathrm{End}(\mathsf{FS}(W))$ via the equality

$$\rho_W(t(u)) = 1+\sum_{l=1}^m \frac{1}{u-y_l}S^{l,\star}$$

In particular, in terms of the generators y_l, we have

$$\rho_W(t_{ij}^{(k)}) = \delta_{ij}+\sum_{l=1}^m y_l^kE_{ji}^l.$$

Proof. We claim by induction on k that

$$\prod_{l=1}^k T^{l,\star}(u-x_l) = 1+\sum_{l=1}^k \frac{1}{u-y_l}S^{l,\star}.$$

The base case $k=1$ is trivial. For the inductive step, noting that $S^{l,\star}S^{k+1,\star} = P^{l,k+1}S^{k+1,\star}$, we have

$$\left(1+\sum_{l=1}^k\frac{1}{u-y_l}S^{l,\star}\right)\left(1+\frac{S^{k+1,\star}}{u-x_{k+1}}\right) = 1+\sum_{l=1}^k\frac{1}{u-y_l}S^{l,\star}+\frac{1}{u-x_{k+1}}\left(1+\sum_{l=1}^k\frac{1}{u-y_l}S^{l,\star}\right)S^{k+1,\star}$$
$$= 1+\sum_{l=1}^k\frac{1}{u-y_l}S^{l,\star}+\frac{1}{u-x_{k+1}}\left(1+\sum_{l=1}^k\frac{1}{u-y_l}P^{l,k+1}\right)S^{k+1,\star}$$
$$= 1+\sum_{l=1}^k\frac{1}{u-y_l}S^{l,\star}+\frac{1}{u-x_{k+1}}\left(1+\sum_{l=1}^kP^{l,k+1}\frac{1}{u-y_{k+1}}\right)S^{k+1,\star}$$
$$= 1+\sum_{l=1}^{k+1}\frac{1}{u-y_l}S^{l,\star}.$$

□

3.5. **Schur-Weyl duality for Yangians.** The upgraded functor FS is known as the *Drinfeld functor*, and an analogue of Theorem 1.5 holds for it.

Theorem 3.7. For $n>m$, the functor $\mathsf{FS}:\mathrm{Rep}(\Lambda_m)\to\mathrm{Rep}(Y(\mathfrak{sl}_n))$ is an equivalence of categories onto the subcategory of $\mathrm{Rep}(Y(\mathfrak{sl}_n))$ generated by representations of weight m.

Proof. We first show essential surjectivity. Viewing any representation W' of $Y(\mathfrak{sl}_n)$ of weight m as a representation of $U(\mathfrak{sl}_n)$, we have by Theorem 1.5 that $W' = \mathsf{FS}(W)$ for some S_m-representation W. We must now extend the S_m-action to an action of Λ_m by defining the action of the y_l. For this, we use that W' is also a representation of $Y(\mathfrak{gl}_n)$ via the quotient map $Y(\mathfrak{gl}_n)\to Y(\mathfrak{sl}_n)$.

Lemma 3.8. We have the following:

(a) if $v \in V^{\otimes m}$ is a vector with non-zero component in each isotypic component of $V^{\otimes m}$ viewed as a $U(\mathfrak{sl}_n)$-representation, the linear map $W \to \mathsf{FS}(W)$ given by $w \mapsto v \cdot w^*$ is injective, where $w^* \in W^*$ is the image of w under the canonical isomorphism $W \simeq W^*$;

(b) if $e_1, \ldots, e_n$ is the standard basis for V, then $v = e_{i_1} \otimes \cdots \otimes e_{i_m} \in V^{\otimes m}$ is such a vector for $i_1, \ldots, i_m$ distinct.

Proof. Theorem 1.3 and reduction to isotypic components of W gives (a), and (b) follows because v is a cyclic vector for $U(\mathfrak{sl}_n)$ in $V^{\otimes m}$. □

Define the special vectors

$$v^{(j)} = e_2 \otimes \cdots \otimes e_j \otimes e_n \otimes e_{j+1} \cdots \otimes e_m \text{ and } w^{(j)} = e_2 \otimes \cdots \otimes e_j \otimes e_1 \otimes e_{j+1} \cdots \otimes e_m.$$

For $w \in W$, the action of $t_{1n}^{(1)}$ on $v^{(j)} \cdot w^*$ lies in $w^{(j)} \cdot W^*$ by $U(\mathfrak{sl}_n)$-weight considerations via (5). By Lemma 3.8, we may define linear maps $\alpha_j \in \mathrm{End}_{\mathbb{C}}(W)$ by

$$t_{1n}^{(1)}(v^{(j)} \cdot w^*) = w^{(j)} \cdot \alpha_j(w)^*.$$

Similarly, we may define maps $\beta_j, \gamma_j \in \mathrm{End}_{\mathbb{C}}(W)$ so that

$$t_{11}^{(1)}(w^{(j)} \cdot w^*) = w^{(j)} \cdot \beta_j(w)^*$$

and

$$t_{1n}^{(2)}(v^{(j)} \cdot w^*) = w^{(j)} \cdot \gamma_j(w)^*.$$

Evaluate the relation $[t_{1n}^{(1)}, t_{11}^{(0)}] - [t_{1n}^{(0)}, t_{11}^{(1)}] = 0$ on $v^{(j)} \cdot w^*$ to find that $\alpha_j(w) - \beta_j(w) = 0$. Now, combining the relations

$$-[t_{1n}^{(2)}, t_{11}^{(0)}] = t_{1n}^{(2)} \text{ and } [t_{1n}^{(2)}, t_{11}^{(0)}] - [t_{1n}^{(1)}, t_{11}^{(1)}] = t_{1n}^{(1)} t_{11}^{(0)} - t_{1n}^{(0)} t_{11}^{(1)},$$

we find that

$$-t_{1n}^{(2)} - [t_{1n}^{(1)}, t_{11}^{(1)}] = t_{1n}^{(1)} t_{11}^{(0)} - t_{1n}^{(0)} t_{11}^{(1)}.$$

Evaluating this on $v^{(j)} \cdot w^*$ implies that $-\gamma_j(w) + \alpha_j^2(w) = 0$.

Lemma 3.9. The formulas for the action of the following Yangian elements

$$t_{1n}^{(1)} = \sum_l \alpha_l E_{1n}^{(l)}, \qquad t_{11}^{(1)} = \sum_l \alpha_l E_{11}^{(l)}, \qquad t_{1n}^{(2)} = \sum_l \alpha_l^2 E_{1n}^{(l)}$$

hold on all of $\mathsf{FS}(W)$.

Proof. For $t_{1n}^{(1)}$, because $t_{ij}^{(0)}$ commutes with $t_{1n}^{(1)}$ for $i, j \notin \{1, n\}$, it suffices by Lemma 3.8(b) to verify the claim on basis vectors $v \in V$ containing $e_2, \ldots, e_{n-1}$ at most once as tensor factors. In fact, for each configuration of e_1's and e_n's which occur, it suffices to verify the claim for a single such basis vector. Similar claims hold for $t_{11}^{(1)}$ and basis vectors containing $e_2, \ldots, e_n$ at most once. Call basis vectors containing r copies of e_1 and s of copies of e_n vectors of type (r, s).

The claim holds for $t_{11}^{(1)}$ for $(0, \star)$ trivially and for $(1, \star)$ because it holds for $w^{(j)}$. Now, we have $[t_{11}^{(1)}, t_{1n}^{(0)}] = t_{1n}^{(1)}$, so this implies that the claim holds for $t_{1n}^{(1)}$ for $(0, \star)$. Now, observe that $[t_{1n}^{(1)}, t_{12}^{(0)}] = 0$, so replacing any v of type (r, s) which does not contain e_2 with v' which has e_2 instead of e_1 in a single tensor coordinate yields

$$t_{1n}^{(1)} v = t_{1n}^{(1)} t_{12}^{(0)} v' = t_{12}^{(0)} t_{1n}^{(1)} v',$$

whence the claim holds for $t_{1n}^{(1)}$ on v if it holds for v'. Induction on r yields the claim for all $t_{1n}^{(1)}$. Now, for $t_{11}^{(1)}$, suppose the claim holds for type $(r-1, 0)$, and choose a v of type $(r, 0)$ with e_1 in coordinates $i_1, \ldots, i_r$, and let v' be the vector containing e_n instead of e_1 in the single tensor coordinate i_r. Then we have $v = t_{1n}^{(0)} v'$, so

$$t_{11}^{(1)} v = t_{11}^{(1)} t_{1n}^{(0)} v' = t_{1n}^{(0)} t_{11}^{(1)} v' + [t_{11}^{(0)}, t_{1n}^{(1)}] v' = t_{1n}^{(0)} \sum_{j=1}^{r-1} \alpha_{i_j} E_{11}^{(i_j)} v' + \alpha_{i_r} v = \Big(\sum_{j=1}^{r-1} \alpha_{i_j} + \alpha_{i_r}\Big) v,$$

which yields the claim for $t_{11}^{(1)}$ by induction on r. The claim for $t_{1n}^{(2)}$ follows from the relation

$$t_{1n}^{(2)} = t_{1n}^{(0)} t_{11}^{(1)} - t_{1n}^{(1)} t_{11}^{(0)} - [t_{1n}^{(1)}, t_{11}^{(1)}].$$

□

To conclude, we claim that the assignment $y_l \mapsto \alpha_l$ extends the S_m-action on $\mathsf{FS}(W)$ to a Λ_m-action. For this, we evaluate relations from $Y(\mathfrak{sl}_n)$ on carefully chosen vectors in $\mathsf{FS}(W)$. To check that $s_i y_i = y_{i+1} s_i$, note that $v^{(i)} \cdot w^* = v^{(i+1)} \cdot (s_i w)^*$, so acting by $t_{1n}^{(1)}$ on both sides gives the desired

$$s_i w^{(i+1)} \cdot \alpha_i(w)^* = w^{(i)} \cdot \alpha_i(w)^* = w^{(i+1)} \cdot \alpha_{i+1}(s_i(w))^*.$$

For the second relation, we evaluate

$$-t_{1n}^{(2)} - [t_{1n}^{(1)}, t_{11}^{(1)}] = t_{1n}^{(1)} t_{11}^{(0)} - t_{1n}^{(0)} t_{11}^{(1)}$$

on

$$\begin{aligned} &e_2 \otimes \cdots \otimes e_i \otimes e_n \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_1 \otimes e_j \otimes \cdots \otimes e_m \cdot w^* \\ &\qquad\qquad = e_2 \otimes \cdots \otimes e_i \otimes e_1 \otimes e_{i+1} \otimes \cdots \otimes e_{j-1} \otimes e_n \otimes e_j \otimes \cdots \otimes e_m \cdot (s_{ij} w)^*, \end{aligned}$$

we find that

$$-(\alpha_j - \alpha_i) s_{ij} w = \alpha_i(\alpha_j(w)) - \alpha_j(\alpha_i(w)),$$

which shows that $[\alpha_i, \alpha_j] = (\alpha_i - \alpha_j) s_{ij}$.

It remains to show that FS is fully faithful. Injectivity on morphisms follows because FS is fully faithful in the classical case. For surjectivity, any map $F : \mathsf{FS}(W) \to \mathsf{FS}(W')$ of $Y(\mathfrak{sl}_n)$-modules is of the form $F = \mathsf{FS}(f)$ for a map $f : W \to W'$ of S_m-modules. Further, viewing W and W' as $Y(\mathfrak{gl}_n)$-modules via the quotient map, F commutes with the full $Y(\mathfrak{gl}_n)$-action because the center acts trivially on both W and W'. Now, because F commutes with the action of $t_{1n}^{(1)}$, we see for all $w \in W$ and $v \in V^{\otimes m}$ that

$$\sum_{l=1}^{m} E_{1n}^{(l)} v \cdot f(y_l w)^* = \sum_{l=1}^{m} E_{1n}^{(l)} v \cdot (y_l f(w))^*.$$

Taking $v = w^{(j)}$ shows that $f(y_j w) = y_j f(w)$, so that f is a map of Λ_m-modules, as needed. □

4. Quantum affine algebras and affine Hecke algebras

4.1. **Definition of the objects.** Our goal in this section will be to extend Corollary 2.7 to the case of $U_q(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{H}_q(m)$. We first define these objects.

Definition 4.1. The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ is the quantum group of the Kac-Moody algebra associated to type $A_{n-1}^{(1)}$, meaning that the Cartan matrix A is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Remark. The obvious embedding $x_i^{\pm} \mapsto x_i^{\pm}$ and $q^{h_i/2} \mapsto q^{h_i/2}$ realizes $U_q(\mathfrak{sl}_n)$ as a Hopf subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$. We say that a $U_q(\widehat{\mathfrak{sl}}_n)$-representation is of weight m if it is of weight m as a $U_q(\mathfrak{sl}_n)$-representation.

Definition 4.2. The affine Hecke algebra $\mathcal{H}_q(m)$ is the associative algebra given by

$$\begin{aligned} \mathcal{H}_q(m) = \Big\langle T_1^{\pm}, \ldots, T_{m-1}^{\pm}, X_1^{\pm}, \ldots, X_m^{\pm} \mid [X_i, X_j] = 0, (T_i - q^{-1})(T_i + q) = 0, T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i X_i T_i = q^2 X_{i+1}, [T_i, T_j] = [T_i, X_j] = 0 \text{ for } |i - j| \neq 1 \Big\rangle. \end{aligned}$$

4.2. **Drinfeld functor and Schur-Weyl duality.** We now give an extension of Corollary 2.7 to the affine setting. The strategy is the analogue of the one we took for Yangians. For variety, we present a construction directly in the Kac-Moody presentation in this case. For a $\mathcal{H}_q(m)$-representation W, define the linear map $\rho_{q,W} : U_q(\widehat{\mathfrak{sl}}_n) \to \mathrm{End}(\mathsf{FS}_q(W))$ by

$$\rho_{q,W}(x_0^{\pm}) = \sum_{l=1}^{m} X_l^{\pm} \otimes (q^{\mp h_\theta/2})^{\otimes l-1} \otimes x_\theta^{\mp} \otimes (q^{\mp h_\theta/2})^{\otimes m-l}, \text{ and}$$
$$\rho_{q,W}(q^{h_0}) = 1 \otimes (q^{-h_\theta})^{\otimes m},$$

where $x_\theta^+ = E_{1n}$ and $x_\theta^- = E_{n1}$ as operators in $\mathrm{End}(V)$, and $q^{h_\theta} = q^{h_1 + \cdots + h_{n-1}}$.

Theorem 4.3. The map $\rho_{q,W}$ defines a representation of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathsf{FS}_q(W)$.

Proof. By a direct computation of the relations of $U_q(\widehat{\mathfrak{sl}}_n)$. For details, the reader may consult [CP96, Theorem 4.2]; note that the coproduct used there differs from our convention, which follows [Jim86]. □

Theorem 4.4. For $n > m$, the functor $\mathsf{FS}_q : \mathrm{Rep}(\mathcal{H}_q(m)) \to \mathrm{Rep}(U_q(\widehat{\mathfrak{sl}}_n))$ is an equivalence of categories onto the subcategory of $\mathrm{Rep}(U_q(\widehat{\mathfrak{sl}}_n))$ generated by representations of weight m.

Proof. The proof of essential surjectivity is analogous to that of Theorem 3.7. The action of $X_i^{\pm}$ is obtained by evaluation on some special basis vectors in $V^{\otimes m}$ and the relations of $\mathcal{H}_q(m)$ are shown to be satisfied for them from the relations of $U_q(\widehat{\mathfrak{sl}}_n)$. For details, see [CP96, Sections 4.4-4.6]. The check that FS_q is fully faithful is again essentially the same as in Theorem 3.7. □

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E-mail address: `yisun@math.mit.edu`