Wakimoto modules

§0. Setup

$\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$

$\cap$

$\mathfrak{g}$ = simple Lie alg. , κ = $\mathfrak{g}$-inv. bilinear form $\rightsquigarrow \hat{\mathfrak{g}}_\kappa$

$U_\kappa(\mathfrak{g}) = U(\hat{\mathfrak{g}}_\kappa)/(\mathbb{1}-1)$

$\tilde{U}_\kappa(\mathfrak{g}) \circlearrowright V_\kappa(\mathfrak{g})$

$\kappa_\mathfrak{g}$ = Killing form , $\kappa_c = -\frac{1}{2}\kappa_\mathfrak{g}$

$\mathfrak{g} = \bigoplus_a \mathbb{C}\cdot J_a$, $\{J_a\}$ is a weighted basis of $\mathfrak{g}$

$$G \circlearrowright G/N_- \supset B_+ \circlearrowleft H \rightsquigarrow \mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \mathrm{Vect}(N_+) \oplus \mathbb{C}[N_+] \otimes \mathfrak{h}$$
$$\simeq \mathrm{Vect}(\mathfrak{n}_+) \oplus \mathbb{C}[\mathfrak{n}_+] \otimes \mathfrak{h}$$
$$\simeq \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{n}_+ \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{h}$$

differential operators

$$\rightsquigarrow U(\mathfrak{g}) \longrightarrow \mathcal{D}(B_+)^H = \mathcal{D}(N_+) \otimes U(\mathfrak{h})$$

$\big\downarrow$ affine analogue

(★) Thm 6.2.1 $\exists$ map of $\mathbb{Z}$-graded VA (satisfying some conditions)

$$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_\mathfrak{g} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})$$

$\underset{\text{enveloping alg}}{\overset{\text{universal}}{\rightsquigarrow}}$ $\tilde{U}_\kappa(\mathfrak{g}) \longrightarrow \tilde{\mathcal{A}}^\mathfrak{g} \hat{\otimes} \tilde{U}_{\kappa-\kappa_c}(\mathfrak{h})$

central ext'n

Def of $M_\mathfrak{g}$ $\quad \hat{\Gamma} \twoheadrightarrow \Gamma = \mathfrak{n}_+((t)) \oplus \mathfrak{n}_+^*((t))\,dt$

$[xf, yw] = \langle x, y\rangle \mathrm{Res}\, fw \cdot \mathbb{1}$

$\cup$

$\Gamma_+ \quad = \mathfrak{n}_+[[t]] \oplus \mathfrak{n}_+^*[[t]]\,dt$

$$\rightsquigarrow \tilde{\mathcal{A}}^\mathfrak{g} = \widetilde{U(\hat{\Gamma})/(\mathbb{1}-1)} \circlearrowright M_\mathfrak{g} = \mathrm{Ind}_{\Gamma_+ \oplus \mathbb{C}\mathbb{1}}^{\hat{\Gamma}} \mathbb{C}|0\rangle$$

$\Gamma_+ \cdot |0\rangle = 0$

$\mathbb{1} \cdot |0\rangle = |0\rangle$

central ext'n given by sym bil. form ν

Def of $V_\nu(\mathfrak{h})$ $\quad \hat{\mathfrak{h}}_\nu \twoheadrightarrow \mathfrak{h}((t))$

$[xf, yg] = -\nu(X, Y)\,\mathrm{Res}\, f dg \cdot \mathbb{1}$

$\cup$

$\mathfrak{h}[[t]]$

$$\rightsquigarrow \tilde{U}_\nu(\mathfrak{h}) = \widetilde{U(\hat{\mathfrak{h}}_\nu)/(\mathbb{1}-1)} \circlearrowright V_\nu(\mathfrak{h}) = \mathrm{Ind}_{\mathfrak{h}[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}_\nu} \mathbb{C}|0\rangle$$

$$\lambda \in \mathfrak{h}^* \rightsquigarrow \pi_\nu^\lambda := \mathrm{Ind}_{\mathfrak{h}[[t]] \oplus \mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}_\nu} \mathbb{C}|\lambda\rangle \in \mathrm{Mod}_{\tilde{U}_\nu(\mathfrak{h})}$$

$b \otimes t^n |\lambda\rangle = \delta_{n,0}\,\lambda(b)\,|\lambda\rangle \quad (b \in \mathfrak{h})$

$\mathbb{1}|\lambda\rangle = |\lambda\rangle$

$$\rightsquigarrow \tilde{U}_\kappa(\mathfrak{g}) \longrightarrow \tilde{\mathcal{A}}^\mathfrak{g} \hat{\otimes} \tilde{U}_{\kappa-\kappa_c}(\mathfrak{h}) \circlearrowright M_\mathfrak{g} \otimes \pi_{\kappa-\kappa_c}^\lambda =: W_{\lambda,\kappa} \in \mathrm{Mod}_{\tilde{U}_\kappa(\mathfrak{g})}$$

this is called Wakimoto module of level κ, highest wt λ.

They will play essential role in the proof of FF center thm.

Example

$\mathfrak{g} = \mathfrak{sl}_2$, $\kappa_0(X, Y) = \mathrm{tr}(XY)$

$\kappa_\mathfrak{g} = 4\kappa_0$, $\kappa_c = -2\kappa_0$, $\kappa = k \cdot \kappa_0$

$\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f \quad \kappa_0(e, f) = 1, \ \kappa_0(h, h) = 2$

copy of e $\qquad$ copy of h

$\mathfrak{n}_+ = \mathbb{C}\cdot a$, $\mathfrak{n}_+^* = \mathbb{C}a^*$, $\mathfrak{h} = \mathbb{C}\cdot b$

$\rightsquigarrow N_+ = \mathrm{Spec}\,\mathbb{C}[a^*]$, $a = \frac{\partial}{\partial a^*} \in \mathrm{Vect}(N_+)$

$$\mathfrak{sl}_2 \longrightarrow \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{n}_+ \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \mathfrak{h} \simeq \mathrm{Vect}(B_+)^H$$
$$e \longmapsto a$$
$$h \longmapsto -2a^*a + b$$
$$f \longmapsto -a^{*2}a + a^*b$$

$([a, a^*] = 1)$

$\hat{\Gamma} \twoheadrightarrow \Gamma = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot a_n \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{C}a_m^* \qquad [a_n, a_m^*] = \delta_{n,-m}\mathbb{1}$

$M_\mathfrak{g} = \mathcal{A}^\mathfrak{g}.|0\rangle$ where $a_n|0\rangle = 0 \quad n \geq 0$

$a_m^*|0\rangle = 0 \quad n \geq 1$

annihilating operators

$\deg a_n = \deg a_n^* = -n$

$[T, a_n] = -n a_{n-1} \quad [T, a_n^*] = -(n-1)a_{n-1}^*$

$Y(a_{-1}|0\rangle, z) = \sum a_n z^{-n-1} =: a(z)$

$Y(a_0^*|0\rangle, z) = \sum a_m^* z^{-m} =: a^*(z)$

$:$ monomial in a_n, a_m^* $:$ = move annihilating operators to the right

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$\hat{\mathfrak{h}}_{k+2} \twoheadrightarrow \mathfrak{h}((t)) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}b_n$

$[b_n, b_m] = 2(k+2)n\,\delta_{n,-m} \cdot \mathbb{1}$

$V_{k+2}(\mathfrak{h}) = U_{k+2}(\mathfrak{h}) \cdot |0\rangle$ where $b_n|0\rangle = 0 \quad n \geq 0$

$Y(b_{-1}|0\rangle, z) = \sum b_n z^{-n-1} =: b(z)$

$[T, b_n] = -n b_{n-1}$

When $\kappa = \kappa_c$, $\widetilde{U}_0(\mathfrak{h}) = \mathrm{Fun}(\mathfrak{h}^*((t)))$

$$w_{\kappa_c} : \widetilde{U}_{\kappa_c}(\mathfrak{g}) \longrightarrow \widetilde{\mathcal{A}}^{\mathfrak{g}} \widehat{\otimes} \mathrm{Fun}(\mathfrak{h}^*((t)))$$

$$\chi(t) \in \mathfrak{h}^*((t)) \rightsquigarrow \widetilde{U}_{\kappa_c}(\mathfrak{g}) \longrightarrow \widetilde{\mathcal{A}}^{\mathfrak{g}} \circlearrowright M_{\mathfrak{g}} =: W_{\chi(t)} \in \mathrm{Mod}_{\widetilde{U}_{\kappa_c}(\mathfrak{g})}$$

called <u>Wakimoto module of critical level</u>

<u>Rmk</u> · $W_{\chi(t)}, W_{\lambda,\kappa} \in \mathcal{O}$

$$\mathcal{O} = \{ V \in \underset{\substack{\text{affine Kac-Moody alg.} \\ \| \\ (\mathbb{C}\mathbb{1} \oplus \mathfrak{g}((t))) \rtimes \mathfrak{g}_{m,rot}}}{\hat{\mathfrak{g}}}\text{-mod} \mid \cdot\ V = \bigoplus_{\lambda \in \underset{\text{Cartan of } \hat{\mathfrak{g}}}{\hat{\mathfrak{h}}^*}} V_\lambda \text{ weight space decomp}, \dim V_\lambda < \infty$$

$$\cdot\ \exists\ \lambda_1, \dots, \lambda_n \in \hat{\mathfrak{h}}^* \text{ s.t. all weights} \in \bigcup_{i=1}^{n} (\lambda_i - \mathbb{Z}_{\geq 0} \underset{\text{positive affine roots}}{\hat{\Phi}_+}) \}$$

· $W_{\chi(t)}$ are simple for some $\chi(t)$

§1. How to construct $V_\kappa(\mathfrak{g}) \longrightarrow V$?

<u>Lem 6.1.1</u>

Let V = $\mathbb{Z}$-graded vertex algebra , the following data are in bijection

- A $\mathbb{Z}$-graded vertex alg. hom. $V_\kappa(\mathfrak{g}) \longrightarrow V$
- $x_a \in V$, $a = 1, \dots, \dim\mathfrak{g}$, $\deg x_a = 1$ s.t.

$$\begin{tikzcd} w \arrow[d, mapsto] \\ x_a = w(J_{a,-1}|0\rangle) \end{tikzcd}$$

$\hat{\mathfrak{g}}_\kappa \longrightarrow \mathrm{End}(V)$ defines a Lie algebra homomorphism

$J_{a,n} \longmapsto x_{a(n)}$

$\mathbb{1} \longmapsto \mathrm{id}$

<u>Proof</u> only "$\Leftarrow$" needs a proof

$$Y(x_a, z)|0\rangle \in V[[z]] \Rightarrow x_{a(n)}|0\rangle = 0 \text{ for } n \geq 0$$

$$\overset{\substack{\text{universal} \\ \text{property} \\ \text{of induced} \\ \text{module}}}{\rightsquigarrow} V_\kappa(\mathfrak{g}) \longrightarrow V \quad \text{linear map}$$

$$J_{a_1,n_1} \cdots J_{a_m,n_m}|0\rangle \longmapsto x_{a_1(n_1)} \cdots x_{a_m(n_m)}|0\rangle$$

<u>Ex</u> Check this is a map of VA □

§2. (☆) for $\mathfrak{sl}_2$

Thm 6.2.1 for $\widehat{\mathfrak{sl}_2}$

$\exists$ $W_k : V_k(\mathfrak{sl}_2) \longrightarrow M_{\mathfrak{sl}_2} \otimes V_{k+2}(\mathfrak{h})$ map of VA s.t.

$$e_{-1}|0\rangle \longmapsto a_{-1}|0\rangle \quad =: \tilde{e}_{-1}|0\rangle \qquad \text{deg } 1,\ \underbrace{\text{wt } 2}_{\text{w.r.t. } h}$$

$$h_{-1}|0\rangle \longmapsto (-2a_0^* a_{-1} + b_{-1})|0\rangle \quad =: \tilde{h}_{-1}|0\rangle \qquad \text{deg } 1,\ \text{wt } 0$$

$$f_{-1}|0\rangle \longmapsto (-a_0^{*2} a_{-1} + a_0^* b_{-1} + k a_{-1}^*)|0\rangle \quad =: \tilde{f}_{-1}|0\rangle \quad \text{deg } 1,\ \text{wt } -2$$

Compare

$$\mathfrak{sl}_2 \longrightarrow \overset{\overset{a^*}{\rotatebox{90}{$\in$}}}{\mathrm{Sym}\,\mathfrak{n}_+^*} \otimes \overset{\overset{a}{\rotatebox{90}{$\in$}}}{\mathfrak{n}_+} \oplus \mathrm{Sym}\,\mathfrak{n}_+^* \otimes \overset{\overset{b}{\rotatebox{90}{$\in$}}}{\mathfrak{h}}$$

$$e \longmapsto a$$

$$h \longmapsto -2a^* a + b$$

$$f \longmapsto -a^{*2} a + a^* b$$

Rmk finite dim'l formulas + deg + wt pin down RHS

Proof Denote $Y(e_{-1}|0\rangle, z) = e(z)$, $Y(\tilde{e}_{-1}|0\rangle, z) = \tilde{e}(z)$

Use thm 6.1.1, sufficies to check commutator relations of $\tilde{e}_n, \tilde{h}_n, \tilde{f}_n$, hence sufficies to check

W_k preserves OPE for $e(z)\cdot f(w)$, $h(z)\cdot f(w)$, $h(z)\cdot e(w)$

Proof for "e.f"

$$\tilde{e}(z)\cdot\tilde{f}(w) \sim \sum_{n\geq 0} \frac{Y(\tilde{e}_{(n)}\tilde{f}_{-1}|0\rangle, w)}{(z-w)^{n+1}}$$

$$= \sum_{n\geq 0} \frac{Y(a_n\cdot(-a_0^{*2}a_{-1} + a_0^* b_{-1} + k a_{-1}^*)|0\rangle, w)}{(z-w)^{n+1}} \quad \in \mathbb{C}[z,w]((\tfrac{z}{w}))$$

only $n = 0, 1$ get non-zero terms

$$= \frac{Y((-2a_0^* a_{-1} + b_{-1})|0\rangle, w)}{z-w} + \frac{k}{(z-w)^2}$$

$$= \frac{\tilde{h}(w)}{z-w} + \frac{k}{(z-w)^2}$$

$$e(z)\cdot f(w) \sim \frac{h(w)}{z-w} + \frac{k}{(z-w)^2}$$

Ex Do the same for $\underbrace{h\cdot f}_{\text{more work}}$, $\underbrace{h\cdot e}_{\text{easy}}$ ☐

§3. Conformal structures in sl_2-case

Assume $k \neq -2$

normalized Sugawara element

Recall $S_k = \frac{1}{2(k+2)}(e_{-1}f_{-1} + f_{-1}e_{-1} + \frac{1}{2}h^2_{-1})|0\rangle \in V_k(sl_2)$ is a conformal vector, $S_k(z) := Y(S_k, z)$

w/ central charge $c_k = \frac{3k}{k+2}$ i.e. $S_k(z)S_k(w) = \frac{c_k/2}{(z-w)^4} + O(\frac{1}{(z-w)^3})$

Prop 6.2.2 $w_k : V_k(sl_2) \longrightarrow M_{sl_2} \otimes V_{k+2}(\mathfrak{h})$ satisfies

$$w_k(S_k) = (a_{-1}a^*_{-1} + \frac{1}{4(k+2)}b^2_{-1} - \frac{1}{2(k+2)}b_{-2}) \cdot |0\rangle$$

Proof $w_k(S_k)$ has deg -2, wt 0 ($\deg a_i = i$, $\deg a^*_i = i$, $\deg b_i = i$

$\Rightarrow w_k(S_k) \in \text{Span}(b^2_{-1}, b_{-2}, a_{-1}a^*_{-1}$ $\quad$ $\text{wt } a_i = 2$, $\text{wt } a^*_i = -2$, $\text{wt } b_i = 0$)

$a_{-2}a^*_0$, $a^2_{-1}a^{*2}_0$, $a_{-1}a^*_0 b_{-1}$)

← only possible monomials s.t. $\deg = -2$, $\text{wt} = 0$

$Y(w_k(S_k), z) =: \sum L_n z^{-n-2}$, $\deg L_n = -n$

Observation 1 $L_n \cdot P(a^*_0)|0\rangle = 0$ for $n \geq 0$, $P(a^*_0) \in \mathbb{C}[a^*_0]$

Proof $n > 0$ true for deg reason

$n = 0$ $\quad L_0 \cdot P(a^*_0)|0\rangle = \frac{1}{2(k+2)}(:e_0f_0: + :f_0e_0: + \frac{1}{2}:h_0^2: + \text{other terms}) \cdot P(a^*_0)|0\rangle$

: $\deg 0$ monomial : $\cdot P(a^*_0)|0\rangle \neq 0 \Rightarrow$ monomial $\in \mathbb{C}[a^*_0, a_0] \Rightarrow$ $\overset{''}{0}$

$$= \frac{1}{2(k+2)}(:a_0 \cdot (-a^{*2}_0 a_0): + :(-a^{*2}_0 a_0) \cdot a_0: + :\frac{1}{2}(-2a^*_0 a_0)^2:)P(a^*_0)|0\rangle$$

$= 0$

abuse of notation
means putting annihilating to the right $\quad \square$

However, the ($)_{(1)}$ part of above monomials acts on $\mathbb{C}[a^*_0] \cdot |0\rangle$ by

$(b^2_{-1})_{(1)}|_{\mathbb{C}[a^*_0]|0\rangle}$ similar for other terms

$\overset{''}{0}$, 0 , 0

$-a^*_0 a_0$, $a^{*2}_0 a^2_0$, 0 $\quad$ viewed as differential operators on $\mathbb{C}[a^*_0]$

$\Rightarrow w_k(S_k) \in \text{Span}(b^2_{-1}, b_{-2}, a_{-1}a^*_{-1}, a_{-1}a^*_0 b_{-1}) \cdot |0\rangle$

wt 2, deg ≥ 0

Observation 2 $L_n \cdot a_{-1}|0\rangle = 0$ $\quad n > 0$

$L_0 \cdot a_{-1}|0\rangle = a_{-1}|0\rangle$

$\overset{''}{}$ $\quad$ $\overset{''}{}$

$w_k(L_0 e_{-1}|0\rangle)$

On the other hand, $(b_{-1}^2)_{(1)} \cdot a_{-1}|0\rangle = 0$

$$(b_{-2})_{(1)} \cdot a_{-1}|0\rangle = 0$$

$$(a_{-1}a_{-1}^*)_{(1)} \cdot a_{-1}|0\rangle = a_{-1}|0\rangle$$

$$(a_{-1}a_0^* b_{-1})_{(1)} \cdot a_{-1}|0\rangle = 0$$

$$\Rightarrow w_k(S_k) \in (a_{-1}a_{-1}^* + \mathrm{Span}(b_{-1}^2 \ , b_{-2} \ , a_{-1}a_0^* b_{-1})) \cdot |0\rangle$$

Observation 3 $L_n w_k(h_{-1}|0\rangle) = \overset{\overset{w_k(L_n h_{-1}|0\rangle)}{\shortparallel}}{0} \qquad n > 0$

$$L_0 \underset{\underset{w_k(L_0 h_{-1}|0\rangle)}{\shortparallel}}{w_k(h_{-1}|0\rangle)} = w_k(h_{-1}|0\rangle)$$

On the other hand,

$$w_k(h_{-1}|0\rangle) = (-2a_0^* a_{-1} + b_{-1})|0\rangle$$

L_0- part $\qquad$ L_1- part

all non-zero terms

$$\underbrace{Y(b_{-1}^2|0\rangle, z)}_{2b_{-1}b_1 z^{-2}} \cdot (-2a_0^* a_{-1} + b_{-1})|0\rangle = 4(k+2) b_{-1}|0\rangle \cdot z^{-2} + 0 \cdot z^{-3} + \cdots$$

$$\underbrace{Y(b_{-2}|0\rangle, z)}_{-2b_1 z^{-3}} \cdot (-2a_0^* a_{-1} + b_{-1})|0\rangle = \qquad -4(k+2)|0\rangle \cdot z^{-3}$$

$$\underbrace{Y(a_{-1}a_{-1}^*|0\rangle, z)}_{-a_{-1}a_1^* z^{-2} - a_0 a_1^* z^{-3}} \cdot (-2a_0^* a_{-1} + b_{-1})|0\rangle = -2a_0^* a_{-1}|0\rangle \cdot z^{-2} - 2 \cdot |0\rangle z^{-3}$$

$$\underbrace{Y(a_{-1}a_0^* b_{-1}|0\rangle, z)}_{(a_{-1}a_0^* b_1 + a_0 a_1^* b_{-1}) z^{-2}} \cdot (-2a_0^* a_{-1} + b_{-1})|0\rangle = (2(k+2)a_0^* a_{-1} + 2b_{-1})|0\rangle z^{-2} + 0 \cdot z^{3} + \cdots$$

$$\Rightarrow w_k(S_k) = (a_{-1}a_{-1}^* + \frac{1}{4(k+2)} b_{-1}^2 - \frac{1}{2(k+2)} b_{-2})|0\rangle \qquad \square$$

<u>Rmk</u> $a_{-1}^* a_{-1}|0\rangle \in M_{sl_2}$ is a conformal vector of M_{sl_2} w/ central charge 2

$\frac{1}{4(k+2)} b_{-1}^2 - \frac{1}{2(k+2)} b_{-2}$ is a conformal vector of $V_{k+2}(\mathfrak{h})$ w/ central charge $\frac{k-4}{k+2}$

§4. (☆) for general $\mathfrak{g}$

$\{\alpha_i\}$ = simple roots

$(a_\alpha = \text{copy of } e_\alpha)$

$\mathbb{C}[a^*_\alpha \mid \alpha \in \Phi_+]$ $\quad \bigoplus_{\alpha\in\Phi_+} \mathbb{C}\cdot a_\alpha$ $\quad \bigoplus_i \mathbb{C} b_i$ $(b_i = \text{copy of } h_i)$

$$\text{Recall}\quad \mathfrak{g} \longrightarrow \mathrm{Vect}(B_+)^H = \underset{\rotatebox{90}{$\subseteq$}}{\mathrm{Sym}\, n_+^*} \otimes \overset{\shortparallel}{n_+} \oplus \mathrm{Sym}\, n_+^* \otimes \overset{\shortparallel}{\mathfrak{h}}$$

$$e_i \longmapsto a_{\alpha_i} + \sum_{\beta\in\Delta_+} P^i_\beta(\underline{a}^*)\, a_\beta$$

$$h_i \longmapsto -\sum_{\beta\in\Delta_+} \beta(h_i)\, a^*_\beta a_\beta + b_i$$

$$f_i \longmapsto \sum_{\beta\in\Delta_+} Q^i_\beta(\underline{a}^*)\cdot a_\beta + a^*_{\alpha_i} b_i$$

Affine $\rightsquigarrow$ analogue **Thm 6.2.1** $\exists$ map of VA

$$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_\mathfrak{g} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})$$

$$e_{i,-1}|0\rangle \longmapsto \Big(a_{\alpha_i,-1} + \sum_{\beta\in\Delta_+} P^i_\beta(\underline{a}^*_0)\, a_{\beta,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } \alpha_i$$

$$h_{i,-1}|0\rangle \longmapsto \Big(-\sum_{\beta\in\Delta_+} \beta(h_i)\, a^*_{\beta,0} a_{\beta,-1} + b_{i,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } 0$$

$$f_{i,-1}|0\rangle \longmapsto \Big(\sum_{\beta\in\Delta_+} Q^i_\beta(\underline{a}^*_0)\, a_{\beta,-1} + a^*_{\alpha_i,0} b_{i,-1} + \underbrace{(c_i + (\kappa-\kappa_c)(e_i, f_i))}_{\uparrow}\, a^*_{\alpha_i,-1}\Big)|0\rangle \qquad \text{deg } 1,\ \text{wt } -\alpha_i$$

Rmk finite dim'l case + deg + wt pin down RHS except

§5. Conformal structures in general case

$$\text{Recall}\quad S_\kappa = \frac{1}{2}\sum \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{J_{a,-1} J^a_{-1}}}\, |0\rangle \quad \text{is a conformal vector of } V_\kappa(\mathfrak{g}) \quad (\kappa \neq \kappa_c)$$

$$Y(S_\kappa|0\rangle, z) =: \sum L_\kappa z^{-\kappa-2} \rightsquigarrow [L_\kappa, J_{a,n}] = -n J_{a,n+m}$$

Prop 6.2.2

For $\kappa \neq \kappa_c$

$w_\kappa : V_\kappa(\mathfrak{g}) \longrightarrow M_\mathfrak{g} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})$ satisfies

$$w_\kappa(S_\kappa) = \Big(\underbrace{\sum_{\alpha\in\Delta_+} a_{\alpha,-1} a^*_{\alpha,-1}}_{\substack{\cap \\ M_\mathfrak{g}\otimes 1}} + \underbrace{\frac{1}{2}\sum_{i=1}^{\ell} \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{b_{i,-1} b^i_{-1}}} - \overset{\text{dual under } \kappa-\kappa_c \text{ to } \rho\in\mathfrak{h}^*}{\rho_{-2}}}_{\substack{\cap \\ 1\otimes V_{\kappa-\kappa_c}(\mathfrak{h})}}\Big)|0\rangle$$

Proof Similar to sl_2-case □

§6. Quasi-conformal structures

$\mathrm{Der}_+\mathcal{O} = \mathbb{C}\cdot\{L_1, L_2, \cdots\}$

$\cap$

Recall $\mathrm{Der}\,\mathcal{O} = \mathbb{C}\cdot\{L_{-1}, L_0, L_1, \cdots\}$, $L_k = -t^{k+1}\partial_t$

$\cap$

$\mathrm{Vir} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}L_i \oplus \mathbb{C}\cdot C$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{m,-n}\cdot C$$

Def A quasi-conformal structure on a $\mathbb{Z}$-graded VA is

$\mathrm{Der}\,\mathcal{O} \curvearrowright V$ s.t.

- $[L_m, A_{(k)}] = \sum_{n\geq -1}\binom{m+1}{n+1}(L_n\cdot A)_{(m+k-n)}$ for all $A\in V$
- $L_{-1} = T$
- $L_0 =$ grading
- $\mathrm{Der}_+\mathcal{O}$ acts nilpotently

e.g. A conformal vector $\omega\in V \rightsquigarrow Y(\omega, z) = \sum L_n z^{-n-2}$

$\rightsquigarrow L_n \in \mathrm{End}_{\mathbb{C}}(V)$ $n = -1, 0, 1, \cdots$

$\rightsquigarrow$ quasi-conformal structure on V

e.g. For $V = V_\kappa(\mathfrak{g})$ $(\kappa \neq \kappa_c)$

$\omega = S_\kappa \rightsquigarrow L_n\cdot J_{a,m}|0\rangle = -mJ_{a,m+n}|0\rangle$ $(*)$

When $\kappa = \kappa_c$, S_{κ_c} doesn't exist , but $(*)$ still makes sense

and defining $\mathrm{Der}\,\mathcal{O} \curvearrowright V_{\kappa_c}(\mathfrak{g})$, which is a quasi-conformal structure

§7. Coordinate independent version

V	Conformal vector $(\kappa\neq\kappa_c)$	$\mathrm{Der}\,\mathcal{O} \circlearrowright V$	Coordinate-free ver
$V_\kappa(\mathfrak{g})$	$S_\kappa = \frac{1}{2}\sum \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{J_{a,-}J^{a}_{-1}}}\vert 0\rangle$	$L_n \cdot J_{a,m}\vert 0\rangle = -mJ_{a,m+n}\vert 0\rangle$	$V_\kappa(\mathfrak{g})_x = \mathrm{Ind}_{\mathfrak{g}_{\mathcal{O}_x}\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_{\kappa,x}}\mathbb{C}\vert 0\rangle$
$\Downarrow$			
$V_{\kappa-\kappa_c}(\mathfrak{h})$	$\omega = \frac{1}{2}\sum_i b_{i,-1}b^{i}_{-1}\vert 0\rangle$	$L_n\cdot b_{i,m}\vert 0\rangle = -m\, b_{i,m+n}\vert 0\rangle$	$V_{\kappa-\kappa_c}(\mathfrak{h})_x = \mathrm{Ind}_{\mathfrak{h}_{\mathcal{O}_x}\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}_{\kappa-\kappa_c,x}}\mathbb{C}\vert 0\rangle$
$V_{\kappa-\kappa_c}(\mathfrak{h})$	$\omega = \left(\frac{1}{2}\sum_i \overset{\text{dual under } \kappa-\kappa_c}{\overbrace{b_{i,-1}b^{i}_{-1}}} - \rho_{-2}\right)\vert 0\rangle$	$L_n\cdot b_{i,m}\vert 0\rangle = (-m\, b_{i,m+n} + n(n+1)\delta_{n,-m})\cdot\vert 0\rangle$	$V_{\kappa-\kappa_c}(\mathfrak{h})_x^{\Omega^{-\rho}} = \mathrm{Ind}_{\mathfrak{h}_{\mathcal{O}_x}\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\kappa-\kappa_c,x}}\mathbb{C}\vert 0\rangle$ ← focus on this!
$M_{\mathfrak{g}}$	$\omega = \sum a_{\alpha,-1}a^{*}_{\alpha,-1}$	$L_n\cdot a_{\alpha,m}\vert 0\rangle = -m\,a_{\alpha,m+n}\vert 0\rangle$ $L_n\cdot a^{*}_{\alpha,m}\vert 0\rangle = -(m+n)a_{\alpha,m+n}\vert 0\rangle$	$M_{\mathfrak{g},x} = \mathrm{Ind}_{\Gamma_{+,x}\oplus\mathbb{C}\mathbb{1}}^{\hat{\Gamma}_x}\mathbb{C}\vert 0\rangle$

same for $\mathfrak{h}$

$$\text{Where}\quad \begin{array}{ccccccc} \hat{\mathfrak{g}}_{\kappa,x} & \twoheadrightarrow & (\mathfrak{g}\otimes K_x) & , & \hat{\Gamma}_x & \twoheadrightarrow & \Gamma_x = (\mathfrak{n}_+\otimes K_x \oplus \mathfrak{n}_+^{*}\otimes\Omega_{K_x}) \\ \cup & & \cup & & \cup & & \cup \\ \mathfrak{g}_{\mathcal{O}_x} & = & \mathfrak{g}\otimes\mathcal{O}_x & & \Gamma_{+,x} & = & \mathfrak{n}_+\otimes\mathcal{O}_x\oplus\mathfrak{n}_+^{*}\Omega_{\mathcal{O}_x} \end{array}$$

Def of $\hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\kappa-\kappa_c,x}$

$$\Omega^{-\rho}_{D_x^\circ} := \Omega_{D_x^\circ}\overset{\mathbb{G}_m,\rho^{-1}}{\times} H^\vee = (\Omega_{D_x^\circ}\times H^\vee)/(\omega t,h)\sim(\omega,\rho^{-1}(t)h) \text{ for all } t\in\mathbb{C}^\times \quad \text{is a } H^\vee\text{-bundle on } D_x^\circ$$

Concretely

For any two coordinates $D_x \overset{t}{\underset{s}{\rightrightarrows}} D$ where $s=\varphi(t)$ for $\varphi\in\mathrm{Aut}\,D$

we get two sections of $\Omega_{D_x^\circ}$: dt and $ds=\varphi'(t)\,dt$

i.e. two trivializations of $\Omega_{D_x^\circ}$ w/ transition function $\varphi'(t)$

$\rightsquigarrow$ two trivializations of $\Omega^{-\rho}_{D_x^\circ}$ w/ transition function $\rho^{-1}(\varphi'(t))$

e.g. Recall Def 3.7 in Katya's notes

For any $(\mathcal{F},\nabla,\mathcal{F}_{B^\vee})\in \mathrm{Op}_{G^\vee}(X)$

i.e. · $\mathcal{F}$ is a $G^\vee$-bundle

· ∇ is a connection on $G^\vee$

· $\mathcal{F}_{B^\vee}$ is a $B^\vee$-reduction of $G^\vee$ s.t. ∇, $\mathcal{F}_{B^\vee}$ are transversal.

i.e. $\overline{\underbrace{\nabla - \nabla'_{B^\vee}}_{\text{any connection of } \mathcal{F}_B}} \in H^0(X, (\mathcal{F}_{B^\vee}\overset{B^\vee}{\times}(\mathfrak{g}^\vee/\mathfrak{b}^\vee)) \otimes \Omega_X)$ gives non-zero sections of

$(\mathcal{F}_{B^\vee}\overset{B^\vee}{\times}\mathfrak{g}_{-\alpha_i}) \otimes \Omega_X$, α_i simple roots

we have $\mathcal{F}_{B^\vee}\overset{B^\vee}{\times} H^\vee \xrightarrow{\sim} \Omega_X^{\rho}$

For $\nabla = \lambda d + A(s)ds \in \mathrm{Conn}_\lambda(\Omega_{D^\circ})$, $s=\varphi(t)$

$\nabla_{\partial_t}(dt) = \nabla_{\partial_t}(\varphi'(t)^{-1}ds) = -\lambda\cdot\frac{\varphi''}{(\varphi')^2}\cdot ds + \varphi'^{-1}\cdot\nabla_{\varphi'\partial_s}ds$

$= -\lambda\frac{\varphi''}{\varphi'}dt + A\cdot ds = -\lambda\frac{\varphi''}{\varphi'}dt + A\cdot\varphi' dt$

$\Rightarrow \nabla = \lambda d - \lambda\cdot\frac{\varphi''}{\varphi'}dt + A\cdot\varphi' dt$

Def $\lambda \in \mathbb{C} \rightsquigarrow \mathrm{Conn}_\lambda(\Omega^{-\rho}_{D_x^\circ}) = \{\lambda\text{-connections on } \Omega^{-\rho}_{D_x^\circ}\}$

Concretely Choose coordinate $D_x \underset{s=\varphi(t)}{\overset{t}{\rightrightarrows}} D$

using corr. t

$\mathrm{Conn}_\lambda(\Omega^{-\rho}_{D_x^\circ}) \simeq \{\lambda d + A(t)dt \mid A(t) \in \mathfrak{h}^\vee((t)) \simeq \mathfrak{h}^*((t))\}$ $\quad \lambda d + \lambda\cdot\rho\cdot\frac{\varphi''(t)}{\varphi'(t)}dt + A(\varphi(t))\varphi'(t)\,dt$

using coor. s $\quad \overset{\sim}{\searrow} \{\lambda d + A(s)ds \mid A(s) \in \mathfrak{h}^*((s))\}$ $\quad \lambda d \uparrow + A(s)\,ds$

In another word, we get $\mathrm{Aut}\,\mathcal{O} \circlearrowright \mathrm{Conn}_\lambda(\Omega^{-\rho}_{D^\circ}) \simeq \mathfrak{h}^*((t))\,dt$

$$\varphi(t)\cdot A(t) = \lambda\rho\cdot\frac{\varphi''(t)}{\varphi'(t)} + A(\varphi(t))\varphi'(t)$$

$\xrightarrow[\lambda]{\text{Varying}} \mathrm{Der}\,\mathcal{O} \circlearrowright \mathrm{Conn}_?(\Omega^{-\rho}_{D^\circ}) \in \mathrm{Vect}_{\mathbb{C}}$ fibered over $\mathbb{C}$

$$L_n\cdot(\lambda d + \underbrace{b^*_{i,m}}) = -(m+n)\,b^*_{i,m+n} - n(n+1)\cdot\lambda\cdot\underbrace{\rho_n}$$

$\mathfrak{h}^* \ni b_i^* \otimes t^{m-1}dt$ $\quad\quad \rho\otimes t^{n-1}dt$

$\updownarrow$ dual $\quad\quad \cap$

$\mathfrak{h} \ni b_i$ $\quad\quad \mathfrak{h}^*$

$$\begin{tikzcd}
 & \mathfrak{h}^*((t))dt \arrow[d, phantom, "\wr"] & \lambda d & \lambda \arrow[l, mapsto] & \\
0 \arrow[r] & \mathrm{Conn}_0(\Omega^{-\rho}_{D^\circ}) \arrow[r] & \mathrm{Conn}_?(\Omega^{-\rho}_{D^\circ}) \arrow[r] & \mathbb{C} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathrm{Conn}_?(\Omega^{-\rho}_{D^\circ})^* \arrow[r] & \mathrm{Conn}_0(\Omega^{-\rho}_{D^\circ})^* \arrow[r] & 0 \\
 & \mathbb{1} & & \mathfrak{h}((t)) \arrow[u, phantom, "\wr"] &
\end{tikzcd}$$

$\{(\ )^*$ (continuous dual)

$\mathbb{1}$ s.t. $\mathbb{1}(\lambda d + \cdots) = \lambda$

Rmk Coordinate t gives splittings of both exact seq. (as vector spaces)

$\rightsquigarrow \mathrm{Der}\,\mathcal{O} \circlearrowright \mathrm{Fun}(\mathrm{Conn}_?(\Omega^{-\rho}_{D^\circ}))$ $\quad\quad$ (1)

$$L_n\cdot(\mu\cdot\mathbb{1} + \underbrace{b_{i,m}}_{b_i\otimes t^m}) = -m\,b_{i,m+n} + \delta_{n,-m}\,n(n+1)\cdot\mathbb{1}$$

Under previously chosen coordinate t

Def $\hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\nu,x} := \hat{\mathfrak{h}}_{\nu,x} \approx \hat{\mathfrak{h}}_\nu$ equippied w/ $\mathrm{Der}\,\mathcal{O}$ action given by

· $L_n\cdot\mathbb{1} = 0$, $L_n\cdot b_{i,m} = -m b_{i,m+n} + \delta_{n,-m}\,n(n+1)\cdot\mathbb{1}$ $\quad\quad$ (2)

Comparing (1), (2), we get a $\mathrm{Der}\,\mathcal{O}$-equivariant isom.

$$\mathrm{Conn}_?(\Omega^{-\rho}_{D_x^\circ})^* \underset{\alpha}{\simeq} \hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\nu,x}$$

Ex The Lie algebra structure on $\mathrm{Conn}_?(\Omega^{-\rho}_{D^\circ})$ induced from α is independent of coordinate t

Note we have an $\mathrm{Der}\,\mathcal{O}$-equivariant map

$$\begin{tikzcd}
\mathfrak{h}_{\mathcal{O}_x} \arrow[r, phantom, "\subset"] \arrow[d, equal] & \hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\nu,x} \arrow[d, equal] \\
\mathrm{Conn}_?(\Omega^{-\rho}_{D_x})^{\perp} \arrow[r, phantom, "\subset"] & \mathrm{Conn}_?(\Omega^{-\rho}_{D_x^\circ})^{*}
\end{tikzcd}$$

$$\rightsquigarrow V_\nu(\mathfrak{h})^{\Omega^{-\rho}}_x := \mathrm{Ind}^{\hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\nu,x}}_{\mathfrak{h}_{\mathcal{O}_x}\oplus\mathbb{C}\mathbb{1}} \cdot \mathbb{C}|0\rangle \quad , \quad \tilde{U}_\nu(\mathfrak{h})^{\Omega^{-\rho}}_x := \widetilde{U(\hat{\mathfrak{h}}^{\Omega^{-\rho}}_\nu)}/(\mathbb{1}-1)$$

With the chosen coordinate t, we have an $\mathrm{Der}\,\mathcal{O}$-equiv. isom.

$$V_\nu(\mathfrak{h})^{\Omega^{-\rho}}_x \xrightarrow{\sim} V_\nu(\mathfrak{h})$$

Cor If $\nu=0$, $\hat{\mathfrak{h}}^{\Omega^{-\rho}}_0 \simeq (\mathrm{Conn}_?(\Omega^{-\rho}_{D_x^\circ}))^*$ are abelian Lie alg.
(at critical level)

$$\tilde{U}_0(\mathfrak{h}^{\Omega^{-\rho}})_x = \mathrm{Fun}(\mathrm{Conn}_?(\Omega^{-\rho}_{D_x^\circ}))/(\mathbb{1}-1) = \mathrm{Fun}(\mathrm{Conn}_1(\Omega^{-\rho}_{D_x^\circ})) \qquad (*)$$

$\mathbb{1}((\nabla,\lambda)) = \lambda$

Relation to Opers (Miura map)

$$\begin{aligned}
\gamma : \mathrm{Conn}_1(\Omega^{-\rho}_{D_x^\circ}) \xrightarrow{\sim} \mathrm{Conn}_1(\Omega^{\rho}_{D_x^\circ}) &\longrightarrow \mathrm{Op}_{G^\vee}(D_x^\circ) \\
\nabla &\longmapsto \left(\mathcal{F} := \Omega^{\rho}_{D_x^\circ} \overset{H^\vee}{\times} G^\vee ,\ \nabla + p_{-1} ,\ \mathcal{F}_{B^\vee} := \Omega^{\rho}_{D_x^\circ} \overset{H^\vee}{\times} B^\vee \right)
\end{aligned}$$

where $p_{-1} \in \mathfrak{g}^\vee_{-1} \otimes K_x \simeq H^0(D_x^\circ, \underbrace{\mathfrak{g}^\vee_{-1} \otimes \Omega^{-1}_{D_x^\circ}}_{} \otimes \Omega_{D_x^\circ}) \simeq H^0(D_x^\circ, \underbrace{(\Omega^{\rho}_{D_x^\circ} \overset{H^\vee}{\times} \mathfrak{g}^\vee_{-1})}_{} \otimes \Omega_{D_x^\circ}) \subset H^0(D_x^\circ, (\mathcal{F} \overset{G^\vee}{\times} \mathfrak{g}^\vee) \otimes \Omega_{D_x^\circ}) = \mathrm{Conn}_0(\mathcal{F})$

$\simeq \quad \Omega_{D_x^\circ} \overset{\mathbb{G}_m,\rho}{\times} \mathfrak{g}^\vee_{-1} \quad \simeq$

$$\rightsquigarrow \mathrm{Fun}(\mathrm{Op}_{G^\vee}(D_x^\circ)) \xrightarrow{\gamma^*} \tilde{U}_0(\mathfrak{h}^{\Omega^{-\rho}})_x$$

Wakimoto modules

We get $\quad w_{\kappa,x} : V_\kappa(\mathfrak{g})_x \longrightarrow M_{\mathfrak{g},x} \otimes V_{\kappa-\kappa_c}(\mathfrak{h})^{\Omega^{-\rho}}_x \circlearrowright \pi^{\lambda,\Omega^{-\rho}}_{\kappa-\kappa_c,x} := \mathrm{Ind}^{\hat{\mathfrak{h}}^{\Omega^{-\rho}}_{\kappa-\kappa_c,x}}_{\mathfrak{h}_{\mathcal{O}_x}\oplus\mathbb{C}\mathbb{1}} \mathbb{C}\cdot|\lambda\rangle =: W_{\lambda,\kappa,x} \in \mathrm{Mod}_{\tilde{U}_\kappa(\mathfrak{g})_x}$

this is the Wakimoto module of level κ, highest weight λ

When $\kappa = \kappa_c$, $(*) \Rightarrow$

$$w_{\kappa_c} \rightsquigarrow \tilde{U}_{\kappa_c}(\mathfrak{g})_x \longrightarrow \tilde{\mathcal{A}}^{\mathfrak{g}}_x \hat{\otimes} \mathrm{Fun}(\mathrm{Conn}_1(\Omega^{-\rho}_{D_x^\circ}))$$

For each $\nabla \in \mathrm{Conn}_1(\Omega^{-\rho}_{D_x^\circ})$, we get

$$\tilde{U}_{\kappa_c}(\mathfrak{g})_x \longrightarrow \tilde{\mathcal{A}}^{\mathfrak{g}}_x \circlearrowright M_{\mathfrak{g},x} =: W_{\nabla,x} \in \mathrm{Mod}_{\tilde{U}_{\kappa_c}(\mathfrak{g})_x}$$

these are Wakimoto modules at critical level