ETINGOF CONJECTURE FOR QUANTIZED QUIVER VARIETIES

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Abstract. We compute the number of finite dimensional irreducible modules for the algebras quantizing Nakajima quiver varieties. We get a lower bound for all quivers and vectors of framing. We provide an exact count in the case when the quiver is of finite type or is of affine type and the framing is the coordinate vector at the extending vertex. The latter case precisely covers Etingof's conjecture on the number of finite dimensional irreducible representations for Symplectic reflection algebras associated to wreath-product groups. We use several different techniques, the two principal ones are categorical Kac-Moody actions and wall-crossing functors.

Contents

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1. Introduction

1.1. **Counting problem.** Studying irreducible representations of algebraic objects, say of associative algebras, is the most fundamental problem in Representation theory. A basic question is how many there are. For most infinite dimensional algebras, the set of all irreducible representations is wild, in particular, the number is infinite. So it makes sense to restrict the class of representations. The most basic choice is to consider only finite dimensional ones. This is a restriction we impose in the present paper.

A classical infinite dimensional algebra appearing in Representation theory is the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let us consider the case when the Lie algebra $\mathfrak{g}$ is semisimple. In this case, the number of finite dimensional irreducible representations is still infinite: they are in a one-to-one correspondence with dominant weights. More precisely, we consider the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and fix a system of simple roots. We say that $\lambda \in \mathfrak{h}^*$ is *dominant* if $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\geqslant 0}$ for any simple root α. Then to λ we can assign the irreducible module with highest weight λ. Those form a complete and irredundant collection of irreducible finite dimensional representations of $\mathfrak{g}$.

However, we can modify the algebra $U(\mathfrak{g})$ to make the counting problem finite. Namely, recall that the center of $U(\mathfrak{g})$ is identified with $S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$ via the Harish-Chandra isomorphism. Here W is the Weyl group acting on $\mathfrak{h}^*$ by $w \bullet \lambda = w(\lambda + \rho) - \rho$, where ρ, as usual, is half the sum of all positive roots. Then, for each $\lambda \in \mathfrak{h}^*/W$, we can consider the corresponding central reduction, $\mathcal{U}_\lambda$. The classification result above can be restated as follows: the algebra $\mathcal{U}_\lambda$ has a single finite dimensional irreducible representation if $\langle \lambda + \rho, \alpha^\vee \rangle$ is a nonzero integer for any root α. Otherwise, there are no finite dimensional representations.

Another classical feature of the algebras $\mathcal{U}_\lambda$ is that they have a very nice underlying geometry. These algebras are filtered and the associated graded algebras $\operatorname{gr} \mathcal{U}_\lambda$ are all identified with $\mathbb{C}[\mathcal{N}]$, where $\mathcal{N}$ stands for the nilpotent cone in $\mathfrak{g}$. Recall the Springer resolution of singularities $\rho : \widetilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}$, where $\widetilde{\mathcal{N}}$ is the cotangent bundle of the flag variety $\mathcal{B}$ of $\mathfrak{g}$. The variety $\widetilde{\mathcal{N}}$ is smooth and symplectic, while $\mathcal{N}$ is a singular Poisson variety. The morphism ρ is therefore a symplectic resolution of singularities.

There is a non-commutative analog of this resolution. Namely, for $\lambda \in \mathfrak{h}^*$, we can consider the sheaf $\mathcal{D}_\lambda$ of λ-twisted differential operators on $\mathcal{B}$. Then $\Gamma(\mathcal{B}, \mathcal{D}_\lambda) = \mathcal{U}_\lambda$, while all higher cohomology of $\mathcal{D}_\lambda$ vanish. So we have the global section functor $\Gamma_\lambda : \mathcal{D}_\lambda\text{-mod} \to \mathcal{U}_\lambda\text{-mod}$ as well as its derived version $R\Gamma_\lambda : D^b(\mathcal{D}_\lambda\text{-mod}) \to D^b(\mathcal{U}_\lambda\text{-mod})$. The former is an equivalence if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leqslant 0}$ for all positive roots α, this is the celebrated Beilinson-Bernstein theorem, [BB1]. Its derived version, [BB3], states that $R\Gamma_\lambda$ is an equivalence if $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all α.

Using the results of the previous paragraph one can give a geometric interpretation of the classification of finite dimensional irreducible representations. Namely, under the abelian Beilinson-Bernstein equivalence, the finite dimensional modules correspond to the $\mathcal{D}_\lambda$-modules whose singular support is contained in $\mathcal{B} \subset \widetilde{\mathcal{N}}$, i.e., to the $\mathcal{O}$-coherent $\mathcal{D}_\lambda$-modules. It is easy to see that such a module exists if and only if λ is integral, in which case it is the line bundle on $\mathcal{B}$ corresponding to λ.

1.2. **Etingof's conjecture.** Another interesting class of associative algebras is Symplectic reflection algebras introduced by Etingof and Ginzburg in [EG]. Those are filtered deformations of the skew-group algebras $S(V)\#\Gamma$, where V is a symplectic vector space and Γ is a finite subgroup of $\mathrm{Sp}(V)$. The symplectic reflection algebras $\mathcal{H}_c$ for the pair (V, Γ) form a family depending on a collection c of complex numbers.

One especially interesting class of groups Γ comes from finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Namely, pick such a subgroup $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$ and form the semidirect product $\Gamma(= \Gamma_n) := \mathfrak{S}_n \ltimes \Gamma_1^n$. The group Γ_n naturally acts on $\mathbb{C}^{2n} = (\mathbb{C}^2)^{\oplus n}$ by symplectomorphisms. Here elements of $\mathfrak{S}_n$ permute the n summands $\mathbb{C}^2$, the n copies of Γ_1 act each on its own summand, and the symplectic form on $(\mathbb{C}^2)^{\oplus n}$ is obtained as the direct sum of the n copies of a Γ_1-invariant symplectic form on $\mathbb{C}^2$. For $n>1$, the algebra $\mathcal{H}_c$ depends on r parameters, where r is the number of conjugacy classes in Γ_1 (for $n=1$, the number of parameters is $r-1$). So one can ask, how many finite dimensional irreducibles does the algebra $\mathcal{H}_c$ have? The answer, of course, should depend on the parameter c.

In [Et, Section 6], Etingof proposed a conjectural answer to this and more general questions. The conjecture takes the following form. Recall that the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are in one-to-one (McKay) correspondence with the affine Dynkin diagrams. Take the affine Dynkin diagram, say Q, corresponding to Γ_1 and form the Kac-Moody algebra $\mathfrak{g}(Q)$ from this diagram. Then Etingof defines a certain subalgebra $\mathfrak{a}\subset\mathfrak{g}(Q)\times\mathfrak{heis}$ depending on c, where $\mathfrak{heis}$ stands for the Heisenberg Lie algebra. Next, he considers the module $\mathbf{V}\otimes\mathcal{F}$, where $\mathbf{V}$ is the basic representation of $\mathfrak{g}(Q)$ (whose highest weight is the fundamental weight corresponding to the extending vertex of Q) and $\mathcal{F}$ is the Fock space representation of $\mathfrak{heis}$. Then Etingof takes an appropriate weight subspace in that representation and considers its intersection with the sum of certain $\mathfrak{a}$-isotypic components. The conjecture is that the number of finite dimensional irreducibles is the dimension of the resulting intersection.

Etingof's conjecture (in fact, its more general version dealing with the number of irreducibles with given support in a category $\mathcal{O}$) was proved in the case when Γ_1 is cyclic by Shan and Vasserot, [SV, Section 6] (under some technical restrictions on c that were removed in [L8, Appendix]). The techniques used in [SV] are based on the representation theory of Rational Cherednik algebras and do not generalize to the case of non-cyclic Γ_1.

The main goal of this paper is to prove Etingof's conjecture on counting finite dimensional irreducibles for all groups Γ_1. But, first, we put it into a more general context: counting finite dimensional irreducible representations over quantizations of symplectic resolutions.

1.3. **Quantizations of symplectic resolutions.** Inside $\mathcal{H}_c$ we can consider the spherical subalgebra, $e\mathcal{H}_ce$, where e is the averaging idempotent. This algebra is a filtered deformation of $S(V)^\Gamma$. By [Et, Theorem 5.5], $e\mathcal{H}_ce$ is Morita equivalent to $\mathcal{H}_c$ if and only if $e\mathcal{H}_ce$ has finite homological dimension (the parameter c is called *spherical* in this case). Under this assumption, the numbers of finite dimensional irreducibles for $\mathcal{H}_c$ and $e\mathcal{H}_ce$ coincide.

When $\Gamma=\Gamma_n$, the variety V/Γ_n can be realized as an affine Nakajima quiver variety and admits a symplectic resolution of singularities that is a smooth Nakajima quiver variety. The algebra $e\mathcal{H}_ce$ can be realized as a quantum Hamiltonian reduction, see [EGGO, L6] and references therein (we briefly recall this below in Section 2.2.6). Also we can quantize the symplectic resolution getting a sheaf of algebras on that symplectic variety. So we again have a nice geometry as in the case of universal enveloping algebras.

There are other algebras that quantize (i.e., are filtered deformations of) affine Poisson varieties admitting symplectic resolutions and it is natural to expect that the counting problems for these algebras have some nice answers that have to do with the geometry of the resolution. There are three known large classes of resolutions giving rise to interesting algebras. First, there are more general Nakajima quiver varieties, the corresponding algebras are obtained as quantum Hamiltonian reductions of algebras of differential operators. Second, there are Slodowy varieties that generalize cotangent bundles to (partial) flag varieties. The corresponding algebras are finite W-algebras generalizing the universal enveloping algebras. The counting problem for W-algebras was studied by the second author and Ostrik in [LO] (in the case of integral central characters), below we will briefly mention how the answer looks like in that case. Third, there are hypertoric varieties that are similar to but much easier than Nakajima quiver varieties, this case is treated in [BLPW1].

In this paper we concentrate on the case of Nakajima quiver varieties. In Section 1.4 we recall necessary definitions.

1.4. **Nakajima quiver varieties and their quantizations.** In this section we briefly recall Nakajima quiver varieties and their quantizations. We will elaborate more on their properties in Section 2.

Let Q be a quiver (=oriented graph, we allow loops and multiple edges). We can formally represent Q as a quadruple (Q_0, Q_1, t, h), where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, $t, h : Q_1 \to Q_0$ are maps that to an arrow a assign its tail and head.

Pick vectors $v, w \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ and vector spaces V_i, W_i with $\dim V_i = v_i, \dim W_i = w_i$. Consider the (co)framed representation space

$$R = R(Q, v, w) := \bigoplus_{a \in Q_1} \operatorname{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in Q_0} \operatorname{Hom}(V_i, W_i).$$

We will also consider the cotangent bundle $T^*R = R \oplus R^*$ that can be identified with

$$\bigoplus_{a \in Q_1} \left(\operatorname{Hom}(V_{t(a)}, V_{h(a)}) \oplus \operatorname{Hom}(V_{h(a)}, V_{t(a)})\right) \oplus \bigoplus_{i \in Q_0} \left(\operatorname{Hom}(V_i, W_i) \oplus \operatorname{Hom}(W_i, V_i)\right).$$

The space T^*R carries a natural symplectic form, denote it by ω. On R we have a natural action of the group $G := \prod_{i \in Q_0} \mathrm{GL}(v_i)$. This action extends to an action on T^*R by linear symplectomorphisms. As any action by linear symplectomorphisms, the G-action on T^*R admits a moment map, i.e., a G-equivariant morphism $\mu : T^*R \to \mathfrak{g}^*$ with the property that $\{\mu^*(x), \bullet\} = x_{T^*R}$ for any $x \in \mathfrak{g}$. Here $\mu^* : \mathfrak{g} \to \mathbb{C}[T^*R]$ denotes the dual map to μ, $\{\bullet, \bullet\}$ is a Poisson bracket on $\mathbb{C}[T^*R]$ induced by ω, and x_{T^*R} is a vector field on T^*R induced by the G-action. Also we consider the dilation action of the one-dimensional torus $\mathbb{C}^\times$ on T^*R given by $t.r = t^{-1}r$. We specify the moment map uniquely by requiring that it is quadratic: $\mu(t.r) = t^{-2}\mu(r)$. In this case $\mu^*(x) = x_R$, where we view x_R, an element of Vect_R, as a function on T^*R.

In what follows, Q and w are often fixed, but v will vary.

Now let us proceed to the definition of Nakajima quiver varieties. Pick a character θ of G (below we will often call θ a *stability condition*) and also an element $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. To θ we associate an open subset $(T^*R)^{\theta-ss}$ of θ-semistable points in T^*R (that may be empty). Recall that a point $r \in T^*R$ is called θ-semistable if there is a $(G, n\theta)$-semiinvariant (with $n > 0$) polynomial $f \in \mathbb{C}[T^*R]$ such that $f(r) \neq 0$.

We can form a GIT quotient $\mathcal{M}_\lambda^\theta(v) := (\mu^{-1}(\lambda) \cap (T^*R)^{\theta-ss})/\!/G$ (we omit the subscript when $\lambda = 0$). This variety is smooth provided (λ, θ) is *generic* (we will explain the precise meaning of this condition in 2.1.1). The variety $\mathcal{M}_\lambda^0(v)$ is affine and there is a projective morphism $\rho : \mathcal{M}_\lambda^\theta(v) \to \mathcal{M}_\lambda^0(v)$. There is a sufficient condition for this morphism to be a resolution of singularities that will be recalled in 2.1.7. We remark that all varieties $\mathcal{M}_\lambda^\theta(v)$ carry natural Poisson structures because they are defined as Hamiltonian reductions. For a generic pair (λ, θ), the variety $\mathcal{M}_\lambda^\theta(v)$ is symplectic. Also we remark that we have an action of $\mathbb{C}^\times$ on $\mathcal{M}^\theta(v)$ that comes from the dilation action on T^*R and so rescales the symplectic form.

Now let us briefly recall Nakajima's construction of a geometric $\mathfrak{g}(Q)$-action on the middle homology groups of the varieties $\mathcal{M}^\theta(v)$, we assume Q has no loops. Consider the space $\bigoplus_v H_{mid}(\mathcal{M}^\theta(v))$, where the subscript "mid" means the middle dimension, i.e., $\dim_{\mathbb{C}} \mathcal{M}^\theta(v)$. We remark that these spaces are naturally identified for different θ, see [Nak1, Section 9], this result is recalled in 2.1.8.

Nakajima, [Nak1], defined an action of $\mathfrak{g}(Q)$ on $\bigoplus_v H_{mid}(\mathcal{M}^\theta(v))$ turning that space into the irreducible integrable $\mathfrak{g}(Q)$-module L_ω with highest weight

$$\omega := \sum_{i \in Q_0} w_i \omega^i, \tag{1.1}$$

where we write ω^i for the fundamental weight corresponding to the vertex i. The individual space $H_{mid}(\mathcal{M}^\theta(v))$ gets identified with the weight space $L_\omega[\nu]$ of weight ν, where

$$\nu := \omega - \sum_{i \in Q_0} v_i \alpha^i \tag{1.2}$$

(we write α^i for the simple root corresponding to i).

Now we proceed to the quantum part of this story. Let us start by constructing quantizations of $\mathcal{M}^\theta(v)$ that will be certain sheaves of filtered algebras on $\mathcal{M}^\theta(v)$. Namely, consider the algebra $D(R)$ of differential operators on R. We can localize this algebra to a *microlocal* (the sections are only defined on $\mathbb{C}^\times$-stable open subsets) sheaf on T^*R denoted by D_R. We have a quantum comoment map $\Phi : \mathfrak{g} \to D(R)$ quantizing the classical comoment map $\mathfrak{g} \to \mathbb{C}[T^*R]$, still $\Phi(x) = x_R$.

Now fix $\lambda \in \mathbb{C}^{Q_0}$. We get the quantum Hamiltonian reduction sheaf

$$\mathcal{A}^\theta_\lambda(v) := \pi_*[D_R/D_R\{\Phi(x) - \langle \lambda, x\rangle | x \in \mathfrak{g}\}|_{(T^*R)^{\theta-ss}}]^G$$

on $\mathcal{M}^\theta(v)$, here π is the quotient morphism $\mu^{-1}(0)^{\theta-ss} \twoheadrightarrow \mathcal{M}^\theta(v)$. This is a sheaf of filtered algebras with $\operatorname{gr} \mathcal{A}^\theta_\lambda(v) = \mathcal{O}_{\mathcal{M}^\theta(v)}$. In fact, because of this, it has no higher cohomology, and $\Gamma(\mathcal{A}^\theta_\lambda(v))$ is an algebra $\mathcal{A}_\lambda(v)$ with $\operatorname{gr} \mathcal{A}_\lambda(v) = \mathbb{C}[\mathcal{M}^\theta(v)]$ (one can show that $\mathcal{A}_\lambda(v)$ is independent of θ, see [BPW, Corollary 3.8]). When μ is flat or λ is Zariski generic, then $\mathcal{A}_\lambda(v)$ coincides with $\mathcal{A}^0_\lambda(v) := [D(R)/D(R)\{\Phi(x) - \langle \lambda, x\rangle | x \in \mathfrak{g}\}]^G$, we will elaborate on this in Section 2.2.

1.5. **Main conjecture.** Our ultimate goal is to compute the number of finite dimensional irreducible representations of $\mathcal{A}_\lambda(v)$ or, equivalently, to compute $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$ (we consider all K_0's over $\mathbb{C}$). For this, we need to relate the categories of modules for $\mathcal{A}_\lambda(v)$ and for $\mathcal{A}^\theta_\lambda(v)$ with generic θ. Consider the category $\mathcal{A}^\theta_\lambda(v)\text{-mod}$ of all coherent $\mathcal{A}^\theta_\lambda(v)$-modules and also its derived analog $D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ (see Section 2.3 for the definitions). Then we have the derived global sections functor $R\Gamma^\theta_\lambda : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to D^b(\mathcal{A}_\lambda(v)\text{-mod})$. As McGerty and Nevins checked (under some technical conditions) in [MN], this functor is an equivalence if and only if the algebra $\mathcal{A}_\lambda(v)$ has finite homological dimension, the inverse of $R\Gamma^\theta_\lambda$ is the derived localization functor $L\operatorname{Loc}^\theta_\lambda := \mathcal{A}^\theta_\lambda(v) \otimes^L_{\mathcal{A}_\lambda(v)} \bullet$. In most interesting cases, the precise locus of λ, where the homological dimension is finite (such λ are called *regular*), is not known. Conjecturally, the regular locus should be the complement of a finite union of hyperplanes, see Section 4.3. In this paper we deal with the counting problem only in the case when λ is regular, and we make a conjecture on the answer in general, Conjecture 11.8.

For an object in $\mathcal{A}^\theta_\lambda(v)$-mod, one can define its (singular) support, a $\mathbb{C}^\times$-stable coisotropic subvariety of $\mathcal{M}^\theta_0(v)$, and the characteristic cycle, $\mathsf{CC}_v(M)$, see Section 2.4. The equivalence $R\Gamma^\theta_\lambda$ identifies the following two categories:

- the full subcategory $D^b_{fin}(\mathcal{A}_\lambda(v)\text{-mod}) \subset D^b(\mathcal{A}_\lambda(v)\text{-mod})$ of all complexes with finite dimensional homology
- and the full subcategory $D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \subset D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ of all complexes whose homology is supported on (i.e., has support contained in) $\rho^{-1}(0)$.

Since $\rho^{-1}(0)$ is an isotropic subvariety in $\mathcal{M}^\theta(v)$, the characteristic cycle of a module supported on $\rho^{-1}(0)$ is a combination of the lagrangian irreducible components of $\rho^{-1}(0)$ (this is a consequence of the Gabber involutivity theorem recalled in Section 2.4), let us denote the set of lagrangian components by comp. So we get a linear map

$$\mathsf{CC}^\lambda_v : K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \to \mathbb{C}^{\mathsf{comp}}. \tag{1.3}$$

The space $\mathbb{C}^{\mathsf{comp}}$ is identified with $H_{mid}(\mathcal{M}^\theta(v)) = L_\omega[\nu]$ and so is canonically independent of θ. We will see below, Section 7.3, that CC^λ_v is actually independent of θ. By an unpublished result of Baranovsky and Ginzburg, [BaGi], the map CC^λ_v is injective (since this result is not published yet, we actually give a proof of this result for quantized quiver varieties). So, to solve our counting problem, we just need to describe the image of $\mathsf{CC}^\lambda = \bigoplus_v \mathsf{CC}^\lambda_v$.

Our conjectural description is inspired by Etingof's conjecture. Namely, consider the subalgebra $\mathfrak{a} \subset \mathfrak{g}(Q)$ constructed from λ as follows: the algebra $\mathfrak{a}$ is generated by the Cartan $\mathfrak{h} \subset \mathfrak{g}(Q)$ and all root subspaces $\mathfrak{g}(Q)_\beta$ for <u>real</u> roots $\beta = \sum_{i\in Q_0} b_i\alpha^i$ with $\sum_{i\in Q_0} b_i\lambda_i \in \mathbb{Z}$. For instance, if λ is generic, then $\mathfrak{a} = \mathfrak{h}$, while if all $\lambda_i \in \mathbb{Z}$, then $\mathfrak{a} = \mathfrak{g}(Q)$ provided Q contains no loops. Let $L^\mathfrak{a}_\omega$ be the $\mathfrak{a}$-submodule of L_ω generated by the weight spaces $L_\omega[\sigma\omega]$ for $\sigma \in W(Q)$, where $W(Q)$ stands for the Weyl group of $\mathfrak{g}(Q)$.

Conjecture 1.1. Assume that Q has no loops. Then we have $\operatorname{Im}\mathsf{CC}^\lambda = L^\mathfrak{a}_\omega$.

Let us point out that the case when Q has a loop is non-interesting for our counting problem: the answer is 0 (provided the dimension in the corresponding vertex is positive, if it is zero, then the loop does not matter anyway). In this case, the algebra $\mathcal{A}_\lambda(v)$ decomposes into the product of $D(\mathbb{C})$, the algebra of differential operators on $\mathbb{C}$, and of another algebra. The former has no finite dimensional representations.

We remark that the dimension vectors v corresponding to $\nu = \sigma\omega$ are precisely those with $\mathcal{M}^\theta(v) = \{\text{pt}\}$ and hence $\mathcal{A}_\lambda(v) = \mathbb{C}$. In particular, if λ is Weil generic (which, by definition, means a parameter lying outside countably many proper subvarieties), then our conjecture predicts that a non-trivial algebra $\mathcal{A}_\lambda(v)$ has no finite dimensional representations, as expected. The other extreme is when λ is integral. Here our conjecture predicts that $\operatorname{Im}\mathsf{CC} = L_\omega$. This follows from the work of Webster, [We1, Section 3], see Section 5.2 for details.

Here is the main result of the present paper.

Theorem 1.2. *Conjecture 1.1 is true*

- *when Q is of finite type,*
- *or when Q is an affine quiver, $v = n\delta, w = \epsilon_0$.*

Here and below we write δ for the indecomposable imaginary root of Q and ϵ_0 for the coordinate vector at the extending vertex.

Let us notice that (ii) precisely covers the algebras of interest for Etingof's conjecture. In fact, that conjecture follows from Theorem 1.2 and results of [GL], we will elaborate on that in Section 11.3.

In a forthcoming paper [L10] the second author proves Conjecture 1.1 for affine type quivers with arbitrary framing.

We also would like to point out that there is a very similar conjecture for finite W-algebras $U(\mathfrak{g}, e)$, see [LO, Theorem 1.1, Conjecture 7.13] (here $\mathfrak{g}$ is a semisimple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element). This conjecture is proved in *loc.cit.* for integral central characters. The role of $\mathcal{M}^\theta(v)$ is played by the Slodowy variety $\widetilde{S}$ that is obtained as follows. We take the transversal *Slodowy slice* S to the G-orbit of e in $\mathfrak{g}$, and for $\widetilde{S}$ take the preimage of S in $\widetilde{\mathcal{N}}$. The zero fiber $\rho^{-1}(0)$ becomes the Springer fiber $\mathcal{B}_e$. Therefore $H_{mid}(\mathcal{B}_e)$ is the Springer representation of the Weyl group $W(\mathfrak{g})$ of $\mathfrak{g}$. So instead of $\mathfrak{g}(Q)$ we need to consider $W(\mathfrak{g})$, and instead of $\mathfrak{a}$ we take the integral Weyl group W' corresponding to a given central character. Then K_0 of the finite dimensional representations is expected to coincide with the sum of certain isotypic components for W', see [LO, Conjecture 7.13] for details.

1.6. **Content of the paper.** The paper is roughly split into two parts. The first part, consisting of Sections 2, 3, 4 is preparatory: there we recall known results (or their generalizations to settings we need) as well as some technical results. In Section 2 we recall preliminaries on Nakajima quiver varieties, their quantizations, coherent and quasi-coherent modules over sheaf quantizations, supports and characteristic cycles and Hamiltonian reduction functors. In Section 3 we recall Harish-Chandra (HC) bimodules, construct restriction functors for HC bimodules in our setting and describe some applications of these functors. In Section 4 we discuss abelian and derived localization theorems.

The proof of Theorem 1.2 starts with Section 5, where we introduce two main families of functors (the Webster functors that were constructed in [We1] in a special case and wall-crossing functors introduced in the present generality – as well as in more general situations – in [BPW]). Then we explain main steps in the proof of Theorem 1.2. We will describe the content of the subsequent sections in the end of Section 5.

1.7. **Notation.** The following table contains various notation used in the paper (we first list the notation starting with Roman letters in, roughly, the alphabetical order and then list the notation starting with a Greek letter).

$\widehat{\otimes}$	the completed tensor product of complete topological vector spaces/ modules.
$\mathcal{A}^{opp}$	the opposite algebra of $\mathcal{A}$.
$(a_1, \ldots, a_k)$	the two-sided ideal in an associative algebra generated by elements $a_1, \ldots, a_k$.
$A^{\wedge_\chi}$	the completion of a commutative (or "almost commutative") algebra A with respect to the maximal ideal of a point $\chi \in \operatorname{Spec}(A)$.
$\mathcal{A}^\theta_\lambda(v)$	$:= [\mathcal{Q}_\lambda\vert_{(T^*R)^{\theta-ss}}]^G$
$\mathcal{A}_\lambda(v)$	$:= \Gamma(\mathcal{A}^\theta_\lambda(v))$.
$\mathcal{A}^\theta_{\lambda,\chi}(v)$	$:= [\mathcal{Q}_\lambda\vert_{(T^*R)^{\theta-ss}}]^{G,\chi}$, where χ is a character of G, and the superscript (G, χ) means taking χ-semiinvariants.
$\mathfrak{a}^\lambda$	the subalgebra in $\mathfrak{g}(Q)$ generated by Cartan $\mathfrak{h}$ and real root subspaces $\mathfrak{g}(Q)_\beta$ with $\beta \cdot \lambda \in \mathbb{Z}$.

$\mathcal{A}^\theta_{\lambda,\chi}(v)$	$:=\Gamma(\mathcal{A}^\theta_{\lambda,\chi}(v))$.
$\mathcal{A}$-mod	the category of finitely generated modules over an associative algebra $\mathcal{A}$.
$\mathsf{AC}(Y)$	the asymptotic cone of a subvariety $Y\subset\mathbb{C}^n$.
$\mathfrak{AL}(v)$	the set of $\lambda\in\mathfrak{P}$ such that Γ^θ_λ is an abelian equivalence.
$\mathrm{Ann}_{\mathcal{A}}(M)$	the annihilator of an $\mathcal{A}$-module M in an algebra $\mathcal{A}$.
$\mathsf{CC}(M)$	the characteristic cycle of a module/sheaf of modules M.
$D^b_{fin}(\mathcal{A}_\lambda(v)\text{-mod})$	$:=\{M\in D^b(\mathcal{A}_\lambda(v)\text{-mod})\mid \dim H_*(M)<\infty\}$.
$D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$	$:=\{M\in D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})\mid \operatorname{Supp} H_*(M)\subset\rho^{-1}(0)\}$.
$D(R)\text{-mod}^{G,\lambda}$	the category of (G,λ)-equivariant finitely generated $D(R)$-modules.
$D(X)$	the algebra of differential operators on a smooth affine variety X.
D_X	the differential operators on a smooth variety X viewed as microlocal sheaf on T^*X.
$\mathrm{Frac}(A)$	the fraction field of a commutative domain A.
G°	the connected component of unit in an algebraic group G.
(G,G)	the derived subgroup of a group G.
G_x	the stabilizer of x in G.
$G*_H V$	the homogeneous bundle on G/H with fiber V.
$\mathfrak{g}(Q)$	the Kac-Moody algebra associated to a quiver Q.
$\operatorname{gr}\mathcal{A}$	the associated graded vector space of a filtered vector space $\mathcal{A}$.
$\mathrm{Irr}(\mathcal{C})$	the set of simple objects in an abelian category $\mathcal{C}$.
$K_0(\mathcal{C})$	the (complexified) Grothendieck group of an abelian category $\mathcal{C}$.
L_ω	the irreducible integrable representation of $\mathfrak{g}(Q)$ with highest weight ω.
$L_\omega[\nu]$	the ν-weight space in L_ω.
$\mathrm{Loc}^\theta_\lambda$	localization functor $\mathcal{A}_\lambda(v)\text{-mod}\to\mathcal{A}^\theta_\lambda(v)\text{-mod}$.
$L\pi^0_\lambda(v)^!$	the derived left adjoint to $\pi^0_\lambda(v)$.
$\mathcal{M}^\theta_\lambda(v)$	$:=\mu^{-1}(\lambda)^{\theta-ss}/\!/G$.
$\mathcal{M}_\lambda(v)$	$:=\mathrm{Spec}(\mathbb{C}[\mathcal{M}^\theta_\lambda(v)])$ for generic θ.
$\mathfrak{p}$	$:=\mathbb{C}^{Q_0}$, the parameter space for classical reduction.
$\mathfrak{P}$	$:=\mathbb{C}^{Q_0}$, the parameter space for quantum reduction.
$\mathfrak{P}^{iso}$	the locus of $\lambda\in\mathfrak{P}$ such that $\mathcal{A}^0_\lambda(v)\xrightarrow{\sim}\mathcal{A}_\lambda(v)$.
$\mathfrak{P}^{ISO}$	the locus of $\lambda\in\mathfrak{P}^{iso}$ of all λ with $\mathrm{Tor}^i_{\mathbb{C}[\mathfrak{P}]}(D(R),\mathbb{C}_\lambda)=0$ for $i>0$.
$\mathcal{Q}_\lambda$	$:=D(R)/D(R)\{\Phi(x)-\langle\lambda,x\rangle, x\in\mathfrak{g}\}$.
R	$(=R(Q,v,w)):=\bigoplus_{a\in Q_1}\mathrm{Hom}(V_{t(a)},V_{h(a)})\oplus\bigoplus_{k\in Q_0}\mathrm{Hom}(V_k,W_k)$, the coframed representation space of a quiver Q with dimension v and framing w.
$R_\hbar(\mathcal{A})$	$:=\bigoplus_{i\in\mathbb{Z}}\hbar^i\,\mathrm{F}_i\,\mathcal{A}$:the Rees $\mathbb{C}[\hbar]$-module of a filtered vector space $\mathcal{A}$.
$\mathfrak{S}_n$	the symmetric group on n letters.
$S(V)$	the symmetric algebra of a vector space V.
$\mathrm{Supp}(M)$	the support of the module/sheaf of modules M.
$\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B})$	$:=\{\lambda\in\mathfrak{P}\mid\mathcal{B}\otimes_{\mathbb{C}[\mathfrak{P}]}\mathbb{C}_\lambda\neq\{0\}$.
$W(Q)$	the Weyl group of $\mathfrak{g}(Q)$.
$\mathfrak{WC}_{\lambda\to\lambda'}$	a wall-crossing functor.
$\tilde{w}_i$	$:=w_i+\sum_{a,t(a)=i}v_{h(a)}$ (for a source $i\in Q_0$).

$X^{\theta-ss}$	the open locus of θ-semistable points for an action of a reductive group G on an affine algebraic variety X, where θ is a character of G.
$X^{\theta-uns}$	$:= X \setminus X^{\theta-ss}$.
$x \cdot y$	$\sum_{i\in Q_0} x_i y_i$.
(x,y)	$= 2\sum_{k\in Q_0} x_k y_k - \sum_{a\in Q_1}(x_{t(a)}y_{h(a)} + x_{h(a)}y_{t(a)})$, the symmetrized Tits form.
Γ_n	$= \mathfrak{S}_n \ltimes \Gamma_1^n$ for a finite subgroup $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$.
Γ^θ_λ	global section functor $\mathcal{A}^\theta_\lambda(v)$-mod $\to \mathcal{A}_\lambda(v)$-mod.
μ	the moment map $T^*R \to \mathfrak{g}$.
ν	the weight of $\mathfrak{g}(Q)$ determined from dimension vector v and framing w.
$\pi^0_\lambda(v)$	the natural functor $D(R)$-mod$^{G,\lambda} \to \mathcal{A}^0_\lambda(v)$-mod (or its derived analog).
$\pi^\theta_\lambda(v)$	the natural functor D_R-mod$^{G,\lambda} \to \mathcal{A}^\theta_\lambda(v)$-mod (or its derived analog).
ρ	the natural projective morphism $\mathcal{M}^\theta_\lambda(v) \to \mathcal{M}^0_\lambda(v)$ or $\mathcal{M}^\theta_\lambda(v) \to \mathcal{M}_\lambda(v)$.
$\sigma \bullet v$	dimension vector corresponding to $\sigma\nu, \sigma \in W(Q)$.
$\sigma \bullet^v \lambda$	the quantization parameter determined by $\sigma \in W(Q), v \in \mathbb{Z}^{Q_0}_{\geqslant 0}, \lambda \in \mathfrak{P}$.
ω	the dominant weight of $\mathfrak{g}(Q)$ determined from framing w.

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2. Preliminaries on quiver varieties and their quantizations

2.1. **Properties of quiver varieties.**

2.1.1. *Generic parameters.* First of all, let us recall the description of generic values of (λ,θ) (i.e. such that the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ is free) due to Nakajima, [Nak1]. Namely, (λ,θ) is generic when there is no $v' \in \mathbb{Z}^{Q_0}_{\geqslant 0}$ such that

- $v' \leqslant v$ (component-wise),
- $\sum_{i\in Q_0} v'_i \alpha^i$ is a root for $\mathfrak{g}(Q)$,
- and $v' \cdot \theta = v' \cdot \lambda = 0$ (where we write $v' \cdot \lambda$ for $\sum_{i\in Q_0} v'_i\lambda_i$).

We say that λ (resp., θ) is generic if $(\lambda, 0)$ (resp., $(0,\theta)$) is generic. The set of non-generic λ's will be denoted by $\mathfrak{p}^{sing}$ (or $\mathfrak{p}^{sing}(v)$ when we need to indicate the dependence on v).

We note that by results of Crawley-Boevey, [CB1, Section 1, Remarks], $\mathcal{M}^\theta_\lambda(v)$ is connected when (λ,θ) is generic.

2.1.2. *Line bundles.* For a character χ of G, we consider the line bundle $\mathcal{O}(\chi)$ on $\mathcal{M}^\theta(v)$ whose sections are given by

$$\Gamma(U, \mathcal{O}(\chi)) = \mathbb{C}[\pi^{-1}(U)]^{G,\chi} := \{f \in \mathbb{C}[\pi^{-1}(U)] | g.f = \chi(g)f, \forall g \in G\}.$$

Here $U \subset \mathcal{M}^\theta(v)$ is an affine open subset, and π stands for the quotient morphism $\mu^{-1}(0)^{\theta-ss} \to \mathcal{M}^\theta(v)$. By the very definition, $\mathcal{O}(\theta)$ is an ample line bundle.

2.1.3. *LMN isomorphisms.* Now let us discuss certain isomorphisms of quiver varieties. For $\sigma \in W(Q)$, we have an isomorphism $\mathcal{M}^\theta_\lambda(v) \cong \mathcal{M}^{\sigma\theta}_{\sigma\lambda}(\sigma \bullet v)$, where we assume that (λ, θ) is generic. Here we write $\sigma \bullet v$ for the dimension vector that produces the weight $\sigma\nu$ by (1.2). For a simple reflection $\sigma = s_k$ we have $(s_k \bullet v)_\ell = v_\ell$ for $\ell \neq k$ and $(s_k \bullet v)_k = w_k + \sum_{a,t(a)=k} v_{h(a)} + \sum_{a,h(a)=k} v_{t(a)} - v_k$.

The existence of such isomorphisms was conjectured by Nakajima in [Nak1] and first proved by Maffei in [Ma] and, independently, by Nakajima, [Nak3], a closely related construction was found by Lusztig, [Lu]. So we call those *LMN isomorphisms.*

Below we will need a (slightly rephrased) construction of LMN isomorphisms due to Maffei. Let us construct an isomorphism corresponding to a simple reflection $s_i \in W(Q)$. We may assume that the vertex i is a source and, since (λ, θ) is generic, that either $\lambda_i \neq 0$ or $\theta_i > 0$ (if $\theta_i < 0$, then we just construct an isomorphism for $s\theta$). Let

$$\tilde{W}_i := W_i \oplus \bigoplus_{a,t(a)=i} V_{h(a)}, \tilde{w}_i := \dim \tilde{W}_i \tag{2.1}$$

so that $v_i + (s_i \bullet v)_i = \tilde{w}_i$. Set

$$\underline{R}(= \underline{R}^i) := \bigoplus_{a,t(a)\neq i} \operatorname{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{j\neq i} \operatorname{Hom}(V_j, W_j), \underline{G} := \prod_{j \neq i} \operatorname{GL}(V_j), \tag{2.2}$$

so that $R = \underline{R} \oplus \operatorname{Hom}(V_i, \tilde{W}_i)$. Consider the Hamiltonian reduction

$$T^*R /\!\!/\!\!/^{\theta_i}_{\lambda_i} \operatorname{GL}(v_i) = T^* \operatorname{Hom}(V_i, \tilde{W}_i) /\!\!/\!\!/^{\theta_i}_{\lambda_i} \operatorname{GL}(v_i) \times T^*\underline{R}.$$

Let us remark that if $\lambda_i = 0$, the reduction $T^* \operatorname{Hom}(V_i, \tilde{W}_i) /\!\!/\!\!/^{\theta_i}_{\lambda_i} \operatorname{GL}(v_i)$ is just $T^* \operatorname{Gr}(v_i, \tilde{w}_i)$.

An easy special case of Maffei's construction is an isomorphism

$$T^* \operatorname{Hom}(V_i, \tilde{W}_i) /\!\!/\!\!/^{\theta_i}_{\lambda_i} \operatorname{GL}(v_i) \xrightarrow{\sim} T^* \operatorname{Hom}(\tilde{W}_i, V_i') /\!\!/\!\!/^{-\theta_i}_{-\lambda_i} \operatorname{GL}(v_i'),$$

where $v_i' = (s_i \bullet v)_i = \tilde{w}_i - v_i$ and V_i' is a vector space of dimension v_i'. When $\lambda_i = 0$, we just have two realizations of $T^* \operatorname{Gr}(v_i, \tilde{w}_i)$ (where $\operatorname{Gr}(v_i, \tilde{w}_i)$ is thought as the variety of v_i-dimensional subspaces in $\mathbb{C}^{\tilde{w}_i}$ and as the variety of $\tilde{w}_i - v_i$-dimensional quotients), while for $\lambda_i \neq 0$, we get two equal twisted cotangent bundles on the Grassmanian. These isomorphisms are clearly symplectomorphisms, $\mathbb{C}^\times$-equivariant when $\lambda_i = 0$.

As a consequence, we get a $\underline{G}$-equivariant symplectomorphism

$$T^*R /\!\!/\!\!/^{\theta_i}_{\lambda_i} \operatorname{GL}(v_i) \xrightarrow{\sim} T^*R' /\!\!/\!\!/^{-\theta_i}_{-\lambda_i} \operatorname{GL}(v_i'), \tag{2.3}$$

where $R' := \operatorname{Hom}(\tilde{W}_i, V_i') \oplus \underline{R}$. According to [Ma, Section 3.1], this isomorphism does not intertwine the moment maps for the $\underline{G}$-actions. Rather, if $\underline{\mu}, \underline{\mu}'$ are the two moment maps, then $\underline{\mu} - \lambda = \underline{\mu}' - s_i\lambda$. The isomorphism does not intertwine the stability conditions either, instead it maps $(T^*R)^{\theta-ss} /\!\!/\!\!/_{\lambda_i} \operatorname{GL}(v_i)$ to $(T^*R')^{s_i\theta-ss} /\!\!/\!\!/_{-\lambda_i} \operatorname{GL}(v_i')$. So, by reducing the $\underline{G}$-action, we do get a symplectomorphism $\mathcal{M}^\theta_\lambda(v) \xrightarrow{\sim} \mathcal{M}^{s_i\theta}_{s_i\lambda}(s_i \bullet v)$. This isomorphism is $\mathbb{C}^\times$-equivariant, if $\lambda = 0$.

We will need a compatibility of the LMN isomorphisms with certain T-actions. Namely, the torus $T := (\mathbb{C}^\times)^{Q_1} \times (\mathbb{C}^\times)^{Q_0}$ naturally acts on R (the copy of $\mathbb{C}^\times$ corresponding to an arrow a acts by scalars on $\operatorname{Hom}(V_{t(a)}, V_{h(a)})$, the copy corresponding to $i \in Q_0$ acts on $\operatorname{Hom}(V_i, W_i)$). The lift of this T-action to T^*R commutes with G and preserves the moment map and so descends to $\mathcal{M}^\theta(v,w)$. An isomorphism s_i is T-equivariant, [We2, Proposition 4.13].

We will also need to understand the behaviour of line bundles under the LMN isomorphism. Namely, by tracking the construction, we see that s_i maps the line bundle $\mathcal{O}(\chi)$ to $\mathcal{O}(s_i\chi)$.

2.1.4. *Properties of $\mathcal{M}^0(v)$.* Now let us turn to the affine quiver varieties $\mathcal{M}^0(v)$. In [CB1] Crawley-Boevey found a combinatorial criterium on v for μ to be flat. Let us state this criterium. Recall the symmetrized Tits form $(\cdot,\cdot)$ for Q: $(v^1,v^2):=2\sum_{k\in Q_0}v^1_kv^2_k-\sum_a(v^1_{t(a)}v^2_{h(a)}+v^1_{h(a)}v^2_{t(a)})$. We set $p(v):=1-\frac{1}{2}(v,v)$ (so v is a root, then $p(v)\geqslant 0$). According to [CB1, Theorem 1.1], the map μ is flat if and only if

$$p(v)+w\cdot v-(w\cdot v^0+\sum_{i=0}^k p(v^i))\geqslant 0 \tag{2.4}$$

for all decompositions $v=v^0+\ldots+v^k$, equivalently, for the decompositions, where $v^1,\ldots,v^k$ are roots. Also if all inequalities for proper decompositions in (2.4) are strict, then $\mu^{-1}(0)$ is irreducible and contains a free closed orbit, [CB1, Theorem 1.2].

We want to analyze condition (2.4) in the case when Q is finite or affine and ν is dominant. When Q is finite, then $p(v^i)=0$ for all $i>0$ and so the left hand side becomes $\frac{1}{2}((v^0,v^0)-(v,v))+w\cdot(v-v^0)=(\nu-\nu^0,\frac{1}{2}(\nu+\nu^0))=\frac{1}{2}(\nu-\nu^0,\nu-\nu^0)+(\nu,\nu^0-\nu)$. Here, in the second and the third expressions, $(\cdot,\cdot)$ is the usual form on $\mathfrak{h}^*$. The first summand is positive if $v\neq v^0$, while the second is non-negative. We conclude that $\mu^{-1}(0)$ is irreducible and contains a free closed orbit.

Now consider the case when Q is affine. Here $p(v^i)=1$ if $v^i=a_i\delta$ and $p(v^i)=0$ else, for $i>0$. The left hand side of (2.4) is minimized when all $a_i=1$ and we will assume this. So the left hand side becomes

$$\frac{1}{2}(\nu-\nu^0,\nu-\nu^0)+(\nu,\nu-\nu^0)-s=\frac{1}{2}(\nu-\nu^0,\nu-\nu^0)+(\nu,\nu-\nu^0-s\delta)+s((\omega,\delta)-1),$$

where $v\geqslant v^0+s\delta$. The first summand is non-negative, it equals 0 if and only if $v-v^0$ is a multiple of δ. The second summand is non-negative, it equals 0 if and only if $\nu=\nu^0+s\delta$. Finally, the third summand is nonnegative, it is 0 if and only if $(\omega,\delta)=1$. So we see that μ is flat. The subvariety $\mu^{-1}(0)$ is irreducible and contains a free closed orbit provided $(\omega,\delta)>1$.

2.1.5. *Families.* Set $\mathfrak{p}:=\mathbb{C}^{Q_0}\cong(\mathfrak{g}^*)^G$ and consider the varieties $\mathcal{M}^0_{\mathfrak{p}}(v):=\mu^{-1}(\mathfrak{g}^{*G})//G$, $\mathcal{M}^\theta_{\mathfrak{p}}(v):=\mu^{-1}(\mathfrak{g}^{*G})^{\theta-ss}//G, \mathcal{M}_{\mathfrak{p}}(v):=\operatorname{Spec}(\mathbb{C}[\mathcal{M}^\theta_{\mathfrak{p}}(v)])$.

For a vector subspace $\mathfrak{p}_0\subset\mathfrak{p}$, we consider the specializations $\mathcal{M}^0_{\mathfrak{p}_0}(v):=\mathfrak{p}_0\times_{\mathfrak{p}}\mathcal{M}^0_{\mathfrak{p}}(v)$, $\mathcal{M}^\theta_{\mathfrak{p}_0}(v),\mathcal{M}_{\mathfrak{p}_0}(v)$.

2.1.6. *Structure of formal neighborhoods.* Pick a point $x\in\mathcal{M}^0_{\mathfrak{p}}(v)$. We need a description of the formal neighborhood $\mathcal{M}^0_{\mathfrak{p}}(v)^{\wedge_x}$ and of the scheme $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_x}:=\mathcal{M}^0_{\mathfrak{p}}(v)^{\wedge_x}\times_{\mathcal{M}^0_{\mathfrak{p}}(v)}\mathcal{M}^\theta_{\mathfrak{p}}(v)$. The description is due to Nakajima, [Nak1, Section 6].

Let $r\in T^*R$ be a point with closed G-orbit mapping to x. Then r is a semisimple representation of the following quiver $\overline{Q}^w$. We first adjoin the vertex ∞ to Q and connect each vertex $i\in Q_0$ to ∞ with w_i arrows. Then we add an opposite arrow to each existing arrow of Q^w. The dimension of r is $(v,1)$. Let us decompose r into the sum $r=r_0\oplus r_1\otimes U_1\oplus\ldots\oplus r_k\otimes U_k$, where r_0 is an irreducible representation with dimension vector of the form $(v^0,1)$, $r_1,\ldots,r_k$ are pairwise non-isomorphic irreducible representations with dimensions $(v^i,0), i=1,\ldots,k$, and U_i is the multiplicity space of r_i. In particular, the stabilizer G_r of r is $\prod_{i=1}^k \mathrm{GL}(U_i)$.

Let us define a new quiver $\hat{Q}$, a dimension vector $\hat{v}$ and a framing $\hat{w}$. For the set of vertices $\hat{Q}_0$ we take $\{1,\ldots,k\}$ and we set $\hat{v}=(\dim U_i)_{i=1}^k$. The number of arrows between $i,j\in\{1,\ldots,k\}$ is determined as follows. The subspace $T_r(Gr)\subset T_r(T^*R)$ is contained in its skew-orthogonal complement $T_r(Gr)^\angle$. So we get a symplectic G_r-module $T_r(Gr)^\angle/T_r(Gr)$. We want the G_r-module T^*R_x, where we write R_x for $R(\hat{Q},\hat{v},\hat{w})$, to be isomorphic to $T_r(Gr)^\angle/T_r(Gr)$. So $T^*R_x\oplus T^*(\mathfrak{g}/\mathfrak{g}_r)=T^*R$.

For $i\neq j$, the multiplicity of the G_r-module $\mathrm{Hom}(U_i,U_j)$ in T^*R equals $\sum_a (v^i_{t(a)}v^j_{h(a)}+v^j_{t(a)}v^i_{h(a)})$, while the multiplicity in $T^*(\mathfrak{g}/\mathfrak{g}_r)$ equals $2\sum_{k\in Q_0} v^i_k v^j_k$. So the multiplicity of $\mathrm{Hom}(U_i,U_j)$ in the G_r-module T^*R_x has to be equal to $-(v^i,v^j)$ if $i\neq j$ and to $2-(v^i,v^i)$ if $i=j$. Hence the number of arrows between i and j in $\hat{Q}$ has to be $-(v^i,v^j)$ if $i\neq j$ and $p(v^i)=1-\frac{1}{2}(v^i,v^i)$ if $i=j$. Similarly, for $\hat{w}_i$ we need to take $w\cdot v^i-(v^0,v^i)$. Finally, we need to add some loops at ∞ but those are just spaces with trivial action of G_r. We will treat them separately: so the symplectic part of the slice module at r can be written as $T^*R_x\oplus R_0$, where $R_x=R(\hat{Q},\hat{v},\hat{w})$ and R_0 is a symplectic vector space with trivial action of G_r. We choose an orientation on $\hat{Q}$ in such a way that the G_r-modules $R_x\oplus\mathfrak{g}/\mathfrak{g}_r$ and R are isomorphic up to a trivial summand. We remark, however, that this choice may violate the condition that the vertex ∞ (corresponding to r^0) in $\hat{Q}$ is a source.

Consider the homogeneous vector bundle $G*_{G_r}(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)$. The symplectic form on the latter comes from a natural identification of that homogenous bundle with $[T^*G\times(T^*R_x\oplus R_0)]/\!/\!/_0 G_r$ (the action of G_r is diagonal with G_r acting on T^*G from the right). The moment map on the homogeneous bundle is given by $[g,(\alpha,\beta,\beta_0)]\mapsto \mathrm{Ad}(g)(\alpha+\hat{\mu}(\beta))$. Here $[g,(\alpha,\beta,\beta_0)]$ stands for the class in $G*_{G_r}(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)$ of a point $(g,\alpha,\beta,\beta_0)\in G\times(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)$, and $\hat{\mu}:T^*R_x\to\mathfrak{g}_r$ is the moment map.

Let $\pi:T^*R\to T^*R/\!/G$ and $\pi':G*_{G_r}(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)\to(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)/\!/G_r$ denote the quotient morphisms. The symplectic slice theorem (see, for example, [L1] where analytic neighborhoods instead of formal ones were used) asserts that there is an isomorphism of formal neighborhoods U of $\pi(r)$ in $T^*R/\!/G$ and U' of $\pi'([1,(0,0,0)])$ in $(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)/\!/G_r$ that lifts to a G-equivariant symplectomorphism $\pi^{-1}(U)\cong\pi'^{-1}(U')$ intertwining the moment maps.

So we see that (compare with [Nak1, Section 6])

$$\mathcal{M}^0_{\mathfrak{p}}(v)^{\wedge_x}=\hat{\mathcal{M}}^0_{\mathfrak{p}}(\hat{v})^{\wedge_0}\times R_0^{\wedge_0} \tag{2.5}$$

(an equality of formal Poisson schemes). Here we use the following conventions. The superscript $\bullet^{\wedge_x}$ means the completion near x. We have the restriction map $\mathfrak{p}=\mathfrak{g}^{*G}\to\hat{\mathfrak{p}}=\mathfrak{g}_r^{*G_r}$. We set $\hat{\mathcal{M}}^0_{\mathfrak{p}}(\hat{v}):=\mathfrak{p}\times_{\hat{\mathfrak{p}}}\hat{\mathcal{M}}^0_{\hat{\mathfrak{p}}}(\hat{v})$.

We have a similar decomposition for smooth quiver varieties. First, observe that

$$G*_{G_r}(\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x\oplus R_0)^{\theta-ss}=G*_{G_r}([\mathfrak{g}/\mathfrak{g}_r\oplus T^*R_x]^{\theta-ss}\oplus R_0),$$

where in the right hand side we slightly abuse the notation and write θ for the restriction of θ to G_r. From here it follows that

$$\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_x}=(\hat{\mathcal{M}}^\theta_{\mathfrak{p}}(\hat{v})\times R_0)^{\wedge_0}, \tag{2.6}$$

where, recall, by definition, $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_x}:=\mathcal{M}^0_{\mathfrak{p}}(v)^{\wedge_x}\times_{\mathcal{M}^0_{\mathfrak{p}}(v)}\mathcal{M}^\theta_{\mathfrak{p}}(v)$.

Moreover, by the construction, the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}^\theta(v)^{\wedge_x} \arrow[r, "\cong"] \arrow[d, "\rho"] & (\hat{\mathcal{M}}^\theta(\hat{v}) \times R_0)^{\wedge_0} \arrow[d, "\hat{\rho}\times \mathrm{id}"] \\
\mathcal{M}^0(v)^{\wedge_x} \arrow[r, "\cong"] & \hat{\mathcal{M}}(\hat{v})^{\wedge_0} \times R_0^{\wedge_0}
\end{tikzcd}$$

Let us finish this discussion with two remarks.

Remark 2.1. It is not true that $(\hat{\mathcal{M}}^\theta_{\mathfrak{p}}(\hat{v}) \times R_0)^{\wedge_0}$ coincides with the product of schemes $\hat{\mathcal{M}}^\theta_{\mathfrak{p}}(\hat{v})^{\wedge_0} \times R_0^{\wedge_0}$. In order to get a product decomposition, we need to work with formal neighborhoods (that are formal schemes rather than schemes): $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_{\rho^{-1}(x)}}$ of $\rho^{-1}(x)$ in $\mathcal{M}^\theta_{\mathfrak{p}}(v)$ and $\hat{\mathcal{M}}^\theta_{\mathfrak{p}}(\hat{v})^{\wedge_{\hat{\rho}^{-1}(0)}} \times R_0^{\wedge_0}$. These two formal schemes are isomorphic, this follows from (2.6).

Remark 2.2. Let us write ϖ for the restriction map $\mathfrak{p} \to \hat{\mathfrak{p}}$. It follows from (2.5) that $\varpi^{-1}(\hat{\mathfrak{p}}^{sing}) \subset \mathfrak{p}^{sing}$.

2.1.7. *Resolution of singularities.*

Proposition 2.3. *The morphism $\rho: \mathcal{M}^\theta(v) \to \mathcal{M}(v)$ is a resolution of singularities.*

Proof. Fix a generic λ and consider the varieties $\mathcal{M}^\theta_{\mathbb{C}\lambda}(v)$ and $\mathcal{M}^0_{\mathbb{C}\lambda}(v)$. Both are schemes over $\mathbb{C}\lambda$. We have a natural morphism $\phi_{\mathbb{C}\lambda}: \mathcal{M}^\theta_{\mathbb{C}\lambda}(v) \to \mathcal{M}^0_{\mathbb{C}\lambda}(v)$ that is an isomorphism over $\mathbb{C}^\times\lambda$. Note that all components of $\mathcal{M}^\theta_{\mathbb{C}\lambda}(v)$ have dimension $\dim T^*R - 2\dim G + 1$.

Let $\bar{\mathcal{M}}_{\mathbb{C}\lambda}(v)$ be the image of $\phi_{\mathbb{C}\lambda}$, this is a closed subvariety in $\mathcal{M}^0_{\mathbb{C}\lambda}(v)$ because $\phi_{\mathbb{C}\lambda}$ is projective. So it coincides with the closure of the preimage of $\mathbb{C}^\times\lambda$ and has dimension $\dim \mathcal{M}^\theta(v) + 1$. Hence the fiber $\bar{\mathcal{M}}^0(v)$ of $\bar{\mathcal{M}}^0_{\mathbb{C}\lambda}(v)$ over 0 has dimension $\dim \mathcal{M}^\theta(v)$ and admits a surjective projective morphism from $\mathcal{M}^\theta(v)$. Applying the Stein decomposition to this morphism we decompose it to the composition of $\rho: \mathcal{M}^\theta(v) \to \mathcal{M}(v)$ and some finite dominant morphism $\mathcal{M}(v) \to \bar{\mathcal{M}}^0(v)$. So ρ has to be a resolution of singularities. $\square$

Corollary 2.4. *The following is true.*

1. *The higher cohomology of $\mathcal{O}_{\mathcal{M}^\theta(v)}$ vanish.*
2. *The algebra $\mathbb{C}[\mathcal{M}^\theta(v)]$ is the specialization to 0 of $\mathbb{C}[\mathcal{M}^\theta_{\mathfrak{p}}(v)]$.*
3. *The algebra $\mathbb{C}[\mathcal{M}^\theta(v)]$ coincides with the associated graded of $\mathbb{C}[\mathcal{M}^0_\lambda(v)]$ for a generic λ and, in particular, is independent of θ.*
4. *The variety $\mathcal{M}(v)$ is Cohen-Macaulay.*

Proof. (1) is a corollary of the Grauert-Riemenschneider theorem, (2) is a corollary of (1), and (3) is a corollary of (2). (4) follows because $\mathcal{M}(v)$ has rational singularities. $\square$

Proposition 2.5. *Suppose μ is flat. Then $\rho^*: \mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)] \to \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]$ is an isomorphism.*

Proof. It is enough to show that $\rho^*: \mathbb{C}[\mathcal{M}^0(v)] \to \mathbb{C}[\mathcal{M}(v)]$ is an isomorphism because $\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)], \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]$ are graded free over $\mathbb{C}[\mathfrak{p}]$ and $\mathbb{C}[\mathcal{M}_0(v)] = \mathbb{C}[\mathcal{M}^{\mathfrak{p}}_0(v)]/(\mathfrak{p}), \mathbb{C}[\mathcal{M}(v)] = \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]/(\mathfrak{p})$. Now note that both $\mathbb{C}[\mathcal{M}_0(v)], \mathbb{C}[\mathcal{M}(v)]$ are identified with the associated graded of $\mathbb{C}[\mathcal{M}^0_\lambda(v)] = \mathbb{C}[\mathcal{M}_\lambda(v)]$ for λ generic and, under this identification, ρ^* becomes the identity. $\square$

2.1.8. *Identification of homology.* The purpose of this part is to establish an identification of the homology groups $H_*(\mathcal{M}^\theta_\lambda(v))$ for different generic (λ,θ).

First, there is a classical way to produce the identification, [Nak1, Section 9]. We can view θ as an element in $\mathbb{R}^{Q_0}$, in this case we define $\mathcal{M}^\theta_\lambda(v)$ as a hyper-Kähler reduction. We get the same varieties as before, the complex structure on $\mathcal{M}^\theta_\lambda(v)$ depends only on the chamber of θ. As we have mentioned in Section 1.4, this shows that all varieties $\mathcal{M}^\theta_\lambda(v)$ with generic (λ,θ) are diffeomorphic as C^∞-manifolds. Consider the generic locus and a bundle $H_*(\mathcal{M}^\theta_\lambda(v))$ on this locus. This is a flat bundle with respect to the Gauss-Manin connection. But the generic locus of (λ,θ) is simply connected so the connection is trivial. Therefore all fibers are canonically identified.

We will need a slightly different description. Pick a generic parameter $\lambda\in\mathfrak{p}$. We remark that the variety $\mathcal{M}^\theta_\lambda(v)$ is independent of θ, it is actually affine. Let D denote the line through λ. The inclusions $\mathcal{M}^\theta(v)\hookrightarrow\mathcal{M}^\theta_D(v),\mathcal{M}^\theta_\lambda(v)\hookrightarrow\mathcal{M}^\theta_D(v)$ induce maps $H^*(\mathcal{M}^\theta_D(v))\to H^*(\mathcal{M}^\theta(v)),H^*(\mathcal{M}^\theta_D(v))\to H^*(\mathcal{M}^\theta_\lambda(v))$. The former is an isomorphism because $\mathcal{M}^\theta_D(v)$ gets contracted to $\mathcal{M}^\theta(v)$ by a $\mathbb{C}^\times$-action. The latter is also an isomorphism because the resulting map $H^*(\mathcal{M}^\theta(v))\to H^*(\mathcal{M}^\theta_\lambda(v))$ is precisely the identification in the previous paragraph.

2.2. **Properties of quantizations.** In this section, we describe some properties of the algebras $\mathcal{A}_\lambda(v)$ and $\mathcal{A}^0_\lambda(v)$.

2.2.1. *Filtrations.* The algebras $\mathcal{A}_\lambda(v),\mathcal{A}^0_\lambda(v):=[D(R)/D(R)\{\Phi(x)-\langle\lambda,x\rangle\}]^G$ can be filtered in different ways, depending on a filtration on $D(R)$ we consider. First of all, there is the *Bernstein filtration* on $\mathcal{A}_\lambda(v),\mathcal{A}^0_\lambda(v)$ that is induced from the eponymous filtration on $D(R)$ (where $\deg R=\deg R^*=1$). Let us write $\mathrm{F}_i\,\mathcal{A}_\lambda(v)$ for the ith filtration component with respect to this filtration. Note that $[\mathrm{F}_i\,\mathcal{A}_\lambda(v),\mathrm{F}_j\,\mathcal{A}_\lambda(v)]\subset\mathrm{F}_{i+j-2}\,\mathcal{A}_\lambda(v)$.

Sometimes, it will be more convenient for us to work with filtrations, where the commutator decreases degrees by 1. Namely, equip $D(R)$ with the filtration by the order of differential operator (where $\deg R^*=0,\deg R=1$). We have induced filtrations on $\mathcal{A}_\lambda(v),\mathcal{A}^0_\lambda(v)$ to be denoted by F^Q_i (the superscript indicates that these filtrations depend on the orientation). Note that $[\mathrm{F}^Q_i\,\mathcal{A}_\lambda(v),\mathrm{F}^Q_j\,\mathcal{A}_\lambda(v)]\subset\mathrm{F}^Q_{i+j-1}\,\mathcal{A}_\lambda(v)$.

The two filtrations are related to each other. Namely, let eu denote the Euler vector field in $D(R)$. Since this element is G-invariant, it descends to $\mathcal{A}_\lambda(v),\mathcal{A}^0_\lambda(v)$, we denote the images again by eu. So we can consider the inner $\mathbb{Z}$-gradings on the algebras of interest by eigenvalues of $[\mathsf{eu},\cdot]$, let us write $\mathcal{A}_\lambda(v):=\bigoplus_i\mathcal{A}_\lambda(v)_i$ for these gradings. The gradings are compatible with the filtrations $\mathrm{F}_i,\mathrm{F}^Q_i$ and we have

$$\mathrm{F}_{[i/2]}\,\mathcal{A}_\lambda(v)=\bigoplus_{k\in\mathbb{Z}}\mathrm{F}^Q_k\,\mathcal{A}_\lambda(v)_{i-k}.$$

Thanks to this equality, the associated graded for the two filtrations are the same.

The same considerations apply to $\mathcal{A}^0_\lambda(v)$.

In Section 1.4 we have mentioned that $\operatorname{gr}\mathcal{A}_\lambda(v)=\mathbb{C}[\mathcal{M}(v)]$ and $H^i(\mathcal{A}^\theta_\lambda(v))=0$. This is because $\operatorname{gr}\mathcal{A}^\theta_\lambda(v)=\mathcal{O}_{\mathcal{M}^\theta(v)}$ and $H^i(\mathcal{O}_{\mathcal{M}^\theta(v)})=0$ for $i>0$.

2.2.2. $\mathcal{A}^0_\lambda(v)$ *vs* $\mathcal{A}_\lambda(v)$*, I.* Now we want to relate the algebra $\mathcal{A}_\lambda(v)$ to $\mathcal{A}^0_\lambda(v)$. We have a natural epimorphism $\mathbb{C}[\mathcal{M}^0(v)]\twoheadrightarrow\operatorname{gr}\mathcal{A}^0_\lambda(v)$ to be denoted by η. Besides, we have a natural homomorphism $\kappa:\mathcal{A}^0_\lambda(v)\to\mathcal{A}_\lambda(v)$ coming to restricting elements of $D(R)$ to $(T^*R)^{\theta-ss}$. It is clear that $\rho^*:\mathbb{C}[\mathcal{M}^0(v)]\to\mathbb{C}[\mathcal{M}^\theta(v)]$ coincides with the composition

$\operatorname{gr}\kappa\circ\eta : \mathbb{C}[\mathcal{M}^0(v)] \twoheadrightarrow \operatorname{gr}\mathcal{A}_\lambda(v)$. It follows that $\mathcal{A}^0_\lambda(v) = \mathcal{A}_\lambda(v)$ and $\operatorname{gr}\mathcal{A}^0_\lambda(v) = \mathbb{C}[\mathcal{M}^0(v)]$ when μ is flat. In particular, $\mathcal{A}_\lambda(v)$ is independent of θ in this case (this is also true in general by [BPW, Proposition 3.8]).

2.2.3. *Variations.* Now let us consider some variations of the notion of a filtered quantization. We can consider the quantization

$$\mathcal{A}^\theta_{\mathfrak{P}}(v) = D_R/\!\!/\!\!/^\theta G := [(D_R/D_R\{x_R, x\in[\mathfrak{g},\mathfrak{g}]\})|_{T^*R^{\theta-ss}}]^G$$

of $\mathcal{M}^\theta_{\mathfrak{p}}(v)$ and its global section $\mathcal{A}_{\mathfrak{P}}(v)$. For an affine subspace $\mathfrak{P}_0\subset\mathfrak{P}$, we consider pullbacks $\mathcal{A}^\theta_{\mathfrak{P}_0}(v) := \mathbb{C}[\mathfrak{P}_0]\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{A}^\theta_{\mathfrak{P}}(v), \mathcal{A}_{\mathfrak{P}_0}(v) := \mathbb{C}[\mathfrak{P}_0]\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{A}_{\mathfrak{P}}(v)$. Those are quantizations of $\mathcal{M}^\theta_{\mathfrak{p}_0}(v), \mathcal{M}_{\mathfrak{p}_0}(v)$, where $\mathfrak{p}_0\subset\mathfrak{p}$ is the vector subspace corresponding to $\mathfrak{P}_0$. Note that $\mathcal{A}_{\mathfrak{P}_0}(v) = \Gamma(\mathcal{A}^\theta_{\mathfrak{P}_0}(v))$. It also makes sense to speak about $\mathcal{A}^0_{\mathfrak{P}_0}(v)$.

We also consider homogenized versions. Namely, we take the Rees sheaf $D_{R,\hbar}$ of D_R (for the filtration by the order of a differential operator) and its reduction $\mathcal{A}^\theta_{\mathfrak{P}}(v)_\hbar$, it is related to $\mathcal{A}^\theta_{\mathfrak{P}}(v)$ via $\mathcal{A}^\theta_{\mathfrak{P}}(v) = \mathcal{A}^\theta_{\mathfrak{P},\hbar}(v)/(\hbar-1)$. Also consider the global sections $\mathcal{A}_{\mathfrak{P}}(v)_\hbar$. This is a graded (with positive grading) deformation of $\mathbb{C}[\mathcal{M}(v)]$ over the space $\mathfrak{p}\oplus\mathbb{C}$. Here we consider the grading coming from the action of $\mathbb{C}^\times$ on T^*R by dilations: $t.(r,\alpha) = (t^{-1}r, t^{-1}\alpha)$ so that the parameter space $\mathfrak{p}\oplus\mathbb{C}$ is in degree 2.

2.2.4. *Quantized LMN isomorphisms.* The LMN isomorphisms discussed in 2.1.3 can be quantized. This was done in [L6] in a special case (but the construction generalizes in a straightforward way). In fact, the quantum isomorphisms can be obtained by the same reduction in stages construction as before. One either quantizes the steps of that argument or argues similarly to [L6, Section 6.4]: for $\theta_i > 0$, isomorphism (2.3) can be regarded as an isomorphism of symplectic schemes $\mathcal{X} := T^*R/\!\!/\!\!/^{\theta_k}\operatorname{GL}(v_k), \mathcal{X}' := T^*R'/\!\!/\!\!/^{-\theta_k}\operatorname{GL}(\tilde{w}_k - v_k)$ over $\mathbb{A}^1$ that gives the multiplication by -1 on the base. Here we write R' for $\operatorname{Hom}(\tilde{W}_k, V_k)\oplus\underline{R}$. So (2.3) extends to an isomorphism of the canonical (=even+ $\mathbb{C}^\times$-equivariant) deformation quantizations $\mathcal{D},\mathcal{D}'$ of the schemes $\mathcal{X},\mathcal{X}'$ that are defined as follows:

$$\mathcal{D} := [D_{R,\hbar}/D_{R,\hbar}\Phi^{sym}_k(\mathfrak{sl}(v_k))|_{T^*R^{\theta_k-ss}}]^{\operatorname{GL}(v_k)},$$
$$\mathcal{D}' := [D_{R',\hbar}/D_{R',\hbar}\Phi^{sym}_k(\mathfrak{sl}(\tilde{w}_k - v_k))|_{T^*R'^{\theta_k-ss}}]^{\operatorname{GL}(\tilde{w}_k-v_k)}$$

Here Φ^{sym}, the *symmetrized quantum comoment map*, stands the composition of $\mathfrak{g}\to\mathfrak{sp}(T^*R)$ and the natural embedding $\mathfrak{sp}(T^*R)\hookrightarrow\mathbf{A}_\hbar(T^*R)$, where $\mathbf{A}$ denotes the Weyl algebra. For the discussion of canonical and even quantizations and connections between them see [L6, Sections 2.2,2.3]. The isomorphism $\mathcal{D}\xrightarrow{\sim}\mathcal{D}'$ does not intertwine the canonical (symmetrized) quantum comoment maps for the $\underline{G}$-actions on $\mathcal{D},\mathcal{D}'$ but rather does the same change as the with the classical comoment maps. For the definition of a symmetrized quantum comoment map, see [L6, Section 5.4].

So we get an isomorphism $\mathcal{A}^\theta_\lambda(v)\xrightarrow{\sim}\mathcal{A}^{\sigma\theta}_{\sigma\bullet^v\lambda}(\sigma\bullet v)$, where the parameter $\sigma\bullet^v\lambda$ is determined as follows. If instead of $\Phi(x) = x_R$ we have used the symmetrized quantum comoment map $\Phi^{sym}(x)$ (the specialization of the previously defined symmetrized quantum comoment map to $\hbar = 1$) then, by the previous paragraph, we would have $\sigma\bullet^v\lambda = \sigma\lambda$ (compare to [L6, Section 6.4]).

For $\Phi(x) = x_R$, the computation is as follows. Let $\varrho(v)$ be the character of $\mathfrak{g}$ equal $-\frac{1}{2}\chi_{\bigwedge^{top}R}$, where $\chi_{\bigwedge^{top}R}$ is the character of the action of $\mathfrak{g}$ on $\bigwedge^{top}R$. Then $\Phi(x)-$

$\Phi^{sym}(x)=\langle\varrho(v),x\rangle$. We remark that $\Phi(x)$ depends on the orientation of Q (while $\Phi^{sym}(x)$ does not) and we have

$$\varrho(v)_k=\frac{1}{2}\Big(\sum_{a,h(a)=k}v_{t(a)}-\sum_{a,t(a)=k}v_{h(a)}-w_k\Big),\quad k\in Q_0. \tag{2.7}$$

When we change an orientation of Q, the character $\varrho(v)$ changes by an element from $\mathbb{Z}^{Q_0}$. We also would like to point out that the quantum LMN isomorphisms are T-equivariant, this also follows from [We2, Proposition 4.13].

Let us compute $s_i\bullet^v\lambda$ in the case when i is a source so that $\rho(v)_i=-\frac{1}{2}\tilde{w}_i$. We have $\sigma\bullet^v\lambda-\varrho(\sigma\bullet v)=\sigma(\lambda-\varrho(v))$ so that $s_i\bullet^v\lambda=s_i\lambda+\varrho(s_i\bullet v)-s_i\varrho(v)$ and what we need to compute is $\varrho(s_i\bullet v)-s_i\varrho(v)$. We have $\rho(s_i\bullet v)_k=(s_i\varrho(v))_k$ when k is different from i and is not adjacent to i. When $k=i$, we have $(s_i\varrho(v))_k=-\varrho(v)_k=-\varrho(s_i\bullet v)_k$. Finally, let us consider the case when k is adjacent to i, say there are q arrows from i to k. Then $(s_i\varrho(v))_k=\varrho(v)_k+q\varrho(v)_i=\varrho(v)_k-\frac{q}{2}\tilde{w}_i$ and $\varrho(s_i\bullet v)_k=\varrho(v)_k+\frac{q}{2}(\tilde{w}_i-v_i-v_i)$. In particular, we deduce that $\varrho(s_i\bullet v)-s_i\varrho(v)\in\mathbb{Z}^{Q_0}$.

Remark 2.6. One conclusion that will be used below is that $\varrho(\sigma\bullet v)-\sigma\varrho(v)$ is integral and hence $\sigma\bullet^v\lambda-\lambda$ is integral if and only if $\sigma\lambda-\lambda$ is.

An important corollary of $\mathcal{A}^\theta_\lambda(v)\xrightarrow{\sim}\mathcal{A}^{\sigma\theta}_{\sigma\bullet\lambda}(\sigma\bullet v)$ is an isomorphism $\mathcal{A}_\lambda(v)\xrightarrow{\sim}\mathcal{A}_{\sigma\bullet\lambda}(\sigma\bullet v)$. This is a special case of [BPW, Proposition 3.10], but a more explicit construction above is useful for our purposes.

2.2.5. *$\mathcal{A}^0_\lambda(v)$ vs $\mathcal{A}_\lambda(v)$, II.* Recall that for a subvariety $Y\subset V$, where V is a vector space, one can define its *asymptotic cone* $\mathsf{AC}(Y)$ as $\operatorname{Spec}(\operatorname{gr}\mathbb{C}[Y])\subset V$, where we take the filtration on $\mathbb{C}[Y]$ induced by the epimorphism $\mathbb{C}[V]\twoheadrightarrow\mathbb{C}[Y]$.

A Zariski open subset $\mathfrak{P}^0\subset\mathfrak{P}$ will be called *asymptotically generic* if $\mathsf{AC}(\mathfrak{P}\setminus\mathfrak{P}^0)\subset\mathfrak{p}^{sing}$.

Recall that we write $\mathfrak{P}^{iso}$ for the set of $\lambda\in\mathfrak{P}$ such that $\mathcal{A}^0_\lambda(v)\to\mathcal{A}_\lambda(v)$ is an isomorphism. The following proposition (to be proved in Section 3.5) should be thought as a quantum analog of the isomorphism $\mathcal{M}^0_\lambda(v)\cong\mathcal{M}_\lambda(v)$ for a generic λ.

Proposition 2.7. *The subvariety $\mathfrak{P}^{iso}\subset\mathfrak{P}$ is Zariski open and asymptotically generic.*

2.2.6. *Spherical symplectic reflection algebras.* Here we will discuss the special case when Q is an affine quiver. Let 0 denote the extending vertex (so that $Q\setminus 0$ is a finite Dynkin quiver), and $w=\epsilon_0$, the coordinate vector at the extending vertex.

It is a classical fact that all weights of the irreducible $\mathfrak{g}(Q)$-module L_{ω_0} (a.k.a. the basic representation) are conjugate to the weights of the form $\omega_0-n\delta, n\in\mathbb{Z}_{\geqslant 0}$, under the action of $W(Q)$. Thanks to the quantum LMN isomorphisms it is enough to consider $v=n\delta$. Here the algebra $\mathcal{A}_\lambda(v)$ is known to be isomorphic to a certain *spherical symplectic reflection algebra*. Let us recall some basics about these algebras.

Let Γ be a finite subgroup in $\operatorname{Sp}(V)$, where V is a symplectic vector space. We choose independent variables $\mathbf{c}=(\mathbf{c}_0,\ldots,\mathbf{c}_r)$, one for each conjugacy class of symplectic reflections in Γ. Then we can consider the algebra $\mathbf{H}$, the quotient of $T(V)\#\Gamma[\mathbf{c}_0,\ldots,\mathbf{c}_r]$ by the relations of the form

$$[u,v]=\omega(u,v)+\sum_{i=0}^r\mathbf{c}_i\sum_{s\in S_i}\omega_s(u,v),u,v\in V.$$

Here $S_1,\ldots,S_r$ are the conjugacy classes of symplectic reflections in Γ, ω is the symplectic form on V, and $\omega_s(u,v)=\omega(\pi_s u,\pi_s v)$, where we write π_s for the s-invariant projection from V to $\mathrm{im}(s-1)$. Inside $\mathbf{H}$ we can consider the spherical subalgebra $e\mathbf{H}e$, where $e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma$. Also, for numerical values of $\mathbf{c}$, say c, we can consider the specializations $\mathcal{H}_c$ of $\mathbf{H}$. Recall that a parameter c is called *spherical* if $e\mathcal{H}_c e$ and $\mathcal{H}_c$ are Morita equivalent (via the bimodule $\mathcal{H}_c e$).

Examples of Γ that are of most interest for us are as follows. Take a finite subgroup $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$ and a positive integer n. Then we can form the group $\Gamma = \Gamma_n := \mathfrak{S}_n \ltimes \Gamma_1^n$ that acts on $V = \mathbb{C}^{2n}$ by linear symplectomorphisms. We have two kinds of symplectic reflections: the conjugacy class S_0 containing transpositions in $\mathfrak{S}_n$, and conjugacy classes $S_1, \ldots, S_r$ containing elements from the n copies of Γ (here r is the number of the nontrivial conjugacy classes in Γ_1). We will use the notation $\mathcal{H}_{\kappa,c}(n)$ for the algebra corresponding to $c_0 = 2\kappa$ and $c_1, \ldots, c_r$.

Now recall that, by the McKay correspondence, to Γ_1 we can assign an affine Dynkin quiver Q. Take $v = n\delta$, where δ is the indecomposable imaginary root, and $w = \epsilon_0$, where 0 stands for the extending vertex of Q. Then we have isomorphisms $e\mathcal{H}_{\kappa,c}(n)e \cong \mathcal{A}_\lambda(v)$, where λ can obtained from c by formulas explained in [EGGO, 1.4]. In particular, $\kappa = \langle\lambda, \delta\rangle$. For example, for $\Gamma_1 = \{1\}$ we just get $e\mathcal{H}_{\kappa,\varnothing}(n)e = \mathcal{A}_\kappa(n)$.

The following lemma gives a characterization of spherical values of $e\mathcal{H}_{\kappa,c}(n)e$.

Lemma 2.8. *The following is true.*

(1) *The parameter (κ, c) is spherical if and only if $e\mathcal{H}_{\kappa,c}(n)e$ has finite homological dimension.*
(2) *The parameter κ of the type A Rational Cherednik algebra $\mathcal{H}_\kappa(n)$ is not spherical if and only if $\kappa = -\frac{s}{m}$ with $1 < m \leqslant n$ and $0 < s < m$.*

(1) follows from [Et, Theorem 5.5] and (2) is proved in [BE, Corollary 4.2].

2.3. **Coherent and quasi-coherent modules.** Let us proceed to defining suitable categories of sheaves of modules over the sheaves of algebras $\mathcal{A}_\lambda^\theta(v)$.

2.3.1. *Coherent modules.* Now let X be a normal Poisson variety (with a $\mathbb{C}^\times$-action rescaling the Poisson bracket) and $\mathcal{D}$ be its filtered quantization. Recall that this means that $\mathcal{D}$ is a filtered sheaf of algebras in the conical topology on X together with an isomorphism $\mathrm{gr}\,\mathcal{D} \cong \mathcal{O}_X$ of sheaves of graded Poisson algebras. We also require that the filtration on $\mathcal{D}$ is complete and separated.

By a morphism $f : (X, \mathcal{D}^X) \to (Y, \mathcal{D}^Y)$ we mean a $\mathbb{C}^\times$-equivariant morphism $f : X \to Y$ together with a morphism $D^Y \to f_\bullet \mathrm{D}^X$ (where $f_\bullet$ is the sheaf-theoretic push-forward) of filtered algebras that gives the homomorphism $\mathcal{O}_Y \to f_\bullet\mathcal{O}_X$ coming from f on the level of associated graded sheaves.

Let M be a sheaf of $\mathcal{D}$-modules in the conical topology.

Definition 2.9. We say that a $\mathcal{D}$-module M is coherent if it is equipped with a global complete and separated filtration such that $\mathrm{gr}\, M$ is a coherent $\mathcal{O}_X$-module (this filtration is called good).

The category of coherent $\mathcal{D}$-modules (where the morphisms are morphisms of sheaves of $\mathcal{D}$-modules) will be denoted by $\mathrm{Coh}(\mathcal{D})$.

The following lemma establishes basic properties of coherent $\mathcal{D}$-modules (that mirror properties of coherent sheaves in Algebraic geometry).

Lemma 2.10. *The following is true.*

(a) *Let X be affine, $\mathcal{D}$ be its quantization, and $\mathcal{A} := \Gamma(\mathcal{D})$. Then the functors $M \mapsto M^{loc} := \mathcal{D}\otimes_{\mathcal{A}} M$ and $N \mapsto \Gamma(N)$ are mutually inverse equivalences between $\mathcal{A}$-*mod *and* $\mathrm{Coh}(\mathcal{D})$.
(b) *A submodule and a quotient of a coherent $\mathcal{D}$-module are coherent.*
(c) *Let f be a morphism $(X, \mathcal{D}^X) \to (Y, \mathcal{D}^Y)$. Then there is a pull-back functor $f^* : \mathrm{Coh}(\mathcal{D}^Y) \to \mathrm{Coh}(\mathcal{D}^X)$ given by $M \mapsto \mathrm{D}^X \otimes_{f^\bullet \mathrm{D}^Y} f^\bullet M$, where $f^\bullet$ is the sheaf theoretic pull-back.*

Proof. Let us prove (a). Note that $\mathrm{gr}(M^{loc})$ is the coherent sheaf on X associated to $\mathrm{gr}\, M$ and $\mathrm{gr}\,\Gamma(N) = \Gamma(\mathrm{gr}\, N)$, the latter is true because $H^1(X, \mathrm{gr}\, N) = 0$. This shows that the natural homomorphisms $M \mapsto \Gamma(M^{loc}), \Gamma(N)^{loc} \to N$ are isomorphisms after passing to the associated graded modules, hence are isomorphisms because all the filtrations involved are complete and separated.

Let us prove (b). Let $M' \subset M$ be a submodule and M be coherent. Then we can restrict the filtration from M to M'. For an open affine subspace U, we have $\Gamma(U, M') \subset \Gamma(U, M)$ and $\Gamma(U, M)$ is a finitely generated $\Gamma(U, \mathcal{D})$-module with a good filtration. It follows that $\Gamma(U, M')$ is closed (compare to the proof of [L2, Lemma 2.4.4]) and from here one deduces that the filtration on $\Gamma(U, M')$ is complete and separated. So the filtration on M' is complete and separated. Besides, $\mathrm{gr}\, M' \subset \mathrm{gr}\, M$ and so $\mathrm{gr}\, M'$ is coherent. So M' is coherent. To show that M/M' is coherent we notice that it inherits a (global) complete and separated filtration and, by (a), $M/M'|_U$ is coherent for every open affine U. From here we deduce that M/M' is coherent.

To prove (c), notice that f^*M comes with a natural global filtration induced from M. Pick an open affine subset $U \subset Y$ so that $M|_U$ is a quotient of $\bigoplus_i \mathcal{D}^Y|_U\langle d_i\rangle$ (where in the triangular brackets we indicate the filtration shift) with the induced filtration. Moreover, $f^*M|_{f^{-1}(U)}$ is the quotient of $\bigoplus_i \mathcal{D}^X|_{f^{-1}(U)}\langle d_i\rangle$ with induced filtration. So we see that the filtration on f^*M is complete and separated. Also $\mathrm{gr}\, f^*M$ is a quotient of $f^*(\mathrm{gr}\, M)$ so is coherent. This shows that f^*M is coherent. $\square$

2.3.2. *Quasi-coherent modules.* Let us proceed to quasi-coherent $\mathcal{D}$-modules. By definition, those are unions of their coherent submodules. Here are their basic properties.

Lemma 2.11. *The following is true.*

(1) *The direct analogs of (a)-(c) of Lemma 2.10 hold.*
(2) *In the notation of (c) of Lemma 2.10, we have the push-forward functor f_* : $\mathrm{QCoh}(\mathcal{D}^X) \to \mathrm{QCoh}(\mathcal{D}^Y)$ (that coincides with the sheaf theoretic push-forward). If f is proper, then this functor restricts to $\mathrm{Coh}(\mathcal{D}^X) \to \mathrm{Coh}(\mathcal{D}^Y)$.*
(3) *The category $\mathrm{QCoh}(\mathcal{D})$ contains enough injectives.*
(4) *The natural functor $D^b(\mathrm{Coh}(\mathcal{D})) \to D^b(\mathrm{QCoh}(\mathcal{D}))$ is a full embedding.*

Proof. Let us prove (1). The analog of (a) of Lemma 2.10 holds because the localization and global section functors commute with taking unions. The analog of (b) is straightforward and (c) follows because tensor products commute with direct limits.

(2) of the present lemma is more involved. The push-forward commutes with taking unions. So it is enough to show that f_*M is quasi-coherent if M is coherent. We can cover X with open subsets X_i such that $f^i := f|_{X_i} : X_i \to Y$ is affine. Then f_*M is the kernel of $\bigoplus_i f^i_*M \to \bigoplus_{i\neq j} f^{ij}_*M$, where f^{ij} is the restriction of f to $X_i \cap X_j$. So in the proof it is enough to assume that f is affine. Moreover, we can assume that $f = \iota \circ g$, where g is a morphism of affine varieties and ι is an open embedding of an affine variety. It is clear that

g_* maps quasi-coherent sheaves to quasi-coherent ones. It remains to show that ι_* maps coherent sheaves to quasi-coherent ones. We note that the sheaves $\iota_* M$ are generated by their global sections, hence are quotients of $(\mathcal{D}^Y)^{\oplus ?}$ and hence are quasi-coherent. This completes the proof of the claim that f_* maps quasi-coherent sheaves to quasi-coherent ones.

Let us prove (3). Recall that the category of modules over a ring contains enough injectives. Now we can cover X with open affine $\mathbb{C}^\times$-stable subsets, $X = \bigcup_k X^k$, let ι^k denote the inclusion $X^k \hookrightarrow X$. Let $\mathcal{I}^k$ be an injective hull of $\Gamma(M|_{X^i})$. Then $\bigoplus_k \iota^k_* \mathcal{I}^k$ is an injective quasi-coherent module admitting an embedding from M.

(4) is a formal corollary of the claim that every quasi-coherent module is the union of its coherent submodules (and so in every complex with coherent homology we can produce a quasi-isomorphic subcomplex with coherent terms). $\square$

2.4. **Supports and characteristic cycles.** Now let us define the supports of objects in $\mathcal{A}_\lambda(v)$-mod (the category of finitely generated $\mathcal{A}_\lambda(v)$-modules) and supports and characteristic cycles for objects in $\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))$. We also define holonomic modules. Below we will write $\mathcal{A}^\theta_\lambda(v)$-mod for $\mathrm{Coh}(\mathcal{A}^\theta_\lambda(v))$.

For $M \in \mathcal{A}_\lambda(v)$-mod we can define the support, $\operatorname{Supp} M$, to be the support of the coherent sheaf $\operatorname{gr} M$ with respect to any good filtration. Similarly, we can define the support of an object in $\mathcal{A}^\theta_\lambda(v)$-mod.

We remark that the support of an $\mathcal{A}^\theta_\lambda(v)$-module (or an $\mathcal{A}_\lambda(v)$-module) M is a coisotropic subvariety in $\mathcal{M}^\theta(v)$ by the Gabber involutivity theorem, [Ga], an easier proof due to Knop can be found in [Gi1, Section 1.2]. If the support of $M \in \mathcal{A}^\theta_\lambda(v)$-mod is lagrangian, then we call M *holonomic*. An object $N \in \mathcal{A}_\lambda(v)$-mod is called holonomic if the intersection of $\operatorname{Supp}(N)$ with every symplectic leaf in $\mathcal{M}(v)$ is isotropic. By [L12, Appendix], this is equivalent to $\rho^{-1}(\operatorname{Supp}(N))$ to be isotropic.

Let us proceed to characteristic cycles.

Now suppose $Y \subset \mathcal{M}^\theta(v)$ is a $\mathbb{C}^\times$-stable isotropic subvariety. Recall that to a coherent sheaf M_0 on $\mathcal{M}^\theta(v)$ supported on Y on can assign its characteristic cycle $\mathsf{CC}(M_0)$ equal to the following formal linear combination of the irreducible components of Y:

$$\mathsf{CC}(M_0) := \sum_{Y' \subset Y} (\mathrm{grk}_{Y'} M_0) Y',$$

where $\mathrm{grk}_{Y'}$ stands for the rank in the generic point of a component Y'. We can define the characteristic cycle CC of a $\mathcal{A}^\theta_\lambda(v)$-module M supported on Y by $\mathsf{CC}(M) := \mathsf{CC}(\operatorname{gr} M)$, this is easily seen to be well-defined. An alternative definition is given in [BPW, 6.2]. Yet another description of $\mathsf{CC}(M)$ is as follows. The object M gives rise to a well-defined class in $K_0(\mathrm{Coh}_Y(\mathcal{M}^\theta(v)))$, that of $\operatorname{gr} M$ (the map $[M] \to [\operatorname{gr} M] : K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_Y) \to K_0(\mathrm{Coh}_Y \mathcal{M}^\theta(v))$ will be called the *degeneration map* in what follows). Applying the Chern character map, we get an element $\mathsf{CC}'(M) \in H^*(\mathcal{M}^\theta(v), \mathcal{M}^\theta(v) \setminus Y) = H^{BM}_*(Y)$. Then $\mathsf{CC}(M)$ coincides with the projection of $\mathsf{CC}'(M)$ to $H^{BM}_{top}(Y)$.

When $Y = \rho^{-1}(0)$, we have $H^{BM}_*(Y) = H_*(Y) = H_*(\mathcal{M}^\theta(v))$. The first equality holds because $\rho^{-1}(0)$ is compact, the second one is true because $\mathcal{M}^\theta(v)$ is contracted onto $\rho^{-1}(0)$ by the $\mathbb{C}^\times$-action (induced by the dilation action on T^*R).

Proposition 2.12 ([BaGi]). *The map* $\mathsf{CC} : K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to H_{mid}(\mathcal{M}^\theta(v))$ *is injective.*

The proof of this proposition has not appeared yet, so we will give an independent proof later in the paper.

2.5. **Various functors.** In this section we will study Hamiltonian reduction functors from the category of (G,λ)-equivariant D-modules on R to the categories of modules over $\mathcal{A}^0_\lambda(v), \mathcal{A}^\theta_\lambda(v)$. We will also study the localization and global section functors and their connection to Hamiltonian reduction functors.

2.5.1. *Twisted equivariant D-modules.* By a (G,λ)-equivariant $D(R)$-module one means a weakly G-equivariant module $\mathcal{M}$ such that $x_{\mathcal{M}}m = \Phi(x)m - \lambda(x)m$ for all $x\in\mathfrak{g}, m\in\mathcal{M}$. We consider the category $D(R)\operatorname{-Mod}^{G,\lambda}$ of all (G,λ)-equivariant modules over $D(R)$ and its full subcategory $D(R)\operatorname{-mod}^{G,\lambda}$ of finitely generated modules. Note that, for $\chi\in\mathbb{Z}^{Q_0}$, the categories $D(R)\operatorname{-Mod}^{G,\lambda}$ and $D(R)\operatorname{-Mod}^{G,\lambda+\chi}$ are equivalent, via $M\mapsto M\otimes\mathbb{C}_{-\chi}: D(R)\operatorname{-Mod}^{G,\lambda}\to D(R)\operatorname{-Mod}^{G,\lambda+\chi}$, where $\mathbb{C}_{-\chi}$ is the one-dimensional G-module corresponding to the character $-\chi$.

2.5.2. *Functors for abelian categories.* Let us write $\mathcal{A}^0_\lambda(v)$ for the category of all $\mathcal{A}^0_\lambda(v)$-modules. We have a functor $\pi^0_\lambda(v): D(R)\operatorname{-Mod}^{G,\lambda}\to\mathcal{A}^0_\lambda(v)\operatorname{-Mod}$ of taking G-invariants that restricts to $D(R)\operatorname{-mod}^{G,\lambda}\to\mathcal{A}^0_\lambda(v)\operatorname{-mod}$. It is a quotient functor, it kills precisely the modules without nonzero G-invariants. It has a left adjoint (and right inverse)

$$\pi^0_\lambda(v)^!:\mathcal{A}^0_\lambda(v)\operatorname{-Mod}\to D(R)\operatorname{-Mod}^{G,\lambda},$$

given by taking the tensor product with the $D(R)$-$\mathcal{A}^0_\lambda(v)$-bimodule $\mathcal{Q}_\lambda := D(R)/D(R)\{\Phi(x)-\langle\lambda,x\rangle, x\in\mathfrak{g}\}$. Note that $\mathcal{Q}_\lambda$ is (G,λ)-equivariant as a $D(R)$-module so $\pi^0_\lambda(v)^!$ indeed maps to (G,λ)-equivariant $D(R)$-modules. The functor $\pi^0_\lambda(v)$ maps finitely generated modules to finitely generated ones.

Recall that we assume that θ is generic. We have a functor $\pi^\theta_\lambda(v)$ from $D_R\operatorname{-Mod}^{G,\lambda}$ to the category of quasi-coherent $\mathcal{A}^\theta_\lambda(v)$-modules (to be denoted by $\mathcal{A}^\theta_\lambda(v)\operatorname{-Mod}$): it first microlocalizes a D-module to the θ-semistable locus and then takes the G-invariants. The image of $D(R)\operatorname{-mod}^{G,\lambda}$ consists of coherent modules.

Proposition 2.13. *The functor $D(R)\operatorname{-Mod}^{G,\lambda}\to\mathcal{A}^\theta_\lambda(v)\operatorname{-Mod}$ is a quotient functor.*

In the case when μ is flat the proof was given in [BPW, Section 5.4] and also announced in [MN, Proposition 4.9]. Below, in Section 4.2, we will give a proof in general.

2.5.3. *Reminder on equivariant derived categories.* We will need derived versions of the reduction functors considered in 2.5.2. We can form the derived categories $D^?(D(R)\operatorname{-mod}^{G,\lambda})\subset D^?(D(R)\operatorname{-Mod}^{G,\lambda})$ (the naive derived categories; here ? stands for $+,-$ or b) but we will also need the equivariant derived categories $D^?_{G,\lambda}(D(R)\operatorname{-mod})\subset D^?_{G,\lambda}(D(R)\operatorname{-Mod})$. Here we recall some basics regarding equivariant derived categories.

Let $\mathcal{A}$ be an associative algebra equipped with a rational action of a connected reductive algebraic group G. Assume that this action is Hamiltonian with quantum comoment map Φ so it makes sense to speak about weakly G-equivariant and G-equivariant $\mathcal{A}$-modules. Then the equivariant derived category $D^b_G(\mathcal{A}\operatorname{-mod})$ is defined as follows. Consider the Chevalley-Eilenberg complex $\bar{U}(\mathfrak{g})$, a standard resolution of the trivial one dimensional $\mathfrak{g}$-module and form the tensor product $\mathcal{A}\otimes\bar{U}(\mathfrak{g})$. This is a differential graded algebra equipped with a Hamiltonian G-action (the diagonal action together with the diagonal quantum comoment map). So it makes sense to speak about G-equivariant differential graded $\mathcal{A}\otimes\bar{U}(\mathfrak{g})$-modules. The category $D^b_G(\mathcal{A}\operatorname{-mod})$ is obtained from the category of those modules by passing to the homotopy category and localizing the quasi-isomorphisms. Consider the natural homomorphism $\varpi : \mathcal{A}\otimes \bar{U}(\mathfrak{g}) \to \mathcal{A}$ of differential graded algebras (taking the 0th homology). The pull-back functor ϖ^* is a natural functor $D^b(\mathcal{A}\text{-mod}^G) \to D^b_G(\mathcal{A}\text{-mod})$. On the other hand, the category of G-equivariant $\mathcal{A}\otimes U(\mathfrak{g})$-modules is the same as the category of weakly G-equivariant $\mathcal{A}$-modules. We have a G-equivariant homomorphism $\iota : \mathcal{A}\otimes U(\mathfrak{g}) \to \mathcal{A}\otimes \bar{U}(\mathfrak{g})$ intertwining the quantum moment maps. This gives a pull-back functor $\iota^* : D^b_G(\mathcal{A}\text{-mod}) \to D^b(\mathcal{A}\otimes U(\mathfrak{g})\text{-mod}^G)$. The composition $\iota^*\circ\varpi^* : D^b(\mathcal{A}\text{-mod}^G) \to D^b(\mathcal{A}\otimes U(\mathfrak{g})\text{-mod}^G)$ comes from the forgetful functor between abelian categories. Besides, we have left adjoints of ι^*, ϖ^*, the functors $\iota_!(\bullet) := \bullet\otimes^L_{U(\mathfrak{g})}\mathbb{C}$ and $\varpi_!(\bullet) := \mathcal{A}\otimes^L_{\mathcal{A}\otimes\bar{U}(\mathfrak{g})}\bullet$.

This discussion implies the following lemma to be used in what follows.

Lemma 2.14. *Let V be a G-module. For $M\in D^b_G(\mathcal{A}\text{-mod})$, we have a natural isomorphism*

$$\operatorname{Hom}_{D^b_G(\mathcal{A}\text{-mod})}((\mathcal{A}\otimes V)\otimes^L_{U(\mathfrak{g})}\mathbb{C}, M) \cong \operatorname{Hom}_G(V, H_0(M)).$$

2.5.4. *Functors for derived categories.* Let us proceed to derived analogs of $\pi^\theta_\lambda(v), \pi^0_\lambda(v)$ and $\pi^0_\lambda(v)^!$.

A natural functor $D^b(D_R\text{-Mod}^{G,\lambda}) \to D^b_{G,\lambda}(D_R\text{-Mod})$ is an isomorphism provided μ is flat, see [BL, Theorem 1.6]. When μ is not necessarily flat we have the following lemma. Consider the subcategories

$$D^b_{\theta-uns}(D_R\text{-Mod}^{G,\lambda}), D^b_{G,\lambda,\theta-uns}(D_R\text{-Mod})$$

of all all objects with (singular) supports of homology contained in $(T^*R)^{\theta-uns}$.

Lemma 2.15. *The induced functor*

$$D^b(D(R)\text{-mod}^{G,\lambda})/D^b_{\theta-uns}(D(R)\text{-mod}^{G,\lambda}) \to D^b_{G,\lambda}(D(R)\text{-mod})/D^b_{G,\lambda,\theta-uns}(D(R)\text{-mod}).$$

is a category equivalence.

Proof. The microlocalization functors intertwine ϖ^* and $\varpi_!$ from 2.5.3. Recall from [BL, Theorem 1.6] that if G acts freely on U, then the functors $\varpi^*_U, \varpi_{U!}$ for the algebra $\mathcal{A} = D_R(U)$ are mutually inverse equivalences. For $\mathcal{A} = D(R)$, this means that the adjunction morphisms $\varpi_!\circ\varpi^* \to \mathrm{id}$ and $\mathrm{id}\to\varpi^*\circ\varpi_!$ have homology supported on $\mu^{-1}(0)^{\theta-uns}$. This implies the claim of the lemma. ☐

So we can extend the functor $\pi^\theta_\lambda(v)$ to a functor $D^b_{G,\lambda}(D_R\text{-Mod}) \twoheadrightarrow D^b(\mathcal{A}^\theta_\lambda(v)\text{-Mod})$. Assuming the former is a quotient functor, so is the latter.

Let us consider a derived version of $\pi^0_\lambda(v)$. This functor extends to $D^b(D(R)\text{-mod}^{G,\lambda}) \twoheadrightarrow D^b(\mathcal{A}^0_\lambda(v)\text{-mod})$ and we have the derived left adjoint functor $L\pi^0_\lambda(v)^! : D^b(\mathcal{A}^0_\lambda(v)\text{-mod}) \to D^b(D(R)\text{-mod}^{G,\lambda})$.

When λ is Zariski generic, we can also lift $\pi^0_\lambda(v)$ to a quotient functor $D^-_{G,\lambda}(D(R)\text{-mod}) \twoheadrightarrow D^-(\mathcal{A}^0_\lambda(v)\text{-mod})$. For this, we need the following proposition.

Proposition 2.16. *The following is true:*

(1) *There is a Zariski open asymptotically generic subset $\mathfrak{P}^{ISO}\subset\mathfrak{P}^{iso}$ such that $\operatorname{Tor}^i_{U(\mathfrak{g})}(D(R), \mathbb{C}_\lambda) = 0$ for all $i > 0$ and $\lambda\in\mathfrak{P}^{ISO}$.*

(2) *For $\lambda\in\mathfrak{P}^{ISO}$, the functor*

$$\pi^0_\lambda(v) := \operatorname{Hom}_{D^b_{G,\lambda}(D(R)\text{-mod})}(\mathcal{Q}_\lambda, \bullet)$$

maps $M \in D(R)\text{-mod}^{G,\lambda}$ *to* $H_0(M)^G$. *It is a quotient functor* $D^b_{G,\lambda}(D_R\text{-mod}) \to D^b(\mathcal{A}^0_\lambda(v)\text{-mod})$ *with a left adjoint and right inverse functor* $L\pi^0_\lambda(v)^!$ *is given by* $\mathcal{Q}_\lambda \otimes^L_{\mathcal{A}^0_\lambda(v)} \bullet$.

Proof. (1) will be proved below, see Section 3.6. (2) follows from (1) and Lemma 2.14 applied to the trivial G-module V. □

2.5.5. *Global section and localization functors.* Since $R\Gamma(\mathcal{A}^\theta_\lambda(v)) = \mathcal{A}_\lambda(v)$, it makes sense to consider the derived global section functor $R\Gamma^\theta_\lambda : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to D^b(\mathcal{A}_\lambda(v)\text{-mod})$ and its left adjoint, the derived localization functor,

$$L\operatorname{Loc}^\theta_\lambda : \mathcal{A}^\theta_\lambda(v) \otimes^L_{\mathcal{A}_\lambda(v)} \bullet : D^-(\mathcal{A}_\lambda(v)\text{-mod}) \to D^-(\mathcal{A}^\theta_\lambda(v)\text{-mod}).$$

We will also consider the abelian versions of these functors: Γ^θ_λ and its left adjoint $\operatorname{Loc}^\theta_\lambda$. One can realize $R\Gamma^\theta_\lambda$ as taking the Čech complex.

Lemma 2.17. *Assume that* $\lambda \in \mathfrak{P}^{ISO}$. *Then* $L\operatorname{Loc}^\theta_\lambda = \pi^\theta_\lambda(v) \circ L\pi^0_\lambda(v)^!$.

Proof. The functor $L\pi^0_\lambda(v)^!$ is the derived tensor product with $\mathcal{Q}_\lambda$. The functor $\pi^\theta_\lambda(v)$ is the composition of three functors, $\pi^\theta_\lambda(v) = \pi_3 \circ \pi_2 \circ \pi_1$, where the functors π_1, π_2, π_3 are as follows. First, we have the quotient functor

$$\pi_1 : D^-_{G,\lambda}(D(R)\text{-mod}) \twoheadrightarrow D^-_{G,\lambda}(D(R)\text{-mod})/D^-_{G,\lambda}(D(R)\text{-mod})_{\theta-uns}.$$

Second, we have the identification

$$\pi_2 : D^b_{G,\lambda}(D_R\text{-Mod})/D^b_{G,\lambda,\theta-uns}(D_R\text{-Mod}) \xrightarrow{\sim} D^b(D_R\text{-Mod}^{G,\lambda})/D^b_{\theta-uns}(D_R\text{-Mod}^{G,\lambda}),$$

see Lemma 2.15. Third, we have the equivalence

$$\pi_3 : D^b(D_R\text{-Mod}^{G,\lambda})/D^b_{\theta-uns}(D_R\text{-Mod}^{G,\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$$

that is realized by taking G-invariants. The functor

$$\pi_2 \circ \pi_1 \circ L\pi^0_\lambda(v)^! : D^b(\mathcal{A}_\lambda(v)\text{-mod}) \to D^b(D_R\text{-Mod}^{G,\lambda})/D^b_{\theta-uns}(D_R\text{-Mod}^{G,\lambda})$$

is isomorphic to $\pi_3^{-1}(\mathcal{Q}_\lambda \otimes^L_{\mathcal{A}_\lambda(v)} \bullet)$. From here we deduce that

$$\pi^\theta_\lambda(v) \circ L\pi^0_\lambda(v)^! = [\mathcal{Q}_\lambda|_{T^*R^{\theta-ss}} \otimes_{\mathcal{A}^0_\lambda(v)} (\bullet)]^G = \mathcal{A}^\theta_\lambda(v) \otimes^L_{\mathcal{A}^0_\lambda(v)} \bullet.$$

But the functor $L\operatorname{Loc}^\theta_\lambda$ is $\mathcal{A}^\theta_\lambda(v) \otimes^L_{\mathcal{A}^0_\lambda(v)} \bullet$, by its definition. □

Let $D^b_Y(\mathcal{A}_\lambda(v)\text{-mod}) \subset D^b(\mathcal{A}_\lambda(v)\text{-mod}), D^b_{\rho^{-1}(Y)}(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \subset D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ denote the full subcategories consisting of all objects whose homology have support contained in $Y, \rho^{-1}(Y)$, respectively. It is clear from the construction that $R\Gamma^\theta_\lambda$ maps $D^b_{\rho^{-1}(Y)}$ to D^b_Y, while $L\operatorname{Loc}^\theta_\lambda$ maps D^-_Y to $D^-_{\rho^{-1}(Y)}$. We write D^b_{fin} instead of $D^b_{\{0\}}$.

3. Harish-Chandra bimodules and restriction functors

Harish-Chandra (shortly, HC) bimodules and restriction functors between the categories of HC bimodules play a crucial role in this paper. In this section we review a definition and basic properties of these bimodules (Section 3.1,3.2). In the remaining sections, we construct restriction functors for HC bimodules over quantized quiver varieties, study their basic properties and provide some applications. In particular, we prove Proposition 2.7 and part (1) of Proposition 2.16.

3.1. **Harish-Chandra bimodules.** Let us start with a general definition of a Harish-Chandra bimodule, compare to [L2, Gi2, L5, BPW]. Let $\mathcal{A}=\bigcup_{i\leqslant 0}\mathcal{A}^{\leqslant i}, \mathcal{A}'=\bigcup_{i=0}\mathcal{A}'^{\leqslant i}$ be $\mathbb{Z}_{\geqslant 0}$-filtered algebras such that the algebras $\operatorname{gr}\mathcal{A}, \operatorname{gr}\mathcal{A}'$ are identified with graded Poisson quotients of the same finitely generated commutative graded Poisson algebra A. Below we will always consider graded Poisson algebras, where the bracket has degree -1.

We will take $\mathcal{A}=\mathcal{A}^0_\lambda(v), \mathcal{A}'=\mathcal{A}^0_{\lambda'}(v)$ or sometimes $\mathcal{A}=\mathcal{A}_\lambda(v), \mathcal{A}'=\mathcal{A}_{\lambda'}(v)$ (the filtration on $\mathcal{A}$ is induced from the differential operator filtration on $D(R)$). In the first case, we take $A:=\mathbb{C}[\mathcal{M}^0_0(v)]$ (where we consider $\mathcal{M}^0_0(v)$ with its natural scheme structure), in the second case put $A:=\mathbb{C}[\mathcal{M}(v)]$ so that $\operatorname{gr}\mathcal{A}=\operatorname{gr}\mathcal{A}'=A$.

3.1.1. *Definition.* By a Harish-Chandra (HC) $\mathcal{A}'$-$\mathcal{A}$-bimodule we mean a bimodule $\mathcal{B}$ that can be equipped with a bimodule $\mathbb{Z}$-filtration bounded from below, $\mathcal{B}=\bigcup_i\mathcal{B}^{\leqslant i}$, such that $\operatorname{gr}\mathcal{B}$ is a finitely generated A-module (meaning, in particular, that the left and the right actions of A coincide). Such a filtration on $\mathcal{B}$ is called *good*. We remark that every HC bimodule is finitely generated both as a left $\mathcal{A}'$-module and as a right $\mathcal{A}$-module. We also remark that, although $\operatorname{gr}\mathcal{B}$ does depend on the choice of a filtration on $\mathcal{B}$, the support of $\operatorname{gr}\mathcal{B}$ in $\operatorname{Spec}(A)$ depends only on $\mathcal{B}$, this support is called the *associated variety* of $\mathcal{B}$ and is denoted by $\operatorname{V}(\mathcal{B})$. We remark that $\operatorname{V}(\mathcal{B})$ is always a Poisson subvariety of $\operatorname{Spec}(A)$.

By a homomorphism of HC bimodules we mean a bimodule homomorphism. Given a homomorphism $\varphi:\mathcal{B}\to\mathcal{B}'$ we can find good filtrations $\mathcal{B}=\bigcup_i\mathcal{B}^{\leqslant i}$ and $\mathcal{B}'=\bigcup_i\mathcal{B}'^{\leqslant i}$ with $\varphi(\mathcal{B}^{\leqslant i})\subset\mathcal{B}'^{\leqslant i}$ for all i. Indeed, if $\operatorname{gr}\mathcal{B}$ is generated by homogeneous elements of degree up to d then we can use any good filtration on $\mathcal{B}'$ such that $\varphi(\mathcal{B}^{\leqslant i})\subset\mathcal{B}'^{\leqslant i}$ for $i\leqslant d$.

For example, both $\mathcal{A}^0_\lambda(v), \mathcal{A}_\lambda(v)$ are HC $\mathcal{A}^0_\lambda(v)$-bimodules. It follows that any HC $\mathcal{A}_{\lambda'}(v)$-$\mathcal{A}_\lambda(v)$-bimodule is HC also when viewed as a $\mathcal{A}^0_{\lambda'}(v)$-$\mathcal{A}^0_\lambda(v)$-bimodule.

3.1.2. *Rees construction.* Starting from $\mathcal{A}$, we can form the Rees algebra $\mathcal{A}_\hbar:=\bigoplus_i\mathcal{A}^{\leqslant i}\hbar^i$ that is graded with $\deg\hbar=1$.

We can introduce a notion of a Harish-Chandra $\mathcal{A}'_\hbar$-$\mathcal{A}_\hbar$-bimodule: those are finitely generated graded $\mathcal{A}'_\hbar$-$\mathcal{A}_\hbar$-bimodules $\mathcal{B}_\hbar$ with $a'm-ma\subset\hbar\mathcal{B}_\hbar$ (for a,a' such that $a+\hbar\mathcal{A}_\hbar, a'+\hbar\mathcal{A}'_\hbar$ are the images of a single element $\tilde{a}\in A$) that are free over $\mathbb{C}[\hbar]$. To pass from HC $\mathcal{A}_\hbar$-bimodules to HC $\mathcal{A}$-bimodules with a fixed good filtration, one mods out $\hbar-1$. To get back, one takes the Rees bimodule.

3.1.3. *Derived categories.* Consider the category $D^-_{HC}(\mathcal{A}'\text{-}\mathcal{A}\text{-bimod})$ consisting of all complexes of $\mathcal{A}'$-$\mathcal{A}$-bimodules whose homology are Harish-Chandra. Similarly to [BPW, Proposition 6.3], the subcategories $D^-_{HC}(\ldots)\subset D^-(\ldots)$ are closed with respect to $\otimes^L_{\mathcal{A}'}$: $D^-(\mathcal{A}''\text{-}\mathcal{A}'\text{-bimod})\times D^-(\mathcal{A}'\text{-}\mathcal{A}\text{-bimod})\to D^-(\mathcal{A}''\text{-}\mathcal{A}\text{-bimod})$. The same argument implies that $R\operatorname{Hom}_{\mathcal{A}}$ sends $D^-_{HC}(\mathcal{A}\text{-}\mathcal{A}'\text{-bimod})\times D^+_{HC}(\mathcal{A}\text{-}\mathcal{A}''\text{-bimod})$ to $D^+_{HC}(\mathcal{A}'\text{-}\mathcal{A}''\text{-bimod})$.

3.1.4. *Translation bimodules.* Let us provide two closely related examples of HC bimodules over the algebras $\mathcal{A}_?(v), \mathcal{A}^0_?(v)$: translation bimodules.

Recall the $D(R)$-$\mathcal{A}^0_\lambda(v)$-bimodule $\mathcal{Q}_\lambda$ from 2.5.2. Pick $\chi\in\mathbb{Z}^{Q_0}$. We can consider the $\mathcal{A}^0_{\lambda+\chi}(v)$-$\mathcal{A}^0_\lambda(v)$ bimodule $\mathcal{A}^0_{\lambda,\chi}(v)=\mathcal{Q}_\lambda^{G,\chi}$. This bimodule is HC, the filtration on $\mathcal{A}^0_{\lambda,\chi}(v)$ induced from the filtration on $D(R)$ by the order of a differential operator is good.

Now consider the restriction $\mathcal{Q}_\lambda|_{T^*R^{\theta-ss}}$ and set $\mathcal{A}^\theta_{\lambda,\chi}(v):=[\mathcal{Q}_\lambda|_{T^*R^{\theta-ss}}]^{G,\chi}$. This is a sheaf on $\mathcal{M}^\theta(v)$ that is an $\mathcal{A}^\theta_{\lambda+\chi}(v)$-$\mathcal{A}^\theta_\lambda(v)$-bimodule. Set $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v):=\Gamma(\mathcal{A}^\theta_{\lambda,\chi}(v))$. That it is HC was demonstrated in [BPW, Section 6.3] but we want to sketch a proof. Namely, notice that $\operatorname{gr}\mathcal{A}^{\theta}_{\lambda,\chi}(v)=\mathcal{O}(\chi)$. Consider the Rees bimodule $\mathcal{A}^{\theta}_{\lambda,\chi}(v)_\hbar$ that is a deformation of $\mathcal{O}(\chi)$. Then $\Gamma(\mathcal{A}^{\theta}_{\lambda,\chi}(v)_\hbar)$ is the Rees bimodule for $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$. But $\Gamma(\mathcal{A}^{\theta}_{\lambda,\chi}(v)_\hbar)/(\hbar)$ embeds into $\Gamma(\mathcal{O}(\chi))$, the latter is a $\mathbb{C}[\mathcal{M}^\theta(v)]$-module rather than just a bimodule. This completes the proof.

We have a natural bimodule homomorphism

$$\mathcal{A}^{0}_{\lambda,\chi}(v)\to\mathcal{A}^{(\theta)}_{\lambda,\chi}(v) \tag{3.1}$$

induced by the restriction map $\mathcal{Q}_\lambda\to\mathcal{Q}_\lambda|_{T^*R^{\theta-ss}}$. A priori, (3.1) is neither injective, not surjective. In Section 4 we will get some sufficient conditions for (3.1) to be an isomorphism.

3.1.5. *Further properties.* Finally, we need some results from [L12]. The next lemma follows from Theorems 1.2, 1.3 or Section 4.3 there.

Lemma 3.1. *Every HC $\mathcal{A}_{\lambda'}(v)$-$\mathcal{A}_\lambda(v)$-bimodule has finite length.*

The following claim is [L12, Lemma 4.2].

Lemma 3.2. *Let $\mathcal{B}$ be a HC $\mathcal{A}_{\lambda'}(v)$-$\mathcal{A}_\lambda(v)$ bimodule and $\mathcal{J}_\ell,\mathcal{J}_r$ be its left and right annihilators. Then $\mathrm{V}(\mathcal{B})=\mathrm{V}(\mathcal{A}_{\lambda'}(v)/\mathcal{J}_\ell)=\mathrm{V}(\mathcal{A}_\lambda(v)/\mathcal{J}_r)$.*

3.2. **Families of Harish-Chandra bimodules.** Recall from 2.1.5 that we have the scheme $\mathcal{M}^\theta_{\mathfrak{p}}(v)=\mu^{-1}(\mathfrak{g}^{*G})^{\theta-ss}/G$ over $\mathfrak{p}$. In Section 2.2 we have introduced the sheaf of $\mathbb{C}[\mathfrak{P}]$-algebras

$$\mathcal{A}^\theta_{\mathfrak{P}}(v):=[\mathcal{Q}_{\mathfrak{P}}|_{T^*R^{\theta-ss}}]^G,$$

where we write $\mathcal{Q}_{\mathfrak{P}}$ for $D(R)/D(R)\Phi([\mathfrak{g},\mathfrak{g}])$, on $\mathcal{M}^\theta_{\mathfrak{p}}(v)$, and the $\mathbb{C}[\mathfrak{P}]$-algebra $\mathcal{A}_{\mathfrak{P}}(v)=\Gamma(\mathcal{A}^\theta_{\mathfrak{P}}(v))$. We also consider the global Hamiltonian reduction $\mathcal{A}^0_{\mathfrak{P}}(v):=[\mathcal{Q}_{\mathfrak{P}}]^G$. Also, for a vector subspace $\mathfrak{p}_0\subset\mathfrak{p}$, we can consider the specialization $\mathcal{M}^\theta_{\mathfrak{p}_0}(v)$ and, for an affine subspace $\mathfrak{P}_0\subset\mathfrak{P}$, we consider the specializations $\mathcal{A}^\theta_{\mathfrak{P}_0}(v),\mathcal{A}_{\mathfrak{P}_0}(v),\mathcal{A}^0_{\mathfrak{P}_0}(v)$.

The algebra $\mathcal{A}_{\mathfrak{P}_0}(v)$ is filtered with commutative associated graded (equal to $\mathbb{C}[\mathcal{M}_{\mathfrak{p}_0}(v)]$, where $\mathfrak{p}_0$ is the vector subspace of $\mathfrak{p}$ parallel to $\mathfrak{P}_0$). The algebra $\mathcal{A}^0_{\mathfrak{P}_0}(v)$ is filtered as well with $\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}_0}(v)]\twoheadrightarrow\operatorname{gr}\mathcal{A}^0_{\mathfrak{P}_0}(v)$. So it makes sense to speak about HC $\mathcal{A}_{\mathfrak{P}_0}(v)$-bimodules or HC $\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodules. Also for two parallel affine subspaces $\mathfrak{P}_0,\mathfrak{P}_0'$ one can speak about HC $\mathcal{A}_{\mathfrak{P}_0'}(v)$-$\mathcal{A}_{\mathfrak{P}_0}(v)$ bimodules or about HC $\mathcal{A}^0_{\mathfrak{P}_0'}(v)$-$\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodules.

For $\mathcal{A}^0_{\mathfrak{P}_0'}(v)$-$\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodules we can still consider the corresponding derived category of all complexes with HC homology. These categories are closed under derived tensor products or under $R\operatorname{Hom}$'s of left or right modules. The proofs are as for $\mathcal{A}^0_{\lambda'}(v)$-$\mathcal{A}^0_\lambda(v)$-bimodules.

3.2.1. *Translation bimodules.* For example, we have the HC $\mathcal{A}^0_{\mathfrak{P}_0'}(v)$-$\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodule $\mathcal{A}^0_{\mathfrak{P}_0,\chi}(v)$ (where $\mathfrak{P}_0'=\chi+\mathfrak{P}_0$). This is the most important family of HC bimodules considered in this paper. Obviously, the specialization of $\mathcal{A}^0_{\mathfrak{P}_0,\chi}(v)$ to $\lambda\in\mathfrak{P}_0$ coincides with $\mathcal{A}^0_{\lambda,\chi}(v)$.

Yet another family that we will need for technical reasons is $\mathcal{A}^{(\theta)}_{\mathfrak{P}_0,\chi}(v)$ defined analogously to $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$. This is a HC $\mathcal{A}_{\mathfrak{P}_0+\chi}(v)$-$\mathcal{A}_{\mathfrak{P}_0}(v)$-bimodule and hence also a HC $\mathcal{A}^0_{\mathfrak{P}_0+\chi}(v)$-$\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodule. An important result here is as follows, [BPW, Proposition 6.23].

Proposition 3.3. *The $\mathcal{A}_{\mathfrak{P}}(v)$-bimodule $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)$ is independent of θ.*

Let us provide the proof for readers convenience.

Proof. Let $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{reg}$ denote the locus where $\mathcal{M}^\theta_{\mathfrak{p}}(v)\to\mathcal{M}_{\mathfrak{p}}(v)$ is an isomorphism, it is independent of θ and coincides with the union of the open symplectic leaves in $\mathcal{M}_\lambda(v), \lambda\in\mathfrak{p}$. The sheaf $\mathcal{A}^\theta_{\mathfrak{P},\chi}|_{\mathcal{M}_{\mathfrak{p}}(v)^{reg}}$ is a quantization of $\mathcal{O}_{\mathfrak{p}}(\chi)|_{\mathcal{M}_{\mathfrak{p}}(v)^{reg}}$.

The 1st cohomology of the structure sheaf of $\mathcal{M}_{\mathfrak{p}}(v)^{reg}$ vanish. This is because $H^1(\mathcal{M}^\theta_{\mathfrak{p}}(v),\mathcal{O})=0$, $\mathcal{M}_{\mathfrak{p}}(v)$ is Cohen-Macaulay, and the complement of $\mathcal{M}_{\mathfrak{p}}(v)^{reg}$ in $\mathcal{M}_{\mathfrak{p}}(v)$ has codimension 3, compare to the proof of Proposition of [BPW, Proposition 3.7]. It follows that there is a unique microlocal deformation of $\mathcal{O}(\chi)|_{\mathcal{M}_{\mathfrak{p}}(v)^{reg}}$. So the restrictions of all $\mathcal{A}^\theta_{\mathfrak{P},\chi}(v)$ to $\mathcal{M}_{\mathfrak{p}}(v)^{reg}$ coincide. Since the codimension of $\mathcal{M}^\theta_{\mathfrak{p}}(v)\setminus\mathcal{M}^\theta_{\mathfrak{p}}(v)^{reg}$ is bigger than 2, we have $\Gamma(\mathcal{M}^\theta_{\mathfrak{p}}(v)^{reg},\mathcal{A}^\theta_{\mathfrak{P},\chi}(v))=\Gamma(\mathcal{M}^\theta_{\mathfrak{p}}(v),\mathcal{A}^\theta_{\mathfrak{P},\chi}(v))$. The left hand side is independent of θ and so we get the claim of the proposition. □

We would like to point out that the specialization $\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)_\lambda$ admits a natural homomorphism to $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$. This homomorphism is injective because Γ is left exact. We do not know if this is an isomorphism in general, but this is so under additional assumptions.

Lemma 3.4. *Let $\mathfrak{P}_0\subset\mathfrak{P}$ be an affine subspace and pick $\chi\in\mathbb{Z}^{Q_0}$. Suppose that one of the following conditions holds:*

(1) $H^1(\mathcal{M}^\theta(v),\mathcal{O}(\chi))=0$.
(2) $(\lambda+\chi,\theta)\in\mathfrak{AL}(v)$.

Then $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)=\mathcal{A}^{(\theta)}_{\mathfrak{P}_0,\chi}(v)_\lambda$.

Proof. We can view $\mathcal{A}^\theta_{\lambda,\chi}(v)$ as a sheaf on $\mathcal{M}^\theta(v)$ quantizing $\mathcal{O}(\chi)$. Let us show that both our assumptions imply that $H^1(\mathcal{A}^\theta_{\lambda,\chi}(v))=0$. Then we can apply [BPW, Proposition 6.26] to deduce $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)=\mathcal{A}^{(\theta)}_{\mathfrak{P}_0,\chi}(v)_\lambda$.

The filtration on $\mathcal{A}^\theta_{\lambda,\chi}(v)$ induces a separated filtration on $H^1(\mathcal{M}^\theta(v),\mathcal{A}^\theta_{\lambda,\chi}(v))$ (the claim that the filtration is separated is proved similarly to the proof of [GL, Lemma 5.6.3]) with $H^1(\mathcal{M}^\theta(v),\mathcal{O}(\chi))\twoheadrightarrow \operatorname{gr} H^1(\mathcal{M}^\theta(v),\mathcal{A}^\theta_{\lambda,\chi}(v))$. So the equality $H^1(\mathcal{M}^\theta(v),\mathcal{O}(\chi))=0$ implies $H^1(\mathcal{M}^\theta(v),\mathcal{A}^\theta_{\lambda,\chi}(v))=0$.

Now assume that $(\lambda+\chi,\theta)\in\mathfrak{AL}(v)$. Then any object in $\mathcal{A}^\theta_\lambda(v)$-mod has no higher cohomology, and we are done. □

3.2.2. *Supports in parameters.* We also have the following elementary but important property. By the right $\mathfrak{P}$-support of an $\mathcal{A}_{\mathfrak{P}}(v)$-bimodule $\mathcal{B}$ (denoted by $\operatorname{Supp}^r_{\mathfrak{P}}(\mathcal{B})$), we mean the set of all $\lambda\in\mathfrak{P}$ such that the specialization $\mathcal{B}_\lambda$ is nonzero. Analogously, we can speak about the left support $\operatorname{Supp}^\ell_{\mathfrak{P}}(\mathcal{B})$.

Lemma 3.5. *For a closed subscheme Y of $\mathfrak{P}$, set $\mathcal{A}^0_Y(v):=\mathbb{C}[Y]\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{A}^0_{\mathfrak{P}}(v)$. Any finitely generated right $\mathcal{A}^0_Y(v)$-module $\mathcal{B}$ is generically free over $\mathbb{C}[Y]$, i.e., there is a non zero divisor $f\in\mathbb{C}[Y]$ such that the localization $\mathcal{B}_f$ is a free $\mathbb{C}[Y]_f$-module.*

Proof. We will need to modify a filtration on $\mathcal{A}^0_Y(v)$ so that $\mathbb{C}[Y]$ lives in degree 0. Consider the Rees algebra $\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$ and its base change $\tilde{\mathcal{A}}^0_{\mathfrak{P}}(v)_\hbar=\mathbb{C}[\mathfrak{P},\hbar]\otimes_{\mathbb{C}[\mathfrak{P},\hbar]}\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$, where the endomorphism of $\mathbb{C}[\mathfrak{P},\hbar]$ is given by $\hbar\mapsto\hbar,\alpha\mapsto\alpha\hbar$ for $\alpha\in\mathfrak{P}^*$ (here we consider $\mathfrak{P}=\mathbb{C}^{Q_0}$ as a vector space, not as an affine space). The algebra $\tilde{\mathcal{A}}^0_{\mathfrak{P}}(v)_\hbar$ is graded with

$\deg \hbar = 1, \deg \mathbb{C}[\mathfrak{P}] = 0$. Also the specializations of $\tilde{\mathcal{A}}^0_{\mathfrak{P}}(v)_\hbar, \mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$ at $\hbar = 1$ are the same and so coincide with $\mathcal{A}^0_{\mathfrak{P}}(v)$. We equip $\mathcal{A}^0_{\mathfrak{P}}(v)$ with the filtration coming from the grading on $\tilde{\mathcal{A}}^0_{\mathfrak{P}}(v)_\hbar$ and we equip the quotient $\mathcal{A}^0_Y(v)$ of $\mathcal{A}^0_{\mathfrak{P}}(v)$ with the induced filtration. We remark that $\operatorname{gr} \mathcal{A}^0_Y(v)$ is now a quotient of $\mathbb{C}[\mathcal{M}^0(v)] \otimes \mathbb{C}[Y]$.

A finitely generated right module $\mathcal{B}$ admits a good filtration. By a general commutative algebra result, [Ei, Theorem 14.4], $\operatorname{gr} \mathcal{B}$ is generically free over $\mathbb{C}[Y]$. So there is a non zero divisor f such that $(\operatorname{gr} \mathcal{B})_f$ is free over $\mathbb{C}[Y]_f$. It follows that $\mathcal{B}_f$ and $(\operatorname{gr} \mathcal{B})_f$ are isomorphic free $\mathbb{C}[Y]_f$-modules, and we are done. □

There is a trivial but very important corollary of this lemma.

Corollary 3.6. *Let $\mathcal{B}$ be a Harish-Chandra $\mathcal{A}^0_{\mathfrak{P}'_0}(v)$-$\mathcal{A}^0_{\mathfrak{P}_0}(v)$-bimodule. Then the following claims hold:*

(1) *There is $f \in \mathbb{C}[\mathfrak{P}_0]$ such that $\mathcal{B}_f$ is a free $\mathbb{C}[\mathfrak{P}_0]_f$-module.*
(2) $\operatorname{Supp}^r_{\mathfrak{P}_0}(\mathcal{B})$ *is a constructible set.*

We also have left-handed analogs of these claims.

Proof. A Harish-Chandra bimodule is finitely generated as a right $\mathcal{A}_{\mathfrak{P}_0}(v)$-module (this was noted in the beginning of Section 3.1). So (1) follows from Lemma 3.5.

To prove (2) we note that the support of any finitely generated right $\mathcal{A}_Y(v)$-module is a constructible subset of Y. This is because, as we have seen in the proof of Lemma 3.5, any such module is the direct limit of finitely generated $\mathbb{C}[Y]$-modules. □

Below, Proposition 3.13, we will see that $\operatorname{Supp}^r_{\mathfrak{P}_0}(\mathcal{B})$ is actually a closed subset.

Here is how we are going to use (2). Let $\mathcal{B}$ be a HC $\mathcal{A}_{\mathfrak{P}_0+\chi}$-$\mathcal{A}_{\mathfrak{P}_0}$-bimodule, where $\mathfrak{P}_0 \subset \mathfrak{P}$ is an affine subspace. Then if $\mathcal{B}_\lambda = 0$ for a Weil generic $\lambda \in \mathfrak{P}_0$ (we say that a parameter is Weil generic if it lies outside of the countable union of algebraic subvarieties), then $\mathcal{B}_\lambda = 0$ for a Zariski generic λ as well. HC $\mathcal{A}_{\lambda+\chi}(v)$-$\mathcal{A}_\lambda(v)$-bimodules for λ Weil generic are easier then for an arbitrary (even Zariski generic) λ. We will use this observation many times in our discussion of short wall-crossing functors through the affine wall, Section 10.

3.3. **Restriction functors: construction.** We want to define restriction functors for Harish-Chandra bimodules over $\mathcal{A}^0_{\mathfrak{P}}(v)$ (or over $\mathcal{A}_{\mathfrak{P}}(v)$) similar to the functors $\bullet_\dagger$ used in [L2, L5]. Those will be exact $\mathbb{C}[\mathfrak{P}]$-linear functors mapping HC bimodules over $\mathcal{A}^0_{\mathfrak{P}}(v)$ to those over $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})$, an algebra defined similarly to $\mathcal{A}^0_{\mathfrak{P}}(v)$ but for the quiver $\hat{Q}$ and vectors $\hat{v}, \hat{w}$ that were constructed in Section 2.1.6 (in fact, we will sometimes need to modify the algebras $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})$, see below).

3.3.1. *Algebras $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})$, etc.* Let us proceed to the construction of $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})$. Let $\hat{\mathfrak{P}} = \hat{\mathfrak{g}}^{\hat{G}*}$ be the parameter space for the quantizations associated to $(\hat{Q}, \hat{v}, \hat{w})$. Let us define an affine map $\hat{r} : \mathfrak{P} \to \hat{\mathfrak{P}}$ whose differential is the restriction map $r : \mathfrak{g}^{G*} \to \hat{\mathfrak{g}}^{\hat{G}*}$. Namely, recall that we have elements $\varrho(v), \hat{\varrho}(\hat{v})$ (the former is defined by (2.7) and the latter is defined analogously). Now set

$$\hat{r}(\lambda) := r(\lambda - \varrho(v)) + \hat{\varrho}(\hat{v}). \tag{3.2}$$

Further, set $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v}) := \mathbb{C}[\mathfrak{P}] \otimes_{\mathbb{C}[\hat{\mathfrak{P}}]} \hat{\mathcal{A}}^0_{\hat{\mathfrak{P}}}(\hat{v})$ and define $\hat{\mathcal{A}}^\theta_{\mathfrak{P}}(\hat{v})$ in a similar way. Here

$$\hat{\mathcal{A}}^0_{\hat{\mathfrak{P}}}(\hat{v}) := [D(\hat{R})/D(\hat{R})\Phi([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])]^{\hat{G}},$$

where $\hat{R} = R(\hat{Q}, \hat{v}, \hat{w})$.

We want to get decompositions similar to (2.5),(2.6) on the quantum level. For this, we consider the Rees sheaves and algebras $\mathcal{A}^\theta_{\mathfrak{P}}(v)_\hbar, \mathcal{A}_{\mathfrak{P}}(v)_\hbar, \mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$ defined for the filtrations by the order of a differential operator. We can complete those at x getting the algebras $\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar, \mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$ with $\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar/(\hbar) = \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)^{\wedge_x}], \mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)^{\wedge_x}] \twoheadrightarrow \mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar/(\hbar)$ and the sheaf of algebras $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$ on $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_x}$ obtained by the $\hbar$-adic completion of

$$\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar \otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar} \mathcal{A}^\theta_{\mathfrak{P}}(v)_\hbar$$

Note that $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar/(\hbar) = \mathcal{O}_{\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_x}}$.

Lemma 3.7. *We have the following decompositions.*

$$\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar = \hat{\mathcal{A}}^0_{\mathfrak{P}}(v)^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbf{A}^{\wedge_0}_\hbar, \tag{3.3}$$

$$\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar = \hat{\mathcal{A}}_{\mathfrak{P}}(v)^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbf{A}^{\wedge_0}_\hbar, \tag{3.4}$$

$$\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar = \left(\hat{\mathcal{A}}^\theta_{\mathfrak{P}}(v)_\hbar \otimes_{\mathbb{C}[[\hbar]]} \mathbf{A}_\hbar\right)^{\wedge_0}. \tag{3.5}$$

Isomorphisms (3.3),(3.5) become (2.5),(2.6) after setting $\hbar = 0$. (3.4) is obtained from (3.5) by taking global sections.

By $\mathbf{A}_\hbar$ we denote the homogenized Weyl algebra of R_0 and we write $\mathbf{A}^{\wedge_0}_\hbar$ for the quantization of the symplectic formal polydisk $R_0^{\wedge_0}$.

Proof. The proof follows that of [L6, Lemma 6.5.2]. We provide it for reader's convenience.

Let U denote the symplectic part of the slice module for r. Then, as we have mentioned in 2.1.6,

$$(T^*R)^{\wedge_{G_r}} \cong ((T^*G \times U)/\!\!/\!\!/ G_r)^{\wedge_{G/G_r}}, \tag{3.6}$$

where G_r acts diagonally on $T^*G \times U$. We can consider the quantization $D_\hbar(R)^{\wedge_{G_r}}$ of $(T^*R)^{\wedge_{G_r}}$ obtained by the completion of the homogenized Weyl algebra on T^*R. Also we can consider the quantization

$$[D_\hbar(G)^{\wedge_G} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbf{A}_\hbar(U)^{\wedge_0}]/\!\!/\!\!/_0 G_r$$

of $([T^*G \times U]/\!\!/\!\!/ G_r)^{\wedge_{G/G_r}}$ (where we use the symmetrized quantum comoment map for G_r). Those are canonical quantizations in the sense of [BezKa1] (for the second quantization this follows from [L6, Section 5.4]) and so they are isomorphic. Consequently, their reductions (both affine and GIT) for the G-action (again, with respect to the symmetrized quantum comoment map Φ^{sym}) are isomorphic. But the reduction of the quantization of the right hand side of (3.6) coincides with

$$\mathbb{C}[[\mathfrak{p}, \hbar]] \widehat{\otimes}_{\mathbb{C}[[\hat{\mathfrak{p}}, \hbar]]} \left[\mathbf{A}^{\wedge_0}_\hbar(U)/\mathbf{A}^{\wedge_0}_\hbar(U)\hat{\Phi}^{sym}([\hat{\mathfrak{g}}, \hat{\mathfrak{g}}])\right]^{\hat{G}}.$$

Since $\Phi - \varrho(v), \hat{\Phi} - \hat{\varrho}(\hat{v})$ are the symmetrized quantum comoment maps, (3.3) and (3.5) follow. (3.4) is obtained from (3.5) after taking global sections on both sides. $\square$

Let us observe that

$$r(\varrho(v)) - \hat{\varrho}(\hat{v}) \in \mathbb{Z}^{\hat{Q}_0}. \tag{3.7}$$

Indeed, by reversing some arrows in $\hat{Q}^{\hat{w}}$ (the quiver obtained from $\hat{Q}$ by adjoining the new vertex ∞, see 2.1.6), we can arrange (by reversing some arrows, perhaps, including arrows coming from ∞) that $R, R_x \oplus \mathfrak{g}/\mathfrak{g}_r$ are isomorphic up to a trivial direct summand. Since

$\mathfrak{g}/\mathfrak{g}_r$ is an orthogonal G_r-module, we see that $\bigwedge^{top} R \cong \bigwedge^{top} R_x$ as G_r-modules. Then we need to turn ∞ back to a sink, so we have to reverse some arrows. Reversing an arrow in a quiver results in adding an integral character to the quantum comoment map, and so (3.7) follows.

3.3.2. *Euler derivations.* The sheaf $\mathcal{A}^\theta_{\mathfrak{P}}(v)_\hbar$ comes with a $\mathbb{C}^\times$-action (that is induced now by the fiberwise dilation action on T^*R) and hence with the Euler derivation eu satisfying $\mathsf{eu}(\hbar) = \hbar$. This derivation extends to the completion $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$. On the other hand, the product $\hat{\mathcal{A}}^\theta_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbf{A}^{\wedge_0}_\hbar$ comes with a $\mathbb{C}^\times$-action, and hence with the Euler derivation $\hat{\mathsf{eu}}$ again satisfying $\hat{\mathsf{eu}}(\hbar) = \hbar$. We want to compare derivations eu and $\hat{\mathsf{eu}}$ of $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$ (and similarly defined derivations of $\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$).

Lemma 3.8. *There is an element $a \in \mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$ such that $\mathsf{eu} - \hat{\mathsf{eu}} = \frac{1}{\hbar}[a, \cdot]$ on $\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$ and on $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$.*

Proof. Consider a more general setting. Let R be a vector space, G be a reductive group acting on R, $v \in \mu^{-1}(0) \subset T^*R$ be a point such that Gv is closed and G_v is connected (for simplicity). Let $\Phi : \mathfrak{g} \to D_\hbar(R)$ be a symmetrized quantum comoment map and let $\theta : G \to \mathbb{C}^\times$ be a character. Consider the quantum Hamiltonian reduction $(D_\hbar(R)/\!\!/\!\!/^\theta_\lambda G)^{\wedge_v}$ (there the completion is taken at the image x of v in $T^*R/\!\!/\!\!/_0 G$). Let d be a G-invariant $\mathbb{C}[\hbar]$-linear derivation of $D_\hbar(R)^{\wedge_{Gv}}$ such that $d \circ \Phi = 0$ so that d induces derivations d^θ on $(D_\hbar(R)/\!\!/\!\!/^\theta_\lambda G)^{\wedge_v}$ and d^0 on $(D_\hbar(R)/\!\!/\!\!/^0_\lambda G)^{\wedge_v}$. We claim that

(*) there is an element $a \in (D_\hbar(R)/\!\!/\!\!/^0_\lambda G)^{\wedge_v}$ such that $d^\theta = \frac{1}{\hbar}[a, \cdot]$ and $d^0 = \frac{1}{\hbar}[a, \cdot]$.

To apply (*) in our situation, we take $d = \mathsf{Eu} - \hat{\mathsf{Eu}}$. Here Eu is the derivation of $D_\hbar(R)^{\wedge_{Gv}}$ induced by the fiberwise $\mathbb{C}^\times$-action on T^*R and $\hat{\mathsf{Eu}}$ is the derivation induced by the fiberwise $\mathbb{C}^\times$-action on $T^*(G *_{G_v} R_x)$.

To prove (*) note that we can replace G with a finite central extension and assume that $G = G_0 \times T$, where T is a torus and G_0 satisfies $G_0 = (G_0, G_0)G_v$. So $D_\hbar(R)^{\wedge_{Gv}} = D_\hbar(T)^{\wedge_T} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} D_\hbar(Y)^{\wedge_{G_0 y}}$, where $Y = G_0 *_{G_r} R_x$ and y is the point $[1, 0] \in Y$. The algebra $D_\hbar(Y)^{\wedge_{G_0 y}}$ is the reduction of $D_\hbar(R)^{\wedge_{Gv}}$ by the action of T and so d descends to $D_\hbar(Y)^{\wedge_{G_0 y}}$. Furthermore, $(D_\hbar(R)/\!\!/\!\!/^\theta_\lambda G)^{\wedge_v} = D_\hbar(Y)^{\wedge_{G_0 y}}/\!\!/\!\!/^\theta_\lambda G_0$. Let us note that $H^1_{DR}(T^*Y) = 0$ because of the assumption $G_0 = (G_0, G_0)G_v$. Modulo $\hbar$, the derivation d is a symplectic vector field on the formal neighborhood of $G_0 y$ in T^*Y. So it is Hamiltonian. From here we deduce that $d = \frac{1}{\hbar}[\tilde{a}, \cdot]$ for some element $\tilde{a} \in D_\hbar(Y)^{\wedge_{G_0 y}}$. This element commutes with $\Phi(\mathfrak{g}_0)$ and hence is G_0-invariant. For a we take its image in $(D_\hbar(R)/\!\!/\!\!/^0_\lambda G)^{\wedge_x}$. It is straightforward to see that this element satisfies (*). $\square$

3.3.3. *Construction of $\bullet_{\dagger,x}$.* Let us now proceed to constructing $\bullet_{\dagger,x} : \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)) \to \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$. Define the category $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)$ as the category of $\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$-bimodules $\mathcal{B}'_\hbar$ that are

- finitely generated as bimodules,
- flat over $\mathbb{C}[[\hbar]]$ and complete and separated in the $\hbar$-adic topology,
- satisfy $[a, b] \in \hbar\mathcal{B}'_\hbar$ for all $a \in \mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar, b \in \mathcal{B}'_\hbar$,
- and come equipped with a derivation Eu compatible with eu on $\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$.

Similarly, we can define the category $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar)$ (we need to have a derivation compatible with $\hat{\mathsf{eu}}$). The categories $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)$ and $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar)$ are equivalent as follows. Using the decomposition (3.4), we view $\mathcal{B}'_\hbar \in \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)$ as a bimodule over $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}^{\wedge_0}_\hbar$. Similarly to [L2, Proposition 3.3.1], this bimodule splits as $\hat{\mathcal{B}}'_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}^{\wedge_0}_\hbar$, where $\hat{\mathcal{B}}'_\hbar$ is an $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar$-bimodule. The derivation $\hat{\mathsf{Eu}} := \mathsf{Eu} - \frac{1}{\hbar}[a,\cdot]$ on $\mathcal{B}'_\hbar$ is compatible with the derivation $\hat{\mathsf{eu}}$ on $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}^{\wedge_0}_\hbar$ and so restricts to $\hat{\mathcal{B}}'_\hbar$ making it an object of $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(v)^{\wedge_0}_\hbar)$. An equivalence $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar) \xrightarrow{\sim} \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar)$ we need maps $\mathcal{B}'_\hbar$ to $\hat{\mathcal{B}}'_\hbar$. A quasi-inverse equivalence sends $\hat{\mathcal{B}}'_\hbar$ to $\hat{\mathcal{B}}'_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}^{\wedge_0}_\hbar$.

Pick $\mathcal{B} \in \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v))$. Choose a good filtration on $\mathcal{B}$ and let $\mathcal{B}_\hbar \in \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)_\hbar)$ be the Rees bimodule. So the completion $\mathcal{B}^{\wedge_x}_\hbar$ is an $\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$-bimodule. By the construction, $\mathcal{B}_\hbar$ comes with the derivation $\mathsf{Eu} := \hbar\partial_\hbar$ compatible with the derivation eu on $\mathcal{A}_{\mathfrak{P}}(v)_\hbar$. The derivation Eu extends to $\mathcal{B}^{\wedge_x}_\hbar$ that makes the latter an object of $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)$. From this object we get $\hat{\mathcal{B}}'_\hbar \in \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_x}_\hbar)$.

By [L2, Proposition 3.3.1], the $\hat{\mathsf{Eu}}$-finite part $\hat{\mathcal{B}}_\hbar$ is dense in $\hat{\mathcal{B}}'_\hbar$. Since $\hat{\mathcal{B}}'_\hbar$ is a finitely generated bimodule over $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar$, and $\hat{\mathcal{B}}_\hbar$ is dense, we can choose generalized $\hat{\mathsf{eu}}$-eigenvectors for generators of $\hat{\mathcal{B}}'_\hbar$. Now it is easy to see that $\hat{\mathcal{B}}_\hbar$ is finitely generated over $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar$. In its turn, this implies that $\hat{\mathcal{B}}_\hbar$ can be made into a graded $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar$-bimodule.

We set $\mathcal{B}_{\dagger,x} := \hat{\mathcal{B}}_\hbar/(\hbar-1)$, it is a Harish-Chandra $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})$-bimodule, a good filtration comes from the $\mathbb{C}^\times$-action on $\hat{\mathcal{B}}_\hbar$. Similarly to [L2, Section 3.4], we see that the assignment $\mathcal{B} \to \mathcal{B}_{\dagger,x}$ is functorial.

Let us note that the functor is independent (up to an isomorphism) of the choice of a (which is defined uniquely up to a summand from $\mathbb{C}[[\mathfrak{P},\hbar]]$). This is because the spaces of $\mathbb{C}^\times$-finite sections arising from a and $a+f$ with $f \in \mathbb{C}[[\mathfrak{P},\hbar]]$ are obtained from one another by applying $\exp([F,\cdot])$, where $F := \frac{1}{\hbar}\int_0^\hbar f d\hbar$.

So we have constructed $\bullet_{\dagger,x} : \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)) \to \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$.

3.3.4. *Variations.* The functor $\bullet_{\dagger,x} : \mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)) \to \mathrm{HC}(\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v}))$ is constructed completely analogously. Similarly to Section 3.1.1, any HC $\mathcal{A}_{\mathfrak{P}}(v)$-bimodule $\mathcal{B}$ is also HC over $\mathcal{A}^0_{\mathfrak{P}}(v)$ and $\mathcal{B}_{\dagger,x}$ does not depend on whether we consider $\mathcal{B}$ as an $\mathcal{A}_{\mathfrak{P}}(v)$-bimodule or as a $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimodule.

Note also that above we have established a functor $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v)^{\wedge_x}_\hbar) \to \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$. Denote it by Ψ. We also have a version of this functor for the $\mathcal{A}^0$-algebras (again, denoted by Ψ).

In the case of affine quivers, we sometimes will need a slight modification of the target category for $\bullet_{\dagger,x}$. Namely, we remark that 0 does not need to be a single symplectic leaf in $\hat{\mathcal{M}}(\hat{v})$. This happens, for example, when the quiver $\hat{Q}$ is a single loop or is a union of such. Let $\mathcal{L}_0$ be a leaf through $0 \in \hat{\mathcal{M}}(\hat{v})$, this is an affine space. So the algebra $\hat{\mathcal{A}}_{\mathfrak{P}}(v)$ splits into the product of the Weyl algebra $\mathbf{A}_0$ quantizing $\mathcal{L}_0$ and of some other algebra $\bar{\mathcal{A}}_{\mathfrak{P}}(\hat{v})$. The latter is obtained by the same reduction but from the space where we replace all summands of the form $\mathrm{End}(\mathbb{C}^{\hat{v}_i})$ with $\mathfrak{sl}_{\hat{v}_i}$. We have a category equivalence $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})) \xrightarrow{\sim} \mathrm{HC}(\bar{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$ sending $\hat{\mathcal{B}}$ to the centralizer $\bar{\mathcal{B}}$ of $\mathbf{A}_0$ in $\hat{\mathcal{B}}$ (so that $\hat{\mathcal{B}} = \mathbf{A}_0\otimes\bar{\mathcal{B}}$). We will view $\bullet_{\dagger,x}$ as a functor with target category $\mathrm{HC}(\bar{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$.

3.4. **Restriction functors: properties.** It is straightforward from the construction that $\bullet_{\dagger,x}$ is exact and $\mathbb{C}[\mathfrak{P}]$-linear, compare to [L2, Section 3.4] or [L7, Section 4.1.4].

Now let us describe the behavior of the functor $\bullet_{\dagger,x}$ on the associated varieties. The following lemma follows straightforwardly from the construction (compare with (4) of [L5, Proposition 3.6.5]).

Lemma 3.9. *Let $\mathcal{B}$ be a HC $\mathcal{A}_{\mathfrak{P}}(v)$-bimodule. Then the associated variety of $\mathcal{B}_{\dagger,x}$ is uniquely characterized by $\mathrm{V}(\mathcal{B}_{\dagger,x})\times\mathcal{L}^{\wedge_x}=\mathrm{V}(\mathcal{B})^{\wedge_x}$, where $\mathcal{L}$ is the symplectic leaf through x. A similar claim holds for HC $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimodules.*

Now let us proceed to the compatibility of $\bullet_{\dagger,x}$ with the Tor's and Ext's.

Lemma 3.10. *We have a functorial isomorphism*

$$\operatorname{Tor}_i^{\mathcal{A}^0_{\mathfrak{P}_0}(v)}(\mathcal{B}^1,\mathcal{B}^2)_{\dagger,x}=\operatorname{Tor}_i^{\hat{\mathcal{A}}_{\mathfrak{P}_0}(\hat{v})}(\mathcal{B}^1_{\dagger,x},\mathcal{B}^2_{\dagger,x}).$$

Here $\mathcal{B}^1\in\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}_0}(v)\text{-}\mathcal{A}^0_{\mathfrak{P}_0'}(v))$ and $\mathcal{B}^2\in\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}_0'}(v)\text{-}\mathcal{A}^0_{\mathfrak{P}_0''}(v))$, where $\mathfrak{P}_0,\mathfrak{P}_0',\mathfrak{P}_0''$ are three parallel affine subspaces in $\mathfrak{P}$. Similarly, we have

$$\operatorname{Ext}^i_{\mathcal{A}^0_{\mathfrak{P}_0}(v)}(\mathcal{B}^1,\mathcal{B}^2)_{\dagger,x}=\operatorname{Ext}^i_{\hat{\mathcal{A}}_{\mathfrak{P}_0}(\hat{v})}(\mathcal{B}^1_{\dagger,x},\mathcal{B}^2_{\dagger,x}),$$

where $\mathcal{B}^1\in\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}_0}(v)\text{-}\mathcal{A}^0_{\mathfrak{P}_0'}(v))$ and $\mathcal{B}^2\in\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}_0}(v)\text{-}\mathcal{A}^0_{\mathfrak{P}_0''}(v))$.

Proof. It is sufficient to prove the lemma in the case when $\mathfrak{P}_0=\mathfrak{P}$. We will do the case of Tor's, the case of Ext's is similar.

Consider the bounded derived category $D^b(\mathcal{A}^0_{\mathfrak{P}}(v))$ of the category $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimod of finitely generated $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimodules and its subcategory $D^b_{HC}(\mathcal{A}^0_{\mathfrak{P}}(v))$ of all complexes with HC homology. Similarly, consider the bounded derived category $D^b(\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar)$ of the category $\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$-grbimod of graded finitely generated graded $\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$-bimodules and its subcategory $D^b_{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar)$ of all complexes whose homology mod $\hbar$ are $\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]$-modules (rather than just arbitrary bimodules). We have a functor $\mathbb{C}_1\otimes_{\mathbb{C}[\hbar]}\bullet:\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$-grbimod $\to$ $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimod whose kernel is the subcategory $\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$-grbimod$_{tor}$ of all bimodules where $\hbar$ acts locally nilpotently. This gives rise to the equivalence

$$\mathbb{C}_1\otimes_{\mathbb{C}[\hbar]}\bullet:D^b(\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar)/D^b_{tor}(\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar)\to D^b(\mathcal{A}^0_{\mathfrak{P}}(v))\tag{3.8}$$

that restricts to an equivalence of the HC subcategories and clearly intertwines the derived tensor product (or Hom) functors.

Let us proceed to the completed setting. Consider the algebra

$$\mathfrak{A}:=\mathbb{C}[\mathsf{eu}]\ltimes(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x,opp}_\hbar),$$

where $[\mathsf{eu},a]=\hbar\partial_\hbar a$ for $a\in\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x,opp}_\hbar$. Any module over $\mathfrak{A}$ is a $\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$-bimodule equipped with an Euler derivation (but not vice versa). Let $D^b(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)\subset D^b(\mathfrak{A}\text{-mod})$ stand for the full subcategory $D^b_{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar)$ of all objects whose homology is a HC $\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$-bimodule. We have the completion functor

$$\bullet^{\wedge_x}:=(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x,opp}_\hbar)\otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar\otimes_{\mathbb{C}[\hbar]}\mathcal{A}^0_{\mathfrak{P}}(v)^{opp}_\hbar}\bullet:\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar\text{-grbimod}\to\mathfrak{A}\text{-mod}$$

We remark that, for a HC bimodule $\mathcal{M}$, we have $\mathcal{M}^{\wedge_x}=\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar\otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar}\mathcal{M}$ because the right hand side is already complete as a right $\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar$-module. The completion functor restricts to a functor

$$\bullet^{\wedge_x}:D^b_{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar)\to D^b_{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar).\tag{3.9}$$

This functor preserves the $\hbar$-torsion subcategories and intertwines the tensor product functors and Hom functors (in the categories of left/right modules). It is t-exact.

Now let us equip $\mathcal{B}^1, \mathcal{B}^2$ with good filtrations and consider the corresponding Rees bimodules $\mathcal{B}^1_\hbar, \mathcal{B}^2_\hbar$. Since $\bullet^{\wedge_x}$ is a t-exact functor, we see that

$$H_i(\mathcal{B}^1_\hbar \otimes^L_{\mathcal{A}^0_{\mathfrak{P}}(v)_\hbar} \mathcal{B}^2_\hbar)^{\wedge_x} = H_i(\mathcal{B}^{1\wedge_x}_\hbar \otimes^L_{\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar} \mathcal{B}^{2\wedge_x}_\hbar), \tag{3.10}$$

the equality of HC $\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar$-bimodules.

Recall the functor $\Psi : \mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar) \to \mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$ from Section 3.3. Applying Ψ to the left hand side of (3.10), we get $H_i(\mathcal{B}^1 \otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)} \mathcal{B}^2)_{\dagger,x}$.

Let us see what happens when we apply Ψ to the right hand side. Note that $\mathcal{B}^{i\wedge_x}_\hbar = \mathbf{A}^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} R_\hbar(\mathcal{B}^i_{\dagger,x})^{\wedge_0}$ that yields

$$\mathcal{B}^{1\wedge_x}_\hbar \otimes^L_{\mathcal{A}^0_{\mathfrak{P}}(v)^{\wedge_x}_\hbar} \mathcal{B}^{2\wedge_x}_\hbar = \mathbf{A}^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \left(R_\hbar(\mathcal{B}^1_{\dagger,x})^{\wedge_0} \otimes^L_{\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar} R_\hbar(\mathcal{B}^2_{\dagger,x})^{\wedge_0} \right).$$

So if we apply Ψ to the right hand side of (3.10) we get $H_i(\mathcal{B}^1_{\dagger,x} \otimes_{\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})} \mathcal{B}^2_{\dagger,x})$. This completes the proof. $\square$

Another important property of the restriction functor is the equality

$$\mathcal{A}^0_{\mathfrak{P},\chi}(v)_{\dagger,x} = \hat{\mathcal{A}}^0_{\mathfrak{P},\chi}(\hat{v}). \tag{3.11}$$

This follows from the decomposition $\mathcal{A}^0_{\mathfrak{P},\chi}(v)^{\wedge_x}_\hbar \cong \hat{\mathcal{A}}^0_{\mathfrak{P},\chi}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbf{A}^{\wedge_0}_\hbar$ that is proved similarly to (3.3).

We finish this section with two remarks.

Remark 3.11. Let us explain why in Lemma 3.10 we deal with Tor's rather than with the derived tensor products. The reason is that we do not have the derived version of the functor $\bullet_{\dagger,x}$. The difficulty here is to pass between the derived version of the category $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar)$ to that of the category $\mathrm{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar)$. For the latter derived version we take the subcategory in the derived category of the category of graded $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar$-bimodules with HC homology. For the former derived version we need to use the subcategory in the derived category of modules over

$$\mathbb{C}[\hat{\mathsf{eu}}] \ltimes \left(\mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} \left(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0, opp}_\hbar \right) \right)$$

with homology that is a localization (from $\mathbb{C}[[\hbar]]$ to $\mathbb{C}((\hbar))$) of a HC bimodule. We need to localize to $\mathbb{C}((\hbar))$ because the operator $\frac{1}{\hbar}[a, \cdot]$ is not defined on an arbitrary $\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar$-bimodule. Of course, we still have a completion functor

$$\begin{aligned} &D^b_{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar\text{-grbimod})/D^b_{HC}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})_\hbar\text{-grbimod})_{tor} \to \\ &D^b_{HC}(\mathbb{C}[\hat{\mathsf{eu}}] \ltimes \mathbb{C}((\hbar)) \otimes_{\mathbb{C}[[\hbar]]} \left[\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0}_\hbar \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})^{\wedge_0, opp}_\hbar \right]) \end{aligned}$$

(here grbimod means graded bimodules). A problem with this functor is that it is not an equivalence, the target category has more Hom's, which has to do with the fact that we do not require the action of a derivation $\hat{\mathsf{eu}}$ to be diagonalizable (and we do not see any way to impose this condition).

Let us point out that this problem does not occur in the W-algebra setting, [L2, L7] because there we have a Kazhdan torus action that fixes a point where we complete. So in that case it is enough to deal with $\mathbb{C}^\times$-equivariant derived categories.

Remark 3.12. Let $\tilde{\mathrm{HC}}(\mathcal{A}^0_{\mathfrak{P}}(v))$ denote the category of *locally HC* $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimodules (i.e., bimodules that are sums of their Harish-Chandra subbimodules), the ind completion of $\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}}(v))$. Then, similarly to [L2, Section 3.4],[L5, Section 3.7], we have a functor $\bullet^{\dagger,x}: \mathrm{HC}(\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})) \to \tilde{\mathrm{HC}}(\mathcal{A}^0_{\mathfrak{P}}(v))$ that is right adjoint to $\bullet_{\dagger,x}$. This functor is automatically $\mathbb{C}[\mathfrak{P}]$-linear. It is likely that the image of $\bullet^{\dagger,x}$ actually lies in $\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}}(v))$ but we do not know the proof of this claim. Below we will see $\underline{\mathcal{B}}^{\dagger,x}$ lies in $\mathrm{HC}(\mathcal{A}^0_{\mathfrak{P}}(v))$ provided $\underline{\mathcal{B}}$ is finitely generated over $\mathbb{C}[\mathfrak{P}]$.

3.5. **Restriction functors: applications.** Our first application will be to $\mathfrak{P}$-supports of HC bimodules.

Proposition 3.13. *Let $\mathcal{B}$ be a HC $\mathcal{A}^0_{\mathfrak{P}}(v)$-bimodule. Then $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B})$ is closed and*

$$\mathsf{AC}(\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B})) = \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}).$$

Recall that AC stands for the asymptotic cone.

Proof. Pick a generic point x in an irreducible component of $\mathrm{V}(\mathcal{B}) \cap \mathcal{M}^0_0(v)$ and consider the HC $\hat{\mathcal{A}}^0_{\mathfrak{P}}(\hat{v})$-bimodule $\mathcal{B}_{\dagger,x}$. By the choice of x, $\mathcal{B}_{\dagger,x}$ is finitely generated over $\mathbb{C}[\mathfrak{P}]$, this follows from Lemma 3.9. Moreover, since $\bullet_{\dagger,x}$ is $\mathbb{C}[\mathfrak{P}]$-linear (and is $\mathbb{C}[\mathfrak{p}]$-linear after passing to the associated graded bimodules) by the construction, we have

$$\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x}) \subset \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}), \quad \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}_{\dagger,x}) \subset \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}).$$

Since $\mathcal{B}_{\dagger,x}$ is finitely generated over $\mathbb{C}[\mathfrak{P}]$, we see that $\mathsf{AC}(\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x})) \subset \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}_{\dagger,x})$. Hence $\mathsf{AC}(\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x})) \subset \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B})$. There is the unique maximal subbimodule $\mathcal{B}' \subset \mathcal{B}$ with $\mathcal{B}'_{\dagger,x} = 0$. Clearly, $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x}) \subset \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')$. On the other hand, let I be the right annihilator of $\mathcal{B}_{\dagger,x}$ in $\mathbb{C}[\mathfrak{P}]$. Then $\mathcal{B}I \subset \mathcal{B}'$ and $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}') \subset \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}I) \subset \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x})$. So we see that $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}_{\dagger,x}) = \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')$ is a closed subvariety in $\mathfrak{P}$ whose asymptotic cone coincides with $\mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}(\mathcal{B}/\mathcal{B}')) = \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}_{\dagger,x})$.

Now let us observe that

$$\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}) = \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}') \cup \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}'). \tag{3.12}$$

The inclusion of the left hand side into the right hand side is clear. Now we just need to show that if $z \in \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}') \setminus \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')$, then $z \in \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B})$. Recall that $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')$ is closed. So if $z \notin \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')$, then $\mathrm{Tor}^1_{\mathbb{C}[\mathfrak{P}]}(\mathcal{B}/\mathcal{B}', \mathbb{C}_z) = 0$. Hence if $z \in \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}')$, then $z \in \mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B})$. Similarly,

$$\mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}) = \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}/\mathcal{B}') \cup \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}\mathcal{B}'). \tag{3.13}$$

The variety $\mathcal{M}^0_0(v)$ has finitely many symplectic leaves. Now our claim follows by induction from (3.12) and (3.13) combined with the claim that $\mathsf{AC}(\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{B}/\mathcal{B}')) = \mathrm{Supp}_{\mathfrak{p}}(\operatorname{gr}(\mathcal{B}/\mathcal{B}'))$. $\square$

Now we are ready to prove Proposition 2.7.

Proof of Proposition 2.7. Consider the natural homomorphism $\mathcal{A}^0_{\mathfrak{P}}(v) \to \mathcal{A}_{\mathfrak{P}}(v)$ and let K, C denote its kernel and cokernel. Both $\mathcal{A}^0_{\mathfrak{P}}(v), \mathcal{A}_{\mathfrak{P}}(v)$ are HC bimodules over $\mathcal{A}^0_{\mathfrak{P}}(v)$ and therefore K, C are HC bimodules as well. The homomorphism $\mathcal{A}^0_{\lambda}(v) \to \mathcal{A}_{\lambda}(v)$ is an isomorphism if and only if $\lambda \notin \mathrm{Supp}^r_{\mathfrak{P}}(K) \cup \mathrm{Supp}^r_{\mathfrak{P}}(C)$. By Proposition 3.13, $\mathrm{Supp}^r_{\mathfrak{P}}(K), \mathrm{Supp}^r_{\mathfrak{P}}(C)$ are closed. The homomorphism is surjective if and only if $\lambda \notin \mathrm{Supp}^r_{\mathfrak{P}}(C)$. Further, if $\lambda \notin \mathrm{Supp}^r_{\mathfrak{P}}(C)$, then K is flat over $\mathfrak{P}$ in a neighborhood of λ (as the kernel of an epimorphism of flat modules). Therefore, modulo $\lambda \not\in \operatorname{Supp}^r_{\mathfrak{P}}(C)$, we get $\mathcal{A}^0_\lambda(v) \xrightarrow{\sim} \mathcal{A}_\lambda(v)$ if and only if $\lambda \not\in \operatorname{Supp}^r_{\mathfrak{P}}(K)$.

Consider the homomorphism $\operatorname{gr}\mathcal{A}^0_{\mathfrak{P}}(v) \to \operatorname{gr}\mathcal{A}_{\mathfrak{P}}(v) = \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]$ and compose it with the epimorphism $\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)] \twoheadrightarrow \operatorname{gr}\mathcal{A}^0_{\mathfrak{P}}(v)$. Let K^0, C^0 denote the kernel and the cokernel of the resulting homomorphism $\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)] \to \mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]$. The latter coincides with ρ^*. It follows that $\operatorname{Supp}_{\mathfrak{p}}(K^0 \oplus C^0) \subset \mathfrak{p}^{sing}$, where, recall, $\mathfrak{p}^{sing}$ denotes the locus of non-generic parameters in $\mathfrak{p}$. Note that $C^0 \twoheadrightarrow \operatorname{gr} C$, while $\operatorname{gr} K$ is a subquotient of K^0. Because of this, we have $\mathsf{AC}(\operatorname{Supp}^r_{\mathfrak{P}}(C)) \subset \operatorname{Supp}_{\mathfrak{p}}(C^0)$ and $\mathsf{AC}(\operatorname{Supp}^r_{\mathfrak{P}}(K)) \subset \operatorname{Supp}_{\mathfrak{p}}(K^0)$. The claim of the proposition follows. □

Next we will show that the algebra $\mathcal{A}_\lambda(v)$ is simple for a Weil generic λ, compare with [L5, Section 4.2]. We will obtain a stronger version of this result using wall-crossing functors below, Proposition 8.6.

Proposition 3.14. *The algebra $\mathcal{A}_\lambda(v)$ is simple for a Weil generic λ.*

Proof. Step 1. Let us show that, for a Weil generic λ, the algebra $\mathcal{A}_\lambda(v)$ has no finite dimensional representations. Let $\mathfrak{P}_d$ denote the set of points $\lambda \in \mathfrak{P}$ such that $\mathcal{A}_\lambda(v)$ has a d-dimensional representation or, in other words, there is a homomorphism $\mathcal{A}_\lambda(v) \to \operatorname{Mat}_d(\mathbb{C})$. Consider the ideal $I^d \subset \mathcal{A}_{\mathfrak{P}}(v)$ generated by the elements

$$\alpha_{2n}(x_1, \ldots, x_{2n}) = \bigoplus_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} \ldots x_{\sigma(2n)}.$$

Any homomorphism $\mathcal{A}_{\mathfrak{P}}(v) \to \operatorname{Mat}_d(\mathbb{C})$ factors through $\mathcal{A}_{\mathfrak{P}}(v)/I^d$, this is the Amitsur-Levitski theorem. The support of $\mathcal{A}_{\mathfrak{P}}(v)/I^d$ in $\mathfrak{P}$ is closed by Proposition 3.13. If a Weil generic element of $\mathfrak{P}$ belongs to $\bigcup_d \operatorname{Supp}_{\mathfrak{P}}(\mathcal{A}_{\mathfrak{P}}(v)/I^d)$, then $\operatorname{Supp}_{\mathfrak{p}}(\mathcal{A}_{\mathfrak{P}}(v)/I^d) = \mathfrak{P}$ for some d. By Proposition 3.13, $\operatorname{Supp}_{\mathfrak{p}}(\mathbb{C}[\mathcal{M}_{\mathfrak{p}}(v)]/\operatorname{gr} I^d) = \mathfrak{p}$. However, this is impossible. Indeed, for a Zariski generic λ, the variety $\mathcal{M}_\lambda(v)$ is symplectic, so its algebra of functions has no Poisson ideals. Since $\operatorname{gr} I^d$ is a Poisson ideal, we get a required contradiction.

Step 2. By the previous step, for a Weil generic λ and all $x \in \mathcal{M}(v) \setminus \mathcal{M}(v)^{reg}$, the algebra $\hat{\mathcal{A}}_\lambda(\hat{v})$ defined from x has no finite dimensional irreducible representations. It follows from Lemma 3.9 that the algebra $\mathcal{A}_\lambda(v)$ has no ideals I such that $\operatorname{V}(\mathcal{A}_\lambda(v)/I)$ is a proper subvariety of $\mathcal{M}(v)$. Indeed, for x that is generic in an irreducible component of $\operatorname{V}(\mathcal{A}_\lambda(v)/I)$, the ideal $I_{\dagger,x} \subset \hat{\mathcal{A}}_\lambda(\hat{v})$ is of finite codimension. On the other hand, if I is a proper ideal, then $\operatorname{V}(\mathcal{A}_\lambda(v)/I)$ is also proper, this is consequence of [BoKr, Corollar 3.6]. The proposition follows. □

3.6. **Applications to derived Hamiltonian reduction.** In this section we prove part (1) of Proposition 2.16. The proof does not have to do with HC bimodules but involves techniques similar to what was used in Sections 3.3-3.5.

We will prove the following claim that implies (1) of Proposition 2.16:

(*) There is an asymptotically generic open affine subset $U \subset \mathfrak{P}^{iso}$ such that $\mathcal{Q}_U := \mathcal{Q}_{\mathfrak{P}} \otimes_{\mathbb{C}[\mathfrak{P}]} \mathbb{C}[U]$ is flat over $\mathbb{C}[U]$ and $\operatorname{Tor}_i^{U(\mathfrak{g})}(D(R), \mathbb{C}[U]) = 0$ for $i > 0$.

Let $r \in T^*R$ be a point with closed G-orbit and let $\hat{R}, R_0$ have the same meaning as in 2.1.6. We need to relate $\operatorname{Tor}_i^{U(\mathfrak{g})}(D(R), \mathbb{C}[\mathfrak{P}])$ to $\operatorname{Tor}_i^{U(\mathfrak{g}_r)}(D(\hat{R}), \mathbb{C}[\mathfrak{P}])$, where $\mathbb{C}[\mathfrak{P}]$ becomes a $U(\mathfrak{g}_r)$-module via the inclusion $U(\mathfrak{g}_r) \hookrightarrow U(\mathfrak{g})$.

Lemma 3.15. *We have a natural $\mathbb{C}[\mathfrak{P},\hbar]$-linear isomorphism*

$$\begin{aligned}&\operatorname{Tor}_i^{U_\hbar(\mathfrak{g})}(D_\hbar(R),\mathbb{C}[\mathfrak{P},\hbar])^{\wedge G_r}\cong\\ &\Gamma\left(G/G_r, G*_{G_r}\operatorname{Tor}_i^{U_\hbar(\mathfrak{g}_r)}\left(D_\hbar(\hat{R}),\mathbb{C}[\mathfrak{P},\hbar]\right)\right)^{\wedge G/G_r}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}_\hbar(R_0)^{\wedge_0}.\end{aligned}\tag{3.14}$$

Proof. Recall the isomorphism

$$D_\hbar(R)^{\wedge G_r}\cong\left([D_\hbar(G)^{\wedge G}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}D_\hbar(\hat{R})^{\wedge_0}]/\!\!/\!\!/_0 G_r\right)\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbf{A}_\hbar(R_0)^{\wedge_0}\tag{3.15}$$

that has appeared in the proof of Lemma 3.7.

Note also that

$$\operatorname{Tor}_i^{U_\hbar(\mathfrak{g})}(D_\hbar(R),\mathbb{C}[\mathfrak{P},\hbar])^{\wedge G_r}\cong\operatorname{Tor}_i^{U_\hbar(\mathfrak{g})}(D_\hbar(R)^{\wedge G_r},\mathbb{C}[\mathfrak{P},\hbar]).$$

So we need to check that the right hand side of (3.14) coincides with the ith Tor of the right hand side of (3.15). This will follow if we check that

$$\begin{aligned}&\operatorname{Tor}_i^{U_\hbar(\mathfrak{g})}(\left([D_\hbar(G)\otimes_{\mathbb{C}[\hbar]}D_\hbar(\hat{R})]/\!\!/\!\!/_0 G_r\right),\mathbb{C}[\mathfrak{P},\hbar])\cong\\ &\Gamma\left(G/G_r, G*_{G_r}\operatorname{Tor}_i^{U_\hbar(\mathfrak{g}_r)}(D_\hbar(\hat{R}),\mathbb{C}[\mathfrak{P},\hbar])\right).\end{aligned}\tag{3.16}$$

Since the actions of G_r and $U_\hbar(\mathfrak{g})$ commute and

$$[D_\hbar(G)\otimes_{\mathbb{C}[\hbar]}D_\hbar(\hat{R})]/\!\!/\!\!/_0 G_r=\left([D_\hbar(G)\otimes_{\mathbb{C}[\hbar]}D_\hbar(\hat{R})]\otimes^L_{U_\hbar(\mathfrak{g}_r)}\mathbb{C}[\mathfrak{P},\hbar]\right)^{G_r},$$

we see that the left hand side of (3.16) coincides with

$$\operatorname{Tor}_i^{U_\hbar(\mathfrak{g}\times\mathfrak{g}_r)}(D_\hbar(G)\otimes_{\mathbb{C}[\hbar]}D_\hbar(\hat{R}),\mathbb{C}[\mathfrak{P},\hbar])^{G_r}.$$

Here $\mathbb{C}[\mathfrak{P},\hbar]$ is viewed as the diagonal $\mathfrak{g}\times\mathfrak{g}_r$-module. But to compute $\operatorname{Tor}_i^{U_\hbar(\mathfrak{g}\times\mathfrak{g}_r)}(D_\hbar(G)\otimes_{\mathbb{C}[\hbar]}D_\hbar(\hat{R}),\mathbb{C}[\mathfrak{P},\hbar])$ we can take the derived tensor product with $U_\hbar(\mathfrak{g})$ and after that the derived tensor product with $U_\hbar(\mathfrak{g}_r)$. What we get is exactly the right hand side of (3.16). ☐

Lemma 3.16. *We have* $\mathsf{AC}\left(\overline{\operatorname{Supp}^r_{\mathfrak{P}}(\operatorname{Tor}_i^{U(\mathfrak{g})}(D(R),\mathbb{C}[\mathfrak{P}]))}\right)\subset\mathfrak{p}^{sing}$ *provided* $i>0$.

Proof. Set $M:=\operatorname{Tor}_i^{U(\mathfrak{g})}(D(R),\mathbb{C}[\mathfrak{P}])$, we view it as a $D(R)\otimes\mathbb{C}[\mathfrak{P}]$-module. It is supported on $\mu^{-1}(\mathfrak{p})\times_{\mathfrak{p}}\mathfrak{p}$. Let N be the maximal submodule of M with the property that $\mathrm{V}(N)\cap(\mu^{-1}(0),0)$ is contained in the nilpotent cone of $\mu^{-1}(0)$, equivalently, $N_\hbar^{\wedge_x}=0$ for all nonzero $x\in\mathcal{M}^0(v)$. Note that we have only finitely many possible $G_r\subset G$ and hence finitely many possible spaces $\hat{\mathfrak{P}}$. Moreover, under the natural projection $\mathfrak{p}\to\hat{\mathfrak{p}}$, the preimage of $\hat{\mathfrak{p}}^{sing}$ lies in $\mathfrak{p}^{sing}$ by Remark 2.2. From this observation combined with Lemma 3.15 and an induction argument, it follows that $\mathsf{AC}(\overline{\operatorname{Supp}^r_{\mathfrak{P}}(M/N)})\subset\mathfrak{p}^{sing}$.

Now set $M^0:=\operatorname{Tor}_i^{U(\mathfrak{g})}(\mathbb{C}[T^*R],\mathbb{C}[\mathfrak{p}])$. The space M is naturally filtered with $\operatorname{gr}M\hookrightarrow M^0$. Inside M^0 we can consider the maximal submodule N^0 defined similarly to $N\subset M$. Note that $\operatorname{Supp}_{\mathfrak{p}}(M^0)\subset\mathfrak{p}^{sing}$ and so $\operatorname{Supp}_{\mathfrak{p}}(M^0/N^0)\subset\mathfrak{p}^{sing}$. From here we deduce that

$$\operatorname{Supp}_{\mathfrak{p}}(N^0)\subset\mathfrak{p}^{sing}.\tag{3.17}$$

Clearly, $\operatorname{gr}N\subset N^0$ (a G-equivariant embedding).

The $D(R)$-module N is weakly G-equivariant and finitely generated. So it is generated by finitely many G-isotypic components, say, corresponding to G-irreps $V_1,\ldots,V_k$. We can assume that the corresponding isotypic components generate N^0 as well (N^0 is also finitely generated). Let $N_V \subset N, N_V^0 \subset N^0$ denote the sum of these isotypic components so that $\operatorname{gr} N_V \subset N_V^0$. We have

$$\operatorname{Supp}^r_{\mathfrak{P}}(N_V) = \operatorname{Supp}^r_{\mathfrak{P}}(N), \operatorname{Supp}_{\mathfrak{p}}(N_V^0) = \operatorname{Supp}^r_{\mathfrak{p}}(N^0). \tag{3.18}$$

Since $\mathrm{V}(N^0) \cap (\mu^{-1}(0), 0)$ lies in the nilpotent cone, we see that any G-isotypic component in N^0 is finitely generated over $\mathbb{C}[\mathfrak{p}]$. A similar claim holds for N. From here and the inclusion $\operatorname{gr} N_V \subset N_V^0$ we deduce that $\mathsf{AC}(\operatorname{Supp}^r_{\mathfrak{P}}(N_V)) \subset \operatorname{Supp}_{\mathfrak{p}}(N_V^0)$. Combining (3.17) with (3.18), we see that $\mathsf{AC}(\overline{\operatorname{Supp}^r_{\mathfrak{P}}(N)}) \subset \mathfrak{p}^{sing}$. Since $\operatorname{Supp}^r_{\mathfrak{P}}(M) \subset \operatorname{Supp}^r_{\mathfrak{P}}(N) \cup \operatorname{Supp}^r_{\mathfrak{P}}(M/N)$, we get $\mathsf{AC}(\overline{\operatorname{Supp}^r_{\mathfrak{P}}(M)}) \subset \mathfrak{p}^{sing}$. $\square$

Now let us show that there is an asymptotically generic $U \subset \mathfrak{P}$ such that $\mathcal{Q}_U$ is flat over U. For this, we consider consider various modules $\operatorname{Tor}_i^{\mathbb{C}[\mathfrak{P}]}(\mathcal{Q}_{\mathfrak{P}}, \mathbb{C}[Y])$, where Y is a closed irreducible subvariety in $\mathfrak{P}$. Since $\mathcal{Q}_{\mathfrak{P}}$ is a finitely generated $D(R)$-module, there is an affine Zariski open subset $U \subset \mathfrak{P}$ (not asymptotically generic, a priori) such that $\mathcal{Q}_U$ is free over U, this is proved analogously to Lemma 3.5. It follows that the Zariski closure of the union of the supports of various $\operatorname{Tor}_i^{\mathbb{C}[\mathfrak{P}]}(\mathcal{Q}_{\mathfrak{P}}, \mathbb{C}[Y])$ is a proper subset, say Z, of $\mathfrak{P}$. We need to check that $\mathsf{AC}(Z) \subset \mathfrak{p}^{sing}$. This is done as in the proof of Lemma 3.16, we need to consider $M := \operatorname{Tor}_i^{\mathbb{C}[\mathfrak{P}]}(\mathcal{Q}_{\mathfrak{P}}, \mathbb{C}[Z])$.

4. Localization theorems and translation bimodules

In this chapter we deal with (abelian and derived) localization theorems that allow to relate the category of modules over $\mathcal{A}_\lambda(v)$ to the category of quasi-coherent sheaves of $\mathcal{A}_\lambda^\theta(v)$. We also study more closely translation bimodules introduced in 3.1.4 that play a crucial role in the abelian localization theorems.

4.1. Abelian and derived localization.

Let θ be a generic stability condition and $\lambda \in \mathfrak{P}$. We say that (λ, θ) satisfies abelian (resp., derived) localization if the functors Γ_λ^θ and $\operatorname{Loc}_\lambda^\theta$ are mutually inverse equivalences between $\mathcal{A}_\lambda(v)$-mod and $\mathcal{A}_\lambda^\theta(v)$-mod (resp., $R\Gamma_\lambda^\theta$ and $L\operatorname{Loc}_\lambda^\theta$ are mutually inverse equivalences between $D^b(\mathcal{A}_\lambda(v)\text{-mod})$ and $D^b(\mathcal{A}_\lambda^\theta(v)\text{-mod})$). We will write $\mathfrak{AL}(v)$ for the set of all (λ, θ) satisfying abelian localization.

First, let us recall the derived localization theorem for $\mathcal{A}_\lambda(v)$.

Proposition 4.1. *Suppose that the moment map μ is flat or Q has finite or affine type. Then (λ, θ) satisfies derived localization if and only if the homological dimension of $\mathcal{A}_\lambda(v)$ is finite.*

Proof. The case when μ is flat follows from [MN, Theorem 1.1].

In general, we can apply a quantum LMN isomorphism and assume that ν is dominant. If Q is of finite or affine type, then by 2.1.4, μ is flat. $\square$

Sufficient conditions (in greater generality) for abelian localization to hold were studied in [BPW, Section 5.3]. Let us recall some results from there. For this we need some terminology. By a *classical wall* for v we mean a hyperplane of the form $\{\theta | \theta \cdot v' = 0\}$, where v' is as in 2.1.1. So θ is generic if and only if it does not lie on a classical wall. By a *classical chamber* we mean the closure of a connected component of the complement to the union of classical walls in $\mathbb{R}^{Q_0}$. Let $C = C_\theta$ be the classical chamber of θ.

Proposition 4.2 (Corollary 5.17 in [BPW])**.** *For every λ and any $\chi \in \mathbb{Z}^{Q_0} \cap \mathsf{int} C$ there is $n_0 \in \mathbb{Z}$ such that the $(\lambda + n\chi, \theta) \in \mathfrak{AL}(v)$ for any $n > n_0$.*

Here we write $\mathsf{int}C$ for the interior of C.

Unfortunately, Proposition 4.2 is not good enough for our purposes, as we will need a stronger version. We will also need to relate abelian localization to the functors $\pi^0_\lambda(v), \pi^\theta_\lambda(v)$. Recall the open subset $\mathfrak{P}^{iso}$ of all parameters λ such that $\mathcal{A}^0_\lambda(v) \xrightarrow{\sim} \mathcal{A}_\lambda(v)$. By Proposition 2.7, for every λ there is $\chi \in \mathbb{Z}^{Q_0}$ such that $\lambda + \chi + (C \cap \mathbb{Z}^{Q_0}) \subset \mathfrak{P}^{iso}$.

Proposition 4.3. *The following is true.*

1. *Suppose $\lambda \in \mathfrak{P}^{iso}$. We have $(\lambda, \theta) \in \mathfrak{AL}(v)$ if and only if the functors $\pi^0_\lambda(v)$ and $\pi^\theta_\lambda(v)$ are isomorphic.*
2. *For every λ, there is $\chi \in \mathbb{Z}^{Q_0}$ such that $(\lambda', \theta) \in \mathfrak{AL}(v)$ for every $\lambda' \in \lambda + \chi + (C \cap \mathbb{Z}^{Q_0})$.*

To prove (2) (that will we discuss wall-crossing functors), we will also need a more technical version of (2), which is the following lemma.

Lemma 4.4. *For every λ, there is $\lambda' \in \mathbb{Z}^{Q_0}$ such that $(\lambda'', \theta) \in \mathfrak{AL}(v)$ for λ'' from a subset in $\lambda' + (C \cap \mathbb{Z}^{Q_0})$ whose intersection with every codimension 1 face of the cone $\lambda' + C$ is Zariski dense in that face.*

4.2. **Translation bimodules.** In this subsection we will apply results from Sections 3.4 and 3.5 to studying translation bimodules $\mathcal{A}^0_{\lambda,\chi}(v), \mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ and a connection between them. In particular, here we will prove Propositions 2.13 and 4.3(1) as well as Lemma 4.4.

The next two propositions investigate when various versions of translations coincide.

Proposition 4.5. *Let $\chi, \chi' \in \mathbb{Z}^{Q_0}$. Then the following subsets of $\mathfrak{P}$ are Zariski open and asymptotically generic.*

1. *The set of λ such that $\mathcal{A}^0_{\lambda,\chi}(v) \to \mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)_\lambda$ is an isomorphism.*
2. *The set of λ such that the multiplication homomorphism $\mathcal{A}^0_{\lambda+\chi,\chi'}(v) \otimes_{\mathcal{A}^0_{\lambda+\chi}(v)} \mathcal{A}^0_{\lambda,\chi}(v) \to \mathcal{A}^0_{\lambda,\chi+\chi'}(v)$ is an isomorphism.*

Proof. Let us prove (1). We have a natural surjection $\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \twoheadrightarrow \operatorname{gr}\mathcal{A}^0_{\mathfrak{P},\chi}(v)$ and a natural inclusion $\operatorname{gr}\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v) \hookrightarrow \mathbb{C}[\mu^{-1}(\mathfrak{p})^{\theta-ss}]^{G,\chi}$. Further, the following diagram is commutative (microlocalization commutes with taking the associated graded)

$$\begin{tikzcd}
\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \arrow[r] \arrow[d] & \mathbb{C}[\mu^{-1}(\mathfrak{p})^{\theta-ss}]^{G,\chi} \\
\operatorname{gr}\mathcal{A}^0_{\mathfrak{P},\chi}(v) \arrow[r] & \operatorname{gr}\mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v) \arrow[u]
\end{tikzcd} \tag{4.1}$$

Now the top horizontal arrow becomes an isomorphism when localized to the generic locus in $\mathfrak{p}$. (1) follows from Proposition 3.13, as in the proof of Proposition 2.7.

Let us prove (2). We have natural epimorphisms $\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \twoheadrightarrow \operatorname{gr}\mathcal{A}^0_{\mathfrak{P},\chi}(v), \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,-\chi} \twoheadrightarrow \operatorname{gr}\mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v)$. Note that

$$\operatorname{gr}\mathcal{Q}_{\mathfrak{P}}|_{\mathcal{M}^0_{\mathfrak{p}reg}(v)} = \mathbb{C}[\mu^{-1}(\mathfrak{p})]|_{\mathcal{M}^0_{\mathfrak{p}reg}(v)} \tag{4.2}$$

as $\mu : \mu^{-1}(\mathfrak{p}) \to \mathfrak{p}$ is flat over $\mathfrak{p}^{reg} := \mathfrak{p} \setminus \mathfrak{p}^{sing}$. So the kernels of these epimorphisms are supported on $\mathfrak{p}^{sing}$. From here we see that the kernel of

$$\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,-\chi} \otimes_{\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]} \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \twoheadrightarrow \operatorname{gr} \mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v) \otimes_{\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]} \operatorname{gr} \mathcal{A}^0_{\mathfrak{P},\chi}$$

is also supported on $\mathfrak{p}^{sing}$. (4.2) also shows that the kernel of

$$\operatorname{gr} \mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v) \otimes_{\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]} \operatorname{gr} \mathcal{A}^0_{\mathfrak{P},\chi} \twoheadrightarrow \operatorname{gr} \left(\mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v) \otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)} \mathcal{A}^0_{\mathfrak{P},\chi}(v) \right)$$

is supported on $\mathfrak{p}^{sing}$. So the kernel of the composition

$$\eta : \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,-\chi} \otimes_{\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]} \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \twoheadrightarrow \operatorname{gr} \left(\mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v) \otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)} \mathcal{A}^0_{\mathfrak{P},\chi}(v) \right)$$

is supported on $\mathfrak{p}^{sing}$. Let ϖ, ϖ^0 denote the natural homomorphisms

$$\mathcal{A}^0_{\mathfrak{P}+\chi,-\chi}(v) \otimes_{\mathcal{A}^0_{\mathfrak{P}}(v)} \mathcal{A}^0_{\mathfrak{P},\chi}(v) \to \mathcal{A}^0_{\mathfrak{P}}(v), \quad \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,-\chi} \otimes_{\mathbb{C}[\mathcal{M}^0_{\mathfrak{p}}(v)]} \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \twoheadrightarrow \mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G}.$$

We have $\varpi^0 = \operatorname{gr} \varpi \circ \eta$. It follows that both the kernel and the cokernel of $\operatorname{gr} \varpi$ are supported on $\mathfrak{p}^{sing}$. Now we can argue as in the proof of Proposition 2.7 to finish the proof of (2). $\square$

Proposition 4.6. *Suppose that χ lies in the interior of the chamber of θ and satisfies $H^1(\mathcal{M}^\theta(v), \mathcal{O}(\chi)) = 0$. Then we have $\mathcal{A}^0_{\mathfrak{P},\chi}(v) \xrightarrow{\sim} \mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)$. Moreover, this isomorphism is filtered and induces an isomorphism $\operatorname{gr} \mathcal{A}^0_{\mathfrak{P},\chi}(v) \xrightarrow{\sim} \operatorname{gr} \mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)$. Both these algebras are identified with $\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi}$.*

Proof. Note that the restriction map $\mathbb{C}[\mu^{-1}(\mathfrak{p})]^{G,\chi} \to \mathbb{C}[\mu^{-1}(\mathfrak{p})^{\theta-ss}]^{G,\chi}$ is injective because χ is in the chamber of θ. We conclude that the left vertical arrow in diagram (4.1) is an isomorphism. From $H^1(\mathcal{M}^\theta(v), \mathcal{O}(\chi)) = 0$ it follows that the right vertical arrow in loc.cit. is an isomorphism. For the same reasons, the same true for the specialization of (4.1) to any value of λ. For λ Zariski generic, the natural map $\mathcal{A}^0_{\lambda,\chi}(v) \to \mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ is an isomorphism. This follows from (1) of Proposition 4.5 combined with Lemma 3.4. Since the induced map of the associated graded modules is an embedding, it is forced to be an isomorphism. So $\mathcal{A}^0_{\lambda,\chi}(v) \to \mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ is an isomorphism for all λ. From here we deduce that $\mathcal{A}^0_{\mathfrak{P},\chi}(v) \hookrightarrow \mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)$ is an isomorphism. The claim about the associated graded follows from here. $\square$

Let us deduce a corollary of the previous proposition.

Corollary 4.7. *Let θ be a generic stability condition. Let χ be generic. Then $\mathcal{A}^0_{\mathfrak{P},\chi}(v) = \mathcal{A}^{(\theta)}_{\mathfrak{P},\chi}(v)$ provided $H^1(\mathcal{M}^\theta(v), \mathcal{O}(\chi)) = 0$.*

Proof. This is a consequence of Propositions 3.3,4.6. $\square$

Proof of Proposition 2.13. By Proposition 4.6, $\mathcal{A}^0_{\lambda+m\chi,n\chi}(v) \xrightarrow{\sim} \mathcal{A}^{(\theta)}_{\lambda+m\chi,n\chi}(v)$ for all $n, m \geqslant 0$. Replacing λ with $\lambda+m\chi$ for some m, we may assume that the $\mathbb{Z}$-algebra $\bigoplus_{n,n'\geqslant 0} \mathcal{A}^0_{\lambda+n\chi,n'\chi}$ is Morita, see [BPW, Section 5.3]. Now the claim that $\pi^\theta_\chi(v)$ is a quotient functor is proved as in [BPW, Section 5.5]. $\square$

Proof of (1) Proposition 4.3. The proof is in several steps.

Step 1. We can choose χ in the interior of the chamber of θ such that:

(i) $H^1(\mathcal{M}^\theta(v), \mathcal{O}(n\chi)) = 0$ for all $n \geqslant 1$.
(ii) $H^1(\mathcal{M}^{-\theta}(v), \mathcal{O}(-n\chi)) = 0$ for all $n \geqslant 1$.

(iii) $\lambda+n\chi\in\mathfrak{P}^{iso}$ and $(\lambda+n\chi,\theta)\in\mathfrak{AL}(v)$ for all $n\geqslant 0$.

Namely, choose χ in the chamber of θ. Multiplying χ by a positive integer, we achieve (i) and (ii). Then we can rescale χ again and achieve (iii) thanks to Proposition 2.7 and Proposition 4.2.

Step 2. By Corollary 4.7, $\mathcal{A}^{(\theta)}_{\mathfrak{P},m\chi}(v)_{\lambda+n\chi}=\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)$ for all $m\geqslant -n$. Since $\lambda+n\chi\in\mathfrak{AL}(v)$, we see that $\mathcal{A}^{(\theta)}_{\mathfrak{P},m\chi}(v)_{\lambda+n\chi}=\mathcal{A}^{(\theta)}_{\lambda+n\chi,m\chi}(v)$ thanks to Lemma 3.4. Therefore $\mathcal{A}^{(\theta)}_{\lambda+n\chi,m\chi}(v)=\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)$ for all $m\geqslant -n$. We are going to deduce (1) of Proposition 4.3 from this equality.

Step 3. Consider the $\mathbb{Z}$-algebra $\mathsf{Z}_{\lambda,\chi}:=\bigoplus_{n,m\geqslant 0}\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)$ and an "extended" $\mathbb{Z}$-algebra $\tilde{\mathsf{Z}}_{\lambda,\chi}:=\bigoplus_{n\geqslant 0,m\geqslant -n}\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)$. For $M\in D(R)\text{-mod}^{G,\lambda}$, the sum $\bigoplus_{n\geqslant 0}M^{G,n\chi}$ is a module over $\tilde{\mathsf{Z}}_{\lambda,\chi}$. But since $(\lambda+n\chi,\theta)\in\mathfrak{AL}(v)$ for all $n\geqslant 0$, all bimodules $\mathcal{A}^{(\theta)}_{\lambda+n\chi,m\chi}(v)=\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)$ are Morita equivalences with inverse $\mathcal{A}^{(\theta)}_{\lambda+(n+m)\chi,-m\chi}(v)=\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)$.

Step 4. We have an isomorphism $\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)\otimes_{\mathcal{A}_{\lambda+(n+m)\chi}(v)}\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)\xrightarrow{\sim}\mathcal{A}_{\lambda+n\chi}(v)$ hence the map $\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)\otimes_{\mathcal{A}_{\lambda+(n+m)\chi}(v)}\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)\otimes_{\mathcal{A}_{\lambda+n\chi}(v)}M^{G,n\chi}\to M^{G,n\chi}$ is an isomorphism as well. But this map comes from taking product by elements of $D(R)$ in M and hence factors as

$$\begin{aligned}&\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)\otimes_{\mathcal{A}_{\lambda+(n+m)\chi}(v)}\mathcal{A}^0_{\lambda+n\chi,m\chi}(v)\otimes_{\mathcal{A}_{\lambda+n\chi}(v)}M^{G,n\chi}\\ &\to\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)\otimes_{\mathcal{A}_{\lambda+(n+m)\chi}(v)}M^{G,(n+m)\chi}\to M^{G,n\chi}.\end{aligned}$$

So we see that the second map is surjective. Similarly, so is the first one. It follows that all maps $\mathcal{A}^0_{\lambda+(n+m)\chi,-m\chi}(v)\otimes_{\mathcal{A}_{\lambda+(n+m)\chi}(v)}M^{G,(n+m)\chi}\to M^{G,n\chi}$ are isomorphisms. A conclusion is that the spaces $M^{G,n\chi}$ are either all zero or all nonzero.

Step 5. As described in [BPW, Section 5.2], the category $\mathcal{A}^\theta_\lambda(v)\text{-mod}$ is equivalent to $\mathsf{Z}_{\lambda,\chi}\text{-mod}$, where the latter stands for the quotient of the category of graded $\mathsf{Z}_{\lambda,\chi}$-modules by the subcategory of all bounded modules. Under this equivalence, the functor $\pi^\theta_\lambda(v)$ becomes $M\mapsto\bigoplus_{n\geqslant 0}M^{G,n\chi}$ by [BPW, Proposition 5.28] (we remark that $\pi^\theta_\lambda(v)\cong\pi^\chi_\lambda(v)$). The conclusion of Step 4 now implies that the kernels of $\pi^0_\lambda(v)$ and of $\pi^\theta_\lambda(v)$ coincide. This proves (1). □

Proof of Lemma 4.4. The proof is in several steps.

Step 1. Pick $\chi\in\mathbb{Z}^{Q_0}$ lying in the interior of the chamber of θ and satisfying $H^1(\mathcal{M}^\theta(v),\mathcal{O}(\chi))=0$. Take $\chi':=-\chi$ in (2) of Proposition 4.5 and consider the asymptotically generic open subset $\mathfrak{P}^0$ from there.

Step 2. We claim that $(\lambda,\theta)\in\mathfrak{AL}(v)$ provided $\lambda+n\chi\in\mathfrak{P}^0$ for all $n\geqslant 0$. As was mentioned in Step 2 of the proof of (1) of Proposition 4.3, $\mathcal{A}^0_{\lambda+m\chi,n\chi}=\mathcal{A}^{(\theta)}_{\lambda+m\chi,n\chi}$ (we will drop "(v)" from the notation). So $\mathcal{A}^{(\theta)}_{\lambda+m\chi,\chi}$ is a Morita equivalence bimodule for all $m\geqslant 0$. By [BPW, Proposition 5.13] what remains to be checked is that the natural homomorphism

$$\mathcal{A}^0_{\lambda+(m+n)\chi,\chi}\otimes_{\mathcal{A}^0_{\lambda+(m+n)\chi}}\mathcal{A}^0_{\lambda+m\chi,n\chi}\to\mathcal{A}^0_{\lambda+m\chi,(n+1)\chi}\tag{4.3}$$

is an isomorphism. Note that we also have a natural homomorphism

$$\mathcal{A}^0_{\lambda+(m+n+1)\chi,-\chi}\otimes_{\mathcal{A}^0_{\lambda+(m+n+1)\chi}}\mathcal{A}^0_{\lambda+m\chi,(n+1)\chi}\to\mathcal{A}^0_{\lambda+m\chi,n\chi}\tag{4.4}$$

Since $\mathcal{A}^0_{\lambda+(m+n)\chi,\chi}$ is a Morita equivalence bimodule with inverse $\mathcal{A}^0_{\lambda+(m+n+1)\chi,-\chi}$, (4.4) gives rise to

$$\mathcal{A}^0_{\lambda+m\chi,(n+1)\chi} \to \mathcal{A}^0_{\lambda+(m+n)\chi,\chi} \otimes_{\mathcal{A}^0_{\lambda+(m+n)\chi}} \mathcal{A}^0_{\lambda+m\chi,n\chi} \tag{4.5}$$

It is easy to see from the construction that (4.5) and (4.3) are mutually inverse to each other. This completes the proof of the claim in the beginning of the step.

Step 3. Let us finish the proof. Pick a codimension 1 face Γ of C. Pick $\chi \in \mathbb{Z}^{Q_0}$ in the interior of C. Finally pick $\lambda' \in \lambda + \mathbb{Z}^{Q_0}$ such that $\lambda' + \Gamma + n\chi \not\subset \mathfrak{P}^0$ for all $n \geqslant 0$. For $r > 0$, let Γ_r denote the set of integral points in Γ of length not exceeding r. Since $\mathfrak{P}^0$ is asymptotically generic, we see that for $n > o(r)$, we have $\lambda' + \Gamma_r + n\chi \subset \mathfrak{P}^0$. It follows that $\lambda' + (\Gamma \cap \mathbb{Z}^{Q_0})$ has a Zariski dense subset of parameters λ'' with $(\lambda'', \theta) \in \mathfrak{AL}(v)$. $\square$

4.3. **Conjectures on localization.** We would like to finish this section by stating conjectures on the precise loci, where abelian and derived localizations hold.

Let us state the main conjecture.

Conjecture 4.8. The following is true:

(1) The locus $\mathfrak{P}^{sing}(v)$ of $\lambda \in \mathfrak{P}$ such that the algebra $\mathcal{A}_\lambda(v)$ has infinite homological dimension is the union of hyperplanes each parallel to $\ker \alpha$, where α is some root of $\mathfrak{g}(Q)$ with $\alpha \leqslant v$.

(2) Let θ be a generic stability condition lying in the classical chamber C. Then $(\lambda, \theta) \in \mathfrak{AL}(v)$ if and only if $(\lambda + (C \cap \mathbb{Z}^{Q_0})) \cap \mathfrak{P}^{sing}(v) = \varnothing$.

Let us give a more detailed conjectural description of $\mathfrak{P}^{sing}(v)$. Assume, for simplicity, that the moment map μ is flat so that $\mathcal{M}(v) = \mathcal{M}^0(v)$.

For a root α, let Σ_α denote the union of hyperplanes parallel to $\ker \alpha$ that are contained in $\mathfrak{P}^{sing}(v)$ so that, according to Conjecture 4.8, $\mathfrak{P}^{sing}(v) = \bigcup_{\alpha \leqslant v} \Sigma_\alpha$. Let us explain how to compute Σ_α.

Pick a generic point $p \in \ker \alpha$ and assume that α is indecomposable. Let k be maximal such that $(v, 1) - k\alpha$ is a root of the quiver Q^w. Then, in the terminology of 2.1.6, we can pick $x \in \mathcal{M}_p(v)$ that corresponds to the decomposition $r = r_0 \oplus r_1 \otimes \mathbb{C}^k$, where $\dim r_1 = \alpha$ and $\dim r_0 = (v^0, 1)$. Then we get the quiver $\hat{Q}$ that has a single vertex and $1 - (\alpha, \alpha)/2$ loops. We consider the dimension $\hat{v} = k$ and the framing $\hat{w} = w \cdot \alpha - (v^0, \alpha)$. Recall the affine map $\hat{r} : \mathfrak{P} \to \hat{\mathfrak{P}} = \mathbb{C}$ from 3.3.1.

Conjecture 4.9. We have $\Sigma_\alpha = \hat{r}^{-1}(\hat{\mathfrak{P}}^{sing}(\hat{v}))$ (where the locus $\hat{\mathfrak{P}}^{sing}(\hat{v})$ is formed for the framing $\hat{w}$).

Conjectures 4.8 and 4.9 reduce the computation of the locus where abelian/derived localization holds to quivers with a single vertex. Let us explain what is known there.

In the case when there are no loops, the algebra $\mathcal{A}_\lambda(v)$ is $D^\lambda(\mathrm{Gr}(v, w))$, the algebra of global λ-twisted differential operators on the grassmanian $\mathrm{Gr}(v, w)$. In this case, analogs of the abelian/derived Beilinson-Bernstein theorems (stated originally for the flag varieties) hold. We have $\mathfrak{P}^{sing}(v) = \{-1, -2, \ldots, 1 - w\}$ and (2) of Conjecture 4.8 holds.

Let us consider the situation when there is one loop. A classical case is when $w = 1$. Here $\mathcal{A}_\lambda(v)$ is the spherical rational Cherednik algebra for $(S_n, \mathbb{C}^n)$, see 2.2.6. The subset $\mathfrak{P}^{sing}(v)$ consists of all rational $\lambda \in (-1, 0)$ with denominator not exceeding n, see e.g. [BE, Corollary 4.2]. Moreover, (2) of Conjecture 4.8 holds, this follows from [GS, KR], the case of half-integer parameters was completed in [BE].

When $w > 1$, we have that $\mathfrak{P}^{sing}(v)$ consists of all rational numbers $\lambda \in (-w, 0)$ with denominator not exceeding v. Moreover, (2) of Conjecture 4.8 holds. These results are obtained in the forthcoming paper [L10] by the second named author.

Finally, let us mention that derived and/or abelian localization is known for some (Q, v, w) with $|Q_0| > 1$. For example, this is the case when Q is of finite Dynkin type A. The case when the corresponding variety $\mathcal{M}^\theta(v)$ is the cotangent bundle to a partial flag variety follows similarly to the Beilinson-Bernstein theorem, while the general case follows as in [Gi2], see also [L14, 3.3.4].

Now let Q be an affine quiver with extending vertex 0. Assume that $w = \epsilon_0$ so that $\mathcal{A}_\lambda(v)$ is the spherical subalgebra in an SRA. In this case there was a conjecture describing the singular (=aspherical) locus in $\mathfrak{P}^{sing}(n\delta)$, [Et, Conjecture 5.3], based on the cyclic case done before that in [DG]. It is easy to see that (after relating the parameterizations) (1) of Conjecture 4.8 reduces to [Et, Conjecture 5.3] for $v = n\delta$. Moreover, part (2) in the cyclic case should follow from results of [L9].

5. Outline and key ingredients for proving the main result

Recall that $\lambda \in \mathfrak{P}$ is chosen in such a way that $\mathcal{A}_\lambda(v)$ has finite homological dimension. By Proposition 4.2 we can always achieve that by replacing λ with some element of $\lambda + \mathbb{Z}^{Q_0}$.

Conjecture 1.1 boils down to the following three claims.

(I) The image of CC^λ contains $L^{\mathfrak{a}}_\omega[\nu]$.

(II) The image of $\mathsf{CC}^\theta_\lambda$ is contained in $L^{\mathfrak{a}}_\omega[\nu]$.

(III) The map $\mathsf{CC}^\theta_\lambda : K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \to L_\omega[\nu]$ is injective.

To prove these claims we will use two different (but related) families of functors. We will have functors categorifying the action of $\mathfrak{a}$. In special cases (for example, when λ is integral and all components of θ are positive) these functors were constructed by Webster in [We1] and our general construction is built on his. So we call these *Webster functors*. The second family of functors was introduced in [BPW, Section 6] under the name of twisting functors. In this paper we call them *wall-crossing functors*. Very roughly speaking, Webster functors are used in (I), they should be thought as induction functors that allow to produce new finite dimensional modules from existing ones, and, to some extent in (III). Again, very roughly speaking, wall-crossing functors are used in (III). In fact, we can prove (I) for any (Q, v, w), while (II) is only proved in the cases described in Theorem 1.2 (the second named author extends (II) to any affine Q in the subsequent paper [L10], an easier proof is found in [L16]). (III) is then deduced from (II).

Let us also point out that Webster functors are related to wall-crossing functors. Namely, the wall-crossing functor through the wall defined by a real root are essentially realized via Rickard complexes (first introduced in a different setting by Chuang and Rouquier in [CR]) constructed from Webster functors corresponding to the real root. This fact is very important for our proofs.

5.1. **Wall-crossing functors.** Here we define wall-crossing functors and study some of their properties. In what follows we assume that the functor $L\operatorname{Loc}^\theta_\lambda$ is a derived equivalence provided the homological dimension of $\mathcal{A}_\lambda(v)$ is finite (this is always the case when the quiver Q is of finite or affine type, see Proposition 4.1).

5.1.1. *Construction of the functor.* Pick $\lambda \in \mathfrak{P}, \chi \in \mathbb{Z}^{Q_0}$. Recall, 3.1.4, the $\mathcal{A}^\theta_{\lambda+\chi}(v)$-$\mathcal{A}^\theta_\lambda(v)$-bimodule $\mathcal{A}^\theta_{\lambda,\chi}(v) := [\mathcal{Q}_\lambda|_{T^*R^{\theta-ss}}]^{G,\chi}$ and its global sections $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$. Note that the functor $\mathcal{T}_{\lambda,\chi}:\mathcal{A}^\theta_{\lambda,\chi}(v)\otimes_{\mathcal{A}^\theta_\lambda(v)}\bullet:\mathcal{A}^\theta_\lambda(v)\text{-mod}\to\mathcal{A}^\theta_{\lambda+\chi}(v)\text{-mod}$ is an equivalence. We remark that

$$\mathcal{T}_{\lambda,\chi}\circ\pi^\theta_\lambda(v)=\pi^\theta_{\lambda+\chi}(v)\circ(\mathbb{C}_{-\chi}\otimes\bullet). \tag{5.1}$$

Now let λ',λ be such that $\chi:=\lambda-\lambda'\in\mathbb{Z}^{Q_0}$, the algebra $\mathcal{A}_{\lambda'}(v)$ has finite homological dimension (and so $L\operatorname{Loc}^\theta_{\lambda'}$ is a derived equivalence), and $(\lambda,\theta)\in\mathfrak{AL}(v)$. Following [BPW, Section 6.4], we define a functor $\mathfrak{WC}_{\lambda'\to\lambda}:D^b(\mathcal{A}_{\lambda'}(v)\text{-mod})\xrightarrow{\sim}D^b(\mathcal{A}_\lambda(v)\text{-mod})$ by

$$\mathfrak{WC}_{\lambda'\to\lambda}:=\Gamma^\theta_\lambda\circ\mathcal{T}_{\lambda',\chi}\circ L\operatorname{Loc}^\theta_{\lambda'}. \tag{5.2}$$

We remark that this functor is right t-exact. If $(\lambda',\theta')\in\mathfrak{AL}(v)$, then we can also consider the functor $\mathfrak{WC}_{\lambda'\to\lambda}=\mathcal{T}_{\lambda',\chi}\circ(R\Gamma^\theta_{\lambda'})^{-1}\circ\Gamma^{\theta'}_{\lambda'}:D^b(\mathcal{A}^{\theta'}_{\lambda'}(v)\text{-mod})\xrightarrow{\sim}D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$. When $(\lambda',\theta')\in\mathfrak{AL}(v)$, we often write $\mathfrak{WC}_{\theta'\to\theta}$ instead of $\mathfrak{WC}_{\lambda'\to\lambda}$. We note that under the identifications $\mathcal{A}^\theta_{\lambda_1}(v)\text{-mod}\xrightarrow{\sim}\mathcal{A}^\theta_\lambda(v)\text{-mod},\mathcal{A}^\theta_{\lambda_1'}(v)\text{-mod}\xrightarrow{\sim}\mathcal{A}^\theta_{\lambda'}(v)\text{-mod}$ with $\lambda_1\in\lambda+\mathbb{Z}^{Q_0},\lambda_1'\in\lambda'+\mathbb{Z}^{Q_0}$, the functor $\mathfrak{WC}_{\theta'\to\theta}$ is independent of the choice of λ_1,λ_1' provided $(\lambda_1,\theta),(\lambda_1',\theta')\in\mathfrak{AL}(v)$.

5.1.2. *Alternative realizations.* Here is another formula for $\mathfrak{WC}_{\theta'\to\theta}$ that holds when $\lambda\in\mathfrak{P}^{ISO}$ (and $(\lambda,\theta)\in\mathfrak{AL}(v)$):

$$\mathfrak{WC}_{\theta'\to\theta}=\pi^\theta_\lambda(v)\circ(\mathbb{C}_{\lambda'-\lambda}\otimes\bullet)\circ L\pi^{\theta'}_{\lambda'}(v)^!. \tag{5.3}$$

This formula follows from (5.1),(5.2) and Lemma 2.17. Here we use the isomorphism $\pi^{\theta'}_{\lambda'}(v)=\pi^0_{\lambda'}(v)$ to produce the functor $L\pi^{\theta'}_{\lambda'}(v)^!$. For us, this formula serves as a motivation for the name "wall-crossing": the functor "crosses the walls" separating the stability conditions θ,θ'.

A connection of $\mathcal{A}^{(\theta)}_{\lambda,\chi}$ to the wall-crossing functor is provided by the following assertion.

Lemma 5.1 (Proposition 6.31 in [BPW])**.** *If $(\lambda,\theta)\in\mathfrak{AL}(v)$, then*

$$\mathfrak{WC}_{\lambda\to\lambda'}(\bullet)=\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)\otimes^L_{\mathcal{A}_\lambda(v)}\bullet.$$

In fact, under the assumptions of Lemma 5.1, $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)=\mathcal{A}^0_{\lambda,\chi}(v)$, as the following proposition shows.

Proposition 5.2. *Suppose that $\lambda,\lambda+\chi\in\mathfrak{P}^{iso}$ and $(\lambda+\chi,\theta)\in\mathfrak{AL}(v)$. Then the natural homomorphism $\mathcal{A}^0_{\lambda,\chi}(v)\to\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ is an isomorphism.*

Proof. By (1) of Proposition 4.3, the functors $\pi^\theta_{\lambda+\chi}(v),\pi^0_{\lambda+\chi}(v)$ are isomorphic (below we suppress v and write $\pi^\theta_{\lambda+\chi}$, etc.). Moreover, $\pi^0_{\lambda+\chi}=\Gamma^\theta_{\lambda+\chi}\circ\pi^\theta_{\lambda+\chi}$. By Lemmas 2.17, 5.1,

$$\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)\otimes_{\mathcal{A}_\lambda(v)}\bullet\cong\Gamma^\theta_{\lambda+\chi}\circ(\mathcal{A}^\theta_{\lambda,\chi}\otimes_{\mathcal{A}^\theta_\lambda}\bullet)\circ\operatorname{Loc}^\theta_\lambda=\Gamma^\theta_{\lambda+\chi}\circ\pi^\theta_{\lambda+\chi}\circ(\mathbb{C}_{-\chi}\otimes\bullet)\circ(\pi^0_\lambda)^!.$$

Also it is easy to see that $\mathcal{A}^0_{\lambda,\chi}(v)\otimes_{\mathcal{A}_\lambda(v)}\bullet=\pi^0_{\lambda+\chi}\circ(\mathbb{C}_{-\chi}\otimes\bullet)\circ(\pi^0_\lambda)^!$. So the functors $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)\otimes_{\mathcal{A}_\lambda(v)}\bullet$ and $\mathcal{A}^0_{\lambda,\chi}(v)\otimes_{\mathcal{A}_\lambda(v)}\bullet$ are isomorphic. It follows that the bimodules $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ and $\mathcal{A}^0_{\lambda,\chi}(v)$ are isomorphic. Let us see why the corresponding isomorphism coincides with (3.1). By the definition, (3.1) coincides with the natural homomorphism

$$\pi^0_{\lambda+\chi}\circ(\mathbb{C}_{-\chi}\otimes\bullet)\circ(\pi^0_\lambda)^!(\mathcal{A}_\lambda(v))\to[\pi^0_{\lambda+\chi}\circ(\pi^\theta_{\lambda+\chi})^*]\circ\pi^\theta_{\lambda+\chi}\circ(\mathbb{C}_{-\chi}\otimes\bullet)\circ(\pi^0_\lambda)^!(\mathcal{A}_\lambda(v)). \tag{5.4}$$

Here

$$(\pi^\theta_\lambda)^*:=\Gamma(\mathcal{Q}_\lambda|_{(T^*R)^{\theta-ss}}\otimes_{\mathcal{A}^\theta_\lambda(v)}\bullet)$$

stands for the right adjoint of π^θ_λ, so the composition in the brackets is $\Gamma^\theta_{\lambda+\chi}$ (this is a corollary of Lemma 2.17). (5.4) is induced by the adjunction $\mathrm{id} \to (\pi^\theta_{\lambda+\chi})^* \circ \pi^\theta_{\lambda+\chi}$ (this adjunction is nothing else but the restriction homomorphism of a module in $D(R)\text{-mod}^{G,\lambda}$ to its sections on the semistable locus). So (5.4) is the homomorphism $\mathcal{A}^0_{\lambda,\chi}(v) \to \mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ constructed before in this proof. □

The importance of this proposition is that the bimodules $\mathcal{A}^0_{\lambda,\chi}(v)$ are better in several aspects than $\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$: for example, the former behave well under restriction functors, (3.11). This will allow to study wall-crossing functors inductively.

5.1.3. *Composition of wall-crossing functors.* It turns out that, under additional restrictions, a composition of wall-crossing functors is again a wall-crossing functor.

More precisely, suppose that we have two generic stability conditions θ, θ'. Suppose that $\theta_i, i = 0, \ldots, q$, are such that $\theta_0 = \theta, \theta_q = \theta'$, θ_i and θ_{i+1} are separated by a single wall and q is minimal with these properties.

Theorem 5.3. *We have an isomorphism of functors*

$$\mathfrak{WC}_{\theta_0\to\theta_q} = \mathfrak{WC}_{\theta_{q-1}\to\theta_q} \circ \ldots \mathfrak{WC}_{\theta_1\to\theta_2} \circ \mathfrak{WC}_{\theta_0\to\theta_1}.$$

Proof. This is established in the proof of [BPW, Theorem 6.35]. □

5.1.4. *Non-essential walls.* Sometimes a wall-crossing functor between two different chambers happens to be an abelian equivalence. We will be interested in the situation when this happens for two chambers sharing a wall.

Namely, we say that a classical wall $\ker\alpha$ is *non-essential* (for the parameter λ) if for every two classical chambers C, C' separated by $\ker\alpha$, the wall-crossing functor $\mathfrak{WC}_{\lambda\to\lambda'}$ is an abelian equivalence for $\theta \in C, \theta' \in C'$ (and $(\lambda, \theta), (\lambda', \theta') \in \mathfrak{AL}(v)$).

Here is an important example of a non-essential wall.

Proposition 5.4. *Suppose α is a real root and $\langle\alpha, \lambda\rangle \notin \mathbb{Z}$. Assume also that the intersections of $\mathfrak{P}^{iso}$ with $\lambda + \ker\alpha, \lambda' + \ker\alpha$ are nonempty. Then the wall $\ker\alpha$ is non-essential for λ.*

Proof. Let $\mathfrak{P}_0 := \lambda + \ker\alpha, \mathfrak{P}_0' := \lambda' + \ker\alpha$. Consider the translation bimodules $\mathcal{B}_{\mathfrak{P}_0} := \mathcal{A}^0_{\mathfrak{P}_0,\chi}(v), \mathcal{B}'_{\mathfrak{P}_0} := \mathcal{A}^0_{\mathfrak{P}_0',-\chi}(v)$ and the algebra $\mathcal{A}_{\mathfrak{P}_0} := \mathcal{A}^0_{\mathfrak{P}_0}(v), \mathcal{A}'_{\mathfrak{P}_0} := \mathcal{A}^0_{\mathfrak{P}_0'}(v)$, where $\chi = \lambda' - \lambda$.

Step 1. We claim that for a Zariski generic $\lambda_1 \in \mathfrak{P}_0$ the specializations $\mathcal{B}_{\lambda_1}, \mathcal{B}'_{\lambda_1}$ are mutually inverse Morita equivalences. Similarly to the proof of Proposition 2.7 (Section 3.5), this amounts to check that the kernels and the cokernels of the natural homomorphisms

$$\mathcal{B}'_{\mathfrak{P}_0} \otimes_{\mathcal{A}'_{\mathfrak{P}_0}} \mathcal{B}_{\mathfrak{P}_0} \to \mathcal{A}_{\mathfrak{P}_0}, \quad \mathcal{B}_{\mathfrak{P}_0} \otimes_{\mathcal{A}_{\mathfrak{P}_0}} \mathcal{B}_{\mathfrak{P}_0'} \to \mathcal{A}'_{\mathfrak{P}_0} \tag{5.5}$$

have proper supports in $\mathfrak{P}_0$. This will be proved in subsequent steps.

Step 2. Pick a Zariski generic $p \in \ker\alpha$ and consider $x \in \mathcal{M}^0_p(v)$. The corresponding representation r of $\overline{Q}^w$ decomposes into irreducibles as $r = r_0 \oplus k r_1$, where r_0 has dimension $(v - k\alpha, 1)$ and r_1 has dimension $(\alpha, 0)$. The corresponding quiver $\hat{Q}$ (see 2.1.6) has one vertex and no loops, $\hat{v} = k$ and $\hat{w} = w \cdot \alpha - (v - k\alpha, \alpha) = (\nu, \alpha) + 2k$. We claim that $\hat{r}(\lambda) = \langle\alpha, \lambda\rangle + s$, where $s \in \mathbb{Z}$. Indeed, this boils down to $\langle\varrho(v), \alpha\rangle - \frac{1}{2}(\nu, \alpha) \in \mathbb{Z}$ that is a straightforward check. So the parameter $\hat{\lambda} := \hat{r}(\lambda)$ is not an integer. Also since $\lambda \in \mathfrak{P}^{iso}$, we see that $\hat{\lambda} \in \hat{\mathfrak{P}}^{iso}$ (this follows from Lemma 3.7). We conclude that the algebra $\hat{\mathcal{A}}^0_{\hat{\lambda}}(\hat{v})$ is the algebra $D^{\hat{\lambda}}(\mathcal{G}(k, \hat{w}))$ of $\hat{\lambda}$-twisted differential operators on the grassmanian $\mathcal{G}(k, \hat{w})$.

Step 3. Let $\hat{\chi} = \langle \chi, \alpha \rangle$, this is an integer. So both $\hat{\lambda}, \hat{\lambda} + \hat{\chi}$ are not integers. A version of the Beilinson-Bernstein abelian localization theorem for twisted differential operators on grassmanians implies that abelian localization holds for $(\hat{\lambda}, \hat{\theta})$ and $(\hat{\lambda} + \hat{\chi}, \hat{\theta})$. Therefore the bimodules $\hat{\mathcal{A}}^{(\hat{\theta})}_{\hat{\lambda},\hat{\chi}}(\hat{v}), \hat{\mathcal{A}}^{(\hat{\theta})}_{\hat{\lambda}+\hat{\chi},-\hat{\chi}}(\hat{v})$ are mutually inverse Morita equivalences. Proposition 5.2 implies that the bimodules $\hat{\mathcal{A}}^0_{\hat{\lambda},\hat{\chi}}(\hat{v}), \hat{\mathcal{A}}^0_{\hat{\lambda}+\hat{\chi},-\hat{\chi}}(\hat{v})$ are Morita equivalences. Equivalently, the kernels and cokernels of the homomorphisms in (5.5) vanish under the functor $\bullet_{\dagger,x}$. In other words, the associated varieties of these kernels and cokernels do not intersect $\mathcal{M}_p(v)$. Since p was chosen to be Zariski generic, the $\mathfrak{P}_0$-supports of the kernels and cokernels are proper. The claim in the beginning of Step 1 follows.

Step 4. Let us finish the proof. By applying integral shifts to $\mathfrak{P}_0, \mathfrak{P}_0'$ (and, in particular, modifying χ), thanks to Lemma 4.4, we may assume that there is $\lambda_0 \in \lambda + (\mathbb{Z}^{Q_0} \cap \ker \alpha)$ such that $(\lambda_1, \theta), (\lambda_1 + \chi, \theta') \in \mathfrak{AL}(v)$ for $\lambda_1 \in \lambda_0 + (C \cap \ker \alpha)$ in some Zariski dense subset of $\ker \alpha$. As was mentioned in 5.1.1, the functor $\mathfrak{WC}_{\lambda_1 \to \lambda_1 + \chi}$ becomes $\mathfrak{WC}_{\lambda \to \lambda + \chi}$ up to some equivalences of abelian categories. Since $\lambda_0 + (C \cap \ker \alpha)$ is Zariski dense, we use Proposition 5.2 together with Lemma 5.1 to see that $\mathfrak{WC}_{\lambda_1 \to \lambda_1 + \chi} = \mathcal{A}^0_{\lambda_1,\chi}(v) \otimes^L_{\mathcal{A}^0_{\lambda_1}} \bullet$ is an abelian equivalence. This completes the proof. $\square$

So the only non-trivial wall-crossing functors corresponding to the walls $\ker \alpha$ with real α are for α that are roots of $\mathfrak{a}^\lambda$. This is the first indication that the representation theory of the algebras $\mathcal{A}_\lambda(v)$ is controlled by the algebras $\mathfrak{a}^\lambda$.

5.2. **Webster functors.**

5.2.1. *Special case: Webster's construction.* In [We1], Webster introduced a quantum categorical version of Nakajima's construction, [Nak1, Section 10]. In the case when all θ_k are positive and for $i \in Q_0$ such that $\lambda_i \in \mathbb{Z}$, he produced functors $F_i : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \leftrightarrows D^b(\mathcal{A}^\theta_\lambda(v + \epsilon_i)\text{-mod}) : E_i$ and studied their properties. We will need the construction so we recall it first.

We start with the simplest possible case when Q is a single vertex without arrows. In this case, $\mathcal{M}^\theta(v) = T^* \operatorname{Gr}(v, w)$ and λ has to be an integer. Our exposition follows [CDK].

Pick $r > 0$ and set $d = w - 2v + r$. Consider the incidence subvariety $C^r(d) := \operatorname{Fl}(v, v - r, w) \subset \operatorname{Gr}(v, w) \times \operatorname{Gr}(v - r, w)$. Consider the δ-function $D_{\operatorname{Gr}(v,w)\times\operatorname{Gr}(v-r,w)}$-module $\delta_{C^r(d)}$ on $C^r(d)$ (the image of the structure sheaf on $C^r(d)$ under the Kashiwara equivalence). Then we consider the following objects:

$$\mathcal{E}^{(r)}(d) = \delta_{C^r(d)}[v(w - v)] \in D^b(D_{\operatorname{Gr}(v,w)\times\operatorname{Gr}(v-r,w)}\text{-mod}),$$
$$\mathcal{F}^{(r)}(d) = \delta_{C^r(d)}[(v - r)(w - v + r)] \in D^b(D_{\operatorname{Gr}(v-r,w)\times\operatorname{Gr}(v,w)}\text{-mod}).$$

The object $\mathcal{E}^{(r)}(d)$ defines a functor $E^{(r)}(d) : D^b(D_{\operatorname{Gr}(v,w)}\text{-mod}) \to D^b(D_{\operatorname{Gr}(v-r,w)}\text{-mod})$ by convolving with $\mathcal{E}^{(r)}(d)$. Similarly, we get a functor $F^{(r)}(d) : D^b(D_{\operatorname{Gr}(v-r,w)}\text{-mod}) \to D^b(D_{\operatorname{Gr}(v,w)}\text{-mod})$. We write $E^{(r)}$ for $\bigoplus_d E^{(r)}(d)$, and $F^{(r)}$ for $\bigoplus_d F^{(r)}(d)$.

The functors $E^{(r)}(d), F^{(r)}(d)$ are adjoint to one another up to homological shifts. Namely, let us write $E^{(r)}(d)_L, E^{(r)}(d)_R$ for the left and right adjoint functors of $E^{(r)}(d)$. We have

$$E^{(r)}(d)_L \cong F^{(r)}(d)[-rd], \quad E^{(r)}(d)_R \cong F^{(r)}(d)[rd]. \tag{5.6}$$

This construction has several extensions. For example, we get functors

$$F : D^b(D_{\mathrm{Gr}(\bullet,w)} \otimes D_{\underline{R}}\text{-mod}) \leftrightarrows D^b(D_{\mathrm{Gr}(\bullet+1,w)} \otimes D_{\underline{R}}\text{-mod}) : E$$

for any vector space $\underline{R}$. Also if H is a reductive group equipped with homomorphisms $H \to \mathrm{GL}(w), \mathrm{GL}(\underline{R})$ and $\underline{\lambda}$ is a character of $\mathfrak{h}$, then we get functors

$$F : D^b_{H,\underline{\lambda}}(D_{\mathrm{Gr}(\bullet,w)} \otimes D_{\underline{R}}\text{-mod}) \leftrightarrows D^b_{H,\underline{\lambda}}(D_{\mathrm{Gr}(\bullet+1,w)} \otimes D_{\underline{R}}\text{-mod}) : E \tag{5.7}$$

Now let us proceed to the case of a general quiver. We assume that $\theta_k > 0$ for all $k \in Q_0$. Let $\underline{R}, \underline{G}$ be as in (2.2), $\underline{\theta}$ be the collection of θ_j with $j \neq i$ and $\underline{\lambda}$ have the similar meaning to $\underline{\theta}$. We reverse arrows if necessary and assume that i is a source in Q.

Since $\theta_i > 0$, we have $R//^{\theta_i} G_i = \mathrm{Gr}(v_i, \tilde{w}_i) \times \underline{R}$, where $\tilde{w}_i$ is defined by (2.1). The group $\underline{G}$ acts on $\mathrm{Gr}(v_i, \tilde{w}_i) \times \underline{R}$ diagonally, the action on $\mathrm{Gr}(v_i, \tilde{w}_i)$ is via a natural action of $\underline{G}$ on $\tilde{W}_i$. Then we have

$$D_R /\!/\!/^{\theta_i}_{\lambda_i} \mathrm{GL}(v_i) = D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}, \quad \mathcal{A}^\theta_\lambda(v) = [D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}] /\!/\!/^{\underline{\theta}}_{\underline{\lambda}} \underline{G}.$$

Let us write $\mathcal{A}^{\theta_i}_{\lambda_i}(v)$ for the former reduction.

It follows from Proposition 2.13 that the category $\mathcal{A}^\theta_\lambda(v)$-mod is the quotient of the category $\mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}^{\underline{G},\underline{\lambda}}$ (of $(\underline{G}, \underline{\lambda})$-equivariant $\mathcal{A}^{\theta_i}_{\lambda_i}(v)$-modules) by the Serre subcategory of all modules whose singular support is contained in the image of $\mu^{-1}(0)^{\theta_i - ss} \setminus \mu^{-1}(0)^{\theta-ss}$ in $T^*R /\!/\!/^{\theta_i} \mathrm{GL}(v_i)$. Lemma 2.15 shows that the same is true on the level of (equivariant) derived categories.

As was checked by Webster, [We1, Section 4], the functors

$$F : D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}) \rightleftarrows D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_i}_{\lambda_i}(v+\epsilon_i)\text{-mod}) : E$$

preserve the subcategories of all complexes whose homology are supported on the image of $\mu^{-1}(0)^{\theta_i - ss} \setminus \mu^{-1}(0)^{\theta-ss}$ in $T^*R /\!/\!/^{\theta_i} \mathrm{GL}(v_i)$ (it is important here that all θ_k are positive). So they descend to endo-functors of $\bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ to be denoted by E_i, F_i.

Remark 5.5. Let $\chi \in \mathbb{Z}^{Q_0}$. By the very definition of the functors E_i, F_i, the equivalences

$$\mathcal{A}^\theta_\lambda(v)\text{-mod} \xrightarrow{\sim} \mathcal{A}^\theta_{\lambda+\chi}(v)\text{-mod}, \quad \mathcal{A}^\theta_\lambda(v+\epsilon_i)\text{-mod} \xrightarrow{\sim} \mathcal{A}^\theta_{\lambda+\chi}(v+\epsilon_i)\text{-mod}$$

intertwine these functors.

5.2.2. *Properties.* Let us explain some properties of the functors E, F (and also of E_i, F_i).

The proof of the next lemma is a part of that for [We1, Theorem 3.1].

Lemma 5.6. *The functors (5.7) define a categorical action of the 2-Kac-Moody algebra $\mathcal{U}(\mathfrak{sl}_2)$ on the category*

$$\bigoplus_{v=0}^{w} D^b_{H,\underline{\lambda}}(D^\lambda_{\mathrm{Gr}(v,w)} \otimes D_{\underline{R}}\text{-mod}). \tag{5.8}$$

The divided power functors are $E^{(r)}, F^{(r)}$.

Moreover, the construction shows that the action factors through a homomorphism ψ of 2-algebras $\mathcal{U}(\mathfrak{sl}_2) \to \mathsf{Q}$. Here Q, where Q is a "single vertex analog" of the 2-category $\mathcal{Q}^\lambda$ introduced in the end of [We1, Section 2] (so that the 1-morphisms, $E^{(r)}, F^{(r)}$ map to $\mathcal{E}^{(r)}, \mathcal{F}^{(r)}$) that we are going to define now. In our definition of Q we will need a version of the Steinberg variety. By definition, this is the subvariety $\mathsf{St} \subset T^*\mathrm{Gr}(v,w) \times T^*\mathrm{Gr}(v',w)$ that is the preimage of the diagonal in $\mathfrak{gl}(w) \times \mathfrak{gl}(w)$ under the moment map $T^*\mathrm{Gr}(v,w) \times T^*\mathrm{Gr}(v',w) \to \mathfrak{gl}(w) \times \mathfrak{gl}(w)$.

To define Q, it is enough to restrict to the case $\lambda = 0$. The collection of objects in that category is $\{0, \ldots, w\}$, and the 1-morphisms from v' to v are the objects from

$$D^b_H(\mathcal{D}_{\mathrm{Gr}(v',w)} \otimes \mathcal{D}_{\mathrm{Gr}(v,w)} \otimes D_{\underline{R}^2})$$

with homology supported (in the sense of the singular support) on $\mathsf{St} \times \mathcal{N}^0_{diag}$, where $\mathcal{N}^0_{diag}$ stands for the conormal bundle to the diagonal $\underline{R} \subset \underline{R}^2$. We remark that Webster's 2-category has, in a sense, more 1-morphisms but what we have above is sufficient for having ψ. The description of 2-morphisms in Q is similar to [We1, Section 2]. The action of Q on (5.8) (as well as the tensor structure on Q) is defined via convolution of D-modules. The grading shift in $\mathcal{U}(\mathfrak{sl}_2)$ corresponds to the homological shift in Q.

Let us point out several other important properties of the functors E_i, F_i that are due to Webster, [We1].

Lemma 5.7. *The following is true:*

(1) *The functors E_i, F_i preserve the subcategory*

$$\bigoplus_v D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \subset \bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}).$$

(2) *Moreover, for $M \in D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$, we have $\mathsf{CC}(E_iM) = e_i\mathsf{CC}(M), \mathsf{CC}(F_iM) = f_i\mathsf{CC}(M)$, where e_i, f_i stand for the Nakajima operators.*

For the proof, see [We1, Corollary 3.4, Proposition 3.5]. In particular, if $\lambda \in \mathbb{Z}^{Q_0}$, we see that $\mathsf{CC} : K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to L_\omega[\nu]$ is surjective. This completes the proof of (I) and (II) in this case.

5.2.3. *General case.* Now let us explain how to generalize Webster's functors to the case when λ is not necessarily integral and θ is not necessarily positive. We will assume that θ lies in the Tits cone (which puts no restrictions for finite type Q and results in $\langle \theta, \delta \rangle > 0$ for affine type Q). We will also assume that θ is generic for all v, in fact, for every given v and every classical chamber there is such an element there.

The element θ defines a Weyl chamber for the algebra $\mathfrak{a}$ and hence a system of simple roots Π^θ for $\mathfrak{a}$ (that consists of positive roots for $\mathfrak{g}(Q)$ because θ lies in the Tits cone). For $\alpha \in \Pi^\theta$ (and all v) we will define functors

$$F_\alpha : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \rightleftarrows D^b(\mathcal{A}^\theta_\lambda(v+\alpha)\text{-mod}) : E_\alpha$$

generalizing the functors constructed by Webster.

Let θ^+ denote a stability condition with all entries positive. Let $\sigma \in W(Q)$ be such that $\sigma\theta^+$ lies in the same Weyl chamber for $\mathfrak{a}$ as θ. The stability conditions $\theta, \sigma\theta^+$ are separated only by non-essential walls of the form $\ker \beta$ for real roots β and so, by Proposition 5.4, we can identify the categories $\mathcal{A}^\theta_\lambda(v)\text{-mod}$ and $\mathcal{A}^{\sigma\theta^+}_\lambda(v)\text{-mod}$ by means of the wall-crossing functor $\mathfrak{WC}_{\theta \to \sigma\theta^+}$ (for all v). Also recall, 2.2.4, that σ gives rise to the quantum LMN isomorphism $\sigma : \mathcal{A}^{\theta^+}_{\lambda'}(v') \xrightarrow{\sim} \mathcal{A}^{\sigma\theta^+}_\lambda(v)$, where $v' := \sigma^{-1} \bullet v, \lambda' := \sigma^{-1} \bullet^v \lambda$, and hence an abelian equivalence $\sigma_* : \mathcal{A}^{\theta^+}_{\lambda'}(v')\text{-mod} \xrightarrow{\sim} \mathcal{A}^{\sigma\theta^+}_\lambda(v)\text{-mod}$.

Now let $\alpha \in \Pi^\theta$. In general, $\sigma^{-1}\alpha$ is not a simple root. However, we can modify θ staying in the same Weyl chamber for $\mathfrak{a}$ and in the Tits cone for $\mathfrak{g}(Q)$ (using a wall-crossing functor through non-essential walls) so that the $\mathfrak{g}(Q)$-chamber of θ is adjacent to a wall for $\mathfrak{a}$. Then we can, in addition, assume $\sigma^{-1}(\alpha)$ is a simple root for $\mathfrak{g}(Q)$, say α^i.

The functors F_α, E_α are, by definition, obtained by transferring Webster's functors F_i, E_i using equivalences

$$\sigma_*: \mathcal{A}^{\theta+}_{\lambda'}(v')\text{-mod} \xrightarrow{\sim} \mathcal{A}^{\sigma\theta+}_{\lambda}(v)\text{-mod},$$
$$\sigma_*: \mathcal{A}^{\theta+}_{\lambda''}(v'+\alpha^i)\text{-mod} \xrightarrow{\sim} \mathcal{A}^{\sigma\theta+}_{\lambda}(v+\alpha)\text{-mod}.$$

Here v' is given by $\sigma^{-1}\bullet v = v'$ and λ', λ'' are given by $\lambda' = \sigma^{-1}\bullet^{v}\lambda, \lambda'' = \sigma^{-1}\bullet^{v+\alpha}\lambda$. Note that $\lambda''-\lambda' \in \mathbb{Z}^{Q_0}$.

More precisely,

$$F_\alpha = \sigma_* \circ F_i \circ \mathcal{T}_{\lambda',\lambda''-\lambda'} \circ \sigma_*^{-1}$$

and E_α is defined in a similar fashion.

By the construction and (1) of Lemma 5.7 the functors E_α, F_α preserve

$$\bigoplus_v D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)})$$

Remark 5.8. Of course, our construction of the functors E_α, F_α depends on the choice of a suitable $\mathfrak{g}(Q)$-chamber in the $\mathfrak{a}$-chamber of θ. In a subsequent paper the second named author plans to check that the functors E_α, F_α are well-defined and, in fact, give a categorical $\mathfrak{a}$-action on $\bigoplus_v D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$, at least when $\mathfrak{a}$ is simply laced. It is expected that this result will allow to compute the Euler characteristics of $R\Gamma(M)$ for $M \in \operatorname{Irr}(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)})$.

5.3. **Outline of proof: lower bound on the image.** We proceed to outlining the key ideas of our proof of Theorem 1.2 starting with (I): $\operatorname{im}\mathsf{CC}^{\lambda} \supset L^{\mathfrak{a}}_{\omega}$ (we will also see that CC^{λ} does not depend on the choice of θ). For this we need to establish an analog of (2) of Lemma 5.7 that will be proved below.

Proposition 5.9. *For $M \in D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)})$, we have $\mathsf{CC}(E_\alpha M) = \pm e_\alpha \mathsf{CC}(M)$ and $\mathsf{CC}(F_\alpha M) = \pm f_\alpha \mathsf{CC}(M)$.*

In fact, we can do even better and prove an analog of this proposition for $K_0(\operatorname{Coh}_{\rho^{-1}(0)}\mathcal{M}^{\theta}(v))$ instead of $H_{mid}(\mathcal{M}^{\theta}(v))$ and the degeneration map $[M] \mapsto [\operatorname{gr} M]$ instead of CC, this is done in Section 7.4.

Let us explain ideas behind a proof of Proposition 5.9 (similar ideas are used to prove a stronger result from Section 7.4, Proposition 7.16). Since the homologies of all $\mathcal{M}^{\theta}(v)$, for generic θ, are identified (see 2.1.8), the LMN isomorphisms give rise to a $W(Q)$-action on L_ω. We also have an action of a suitable extension (by a 2-torsion group) of $W(Q)$ on L_ω coming from the $\mathfrak{g}(Q)$-action. If we knew that on each weight space the two actions coincide up to a sign, then we would have $\sigma_*^{-1}\mathsf{CC}(E_\alpha)\sigma_* = c_v e_{\sigma^{-1}(\alpha^i)} = c_v e_\alpha$ (for $c_v = \pm 1$) and similarly for F's. So, to prove Proposition 5.9, we need to establish the coincidence of the two group actions. Of course, it is enough to check the equality for simple reflections, s_i.

We will check that rather indirectly: on a categorical level. Namely, assume that $\theta_k > 0$ for all k and λ_i is integral. Recall that the functors E_i, F_i give rise to a categorical $\mathfrak{sl}_2$-action and hence produce derived equivalences (convolutions with Rickard complexes) $\Theta_i: D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}^{\theta}_{\lambda}(s_i\bullet v)\text{-mod})$. On the other hand, suppose that $(\lambda,\theta) \in \mathfrak{AL}(s_i\bullet v)$ (and hence $(\lambda', s_i\theta) \in \mathfrak{AL}(v)$ for $\lambda' = s_i \bullet^{s_i\bullet v}\lambda$). Then we can consider the wall-crossing functor

$$\mathfrak{WC}_{\lambda\to\lambda'}: D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}^{s_i\theta}_{\lambda'}(v)\text{-mod}).$$

Theorem 5.10. *Assume that $(\lambda,\theta)\in \mathfrak{AL}(v), \lambda_i\in \mathbb{Z}_{\geqslant 0}$ and $\theta_k>0$ for all $k\in Q_0$. Then we have an isomorphism of functors $\Theta_i = s_{i*}\circ \mathfrak{WC}_{\lambda\to\lambda'}$.*

We are going to use Theorem 5.10 to show that $\mathsf{CC}(E_\alpha)$ acts as $\pm e_\alpha$ on $\operatorname{Im}\mathsf{CC}_v$ and that $\mathsf{CC}(F_\alpha)$ acts as $\pm f_\alpha$ on $\operatorname{Im}\mathsf{CC}_{v+\epsilon_i}$. For this we need to check that $\mathsf{CC}(\Theta_i)$ acts by $\operatorname{Im}\mathsf{CC}_v$ by $\pm s_i$ (by s_i we denote an operator on L_ω induced by the $\mathfrak{g}(Q)$-action, it is defined up to a sign). This follows from (2) of Lemma 5.7 when we use the stability condition θ^+ but not straightforward for $\sigma\theta$. On the other hand, we need to show that $\mathsf{CC}(s_{i*}\circ \mathfrak{WC}_{\lambda\to\lambda'})$ coincides with s_{i*} (the operator of the $W(Q)$-action on the middle homology). Both claims follow from the next proposition. Let us write $\mathsf{CC}_v^{\lambda,\theta}$ for the characteristic cycle map $K_0(\mathcal{A}_\lambda(v)\operatorname{-mod}_{fin})\to L_\omega[\nu]$ defined using the stability condition θ.

Proposition 5.11. *If the homological dimension of $\mathcal{A}_\lambda(v)$ is finite and $\lambda\in\mathfrak{P}^{ISO}$, then the map $\mathsf{CC}_v^{\lambda,\theta}$ is independent of θ. For $M\in D^b_{\rho^{-1}(0)}(\mathcal{A}^{\theta'}_{\lambda'}(v)\operatorname{-mod})$ we have $\mathsf{CC}(M)=\mathsf{CC}(\mathfrak{WC}_{\lambda'\to\lambda}M)$.*

Again, we have an analog of this proposition (and of the equality $s_i=\pm[s_{i*}]$) on $\bigoplus_v \operatorname{Coh}_{\rho^{-1}}(0)(\mathcal{M}^\theta(v))$, see Sections 7.3 and 7.4.

We will deduce the coincidence of the actions from Theorem 5.10 and Proposition 5.11. Also Theorem 5.10 that can be regarded as a formula for $\mathfrak{WC}_{\lambda\to s_i\bullet\lambda}$ will play an important role in proving (II), see the next section.

5.4. **Outline of proof: upper bound on the image.** We prove (II), the inclusion $\operatorname{im}\mathsf{CC}^\lambda\subset L_\omega^{\mathfrak{a}}$, using wall-crossing functors.

More precisely, we will prove the following claim. Let $\mathcal{C}$ denote the full subcategory of $\bigoplus_v \mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)}$ consisting of all modules that appear in the homology of complexes of the form $\mathcal{F}L_0$, where $\mathcal{F}$ is a monomial in the functors $E_\alpha, F_\alpha, \alpha\in\Pi^\theta$, (here, as usual, Π^θ is a simple root system for $\mathfrak{a}$) and $L_0\in\mathcal{A}^\theta_\lambda(\sigma\bullet w)$, where $\sigma\in W(Q)$ is such that $\sigma\omega$ is dominant for $\mathfrak{a}$.

Proposition 5.12. *We have $\mathcal{C}=\bigoplus_v \mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)}$.*

This proposition will be proved in Section 11. In Section 9, we will see that Proposition 5.12 implies (II). Conversely, one can show that, modulo (III), (II) implies Proposition 5.12.

A basic strategy of proving Proposition 5.12 is as follows (we will elaborate on the strategy below in this section).

(1) Characterize the finite dimensional modules using a "long wall-crossing functor".
(2) Reduce the study of the "long wall-crossing functor" to the study of "short wall-crossing functors" – between chambers sharing an essential wall.
(3) Study short wall-crossing functors using Theorem 5.10 and categorical $\mathfrak{sl}_2$-actions.
(4) In the case of affine quivers, study the wall-crossing functor crossing the affine wall $\ker\delta$.

5.4.1. *Long wall-crossing functor.* Our first goal is to characterize the dimension of support of a simple $\mathcal{A}_\lambda(v)$-module in terms of a functor. It turns out that the functor we need is a "long wall-crossing functor" defined as follows. Let λ,θ be such that $(\lambda+k\theta,\theta)\in\mathfrak{AL}(v)$ for any $k\in\mathbb{Z}_{\geqslant 0}$. Thanks to Lemma 4.3, we can find $\lambda^-\in\lambda+\mathbb{Z}^{Q_0}$ such that $(\lambda^- - k\theta,-\theta)\in\mathfrak{AL}(v)$ for any $k\geqslant 0$. By a long wall-crossing functor we mean $\mathfrak{WC}_{\lambda\to\lambda^-}: D^b(\mathcal{A}_\lambda(v)\operatorname{-mod})\xrightarrow{\sim} D^b(\mathcal{A}_{\lambda^-}(v)\operatorname{-mod})$.

We have recalled the definition of holonomic modules for $\mathcal{A}_\lambda(v), \mathcal{A}^\theta_\lambda(v)$ in Section 2.4. In particular, modules from $\mathcal{A}_\lambda(v)\operatorname{-mod}_{fin}, \mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)}$ are holonomic.

The following proposition is inspired by a similar result on the BGG category $\mathcal{O}$, see [BFO, Proposition 4.7].

Proposition 5.13. *Let M be a holonomic $\mathcal{A}_\lambda(v)$-module. Then*

(1) $H_i(\mathfrak{WC}_{\lambda\to\lambda^-}M) = 0$ *if* $i < \frac{1}{2}\dim\mathcal{M}^\theta(v) - \dim\operatorname{Supp} M$ *or* $i > \frac{1}{2}\dim\mathcal{M}^\theta(v)$. *Moreover,* $H_i(\mathfrak{WC}_{\lambda\to\lambda^-}M) \neq 0$ *for* $i = \frac{1}{2}\dim\mathcal{M}^\theta(v) - \dim\operatorname{Supp} M$.

(2) *The functor* $\mathfrak{WC}_{\lambda\to\lambda^-}[-\frac{1}{2}\dim\mathcal{M}^\theta(v)]$ *is an abelian equivalence* $\mathcal{A}_\lambda(v)\operatorname{-mod}_{fin} \xrightarrow{\sim} \mathcal{A}_{\lambda^-}(v)\operatorname{-mod}_{fin}$.

The minimal number i such that $H_i(\mathfrak{WC}_{\lambda\to\lambda^-}L) \neq 0$ will be called the *homological shift* (of L under the functor $\mathfrak{WC}_{\lambda\to\lambda^-}$).

The proof to be given below, Section 8, is based on using abelian localization as well as a connection between the long wall-crossing functor and a homological duality functor.

One problem with the wall-crossing functor is that it is quite hard to study it (in particular, computing the homological shifts) directly. Instead, we will decompose $\mathfrak{WC}_{\lambda\to\lambda^-}$ into a composition of short wall-crossing functors (i.e., functors crossing a single wall between two neighboring chambers) using Theorem 5.3. Of course, it does not even make sense to say that the homological shifts add up under the composition. However, the information about homological shifts under the short wall-crossing functors together with Theorem 5.3 is enough to establish (II) for finite and affine quivers Q (in this paper we only deal with very special framing in the affine case but this can be generalized to arbitrary framing by suitably generalizing techniques that we use here, see the subsequent papers [L10, L16]). The case of wild quivers poses some additional essential difficulties, we will discuss why in 5.4.3.

5.4.2. *Short wall-crossing through real wall.* We want to characterize objects that are not homologically shifted under the short wall-crossing functor through a wall defined by a real root.

Let $\alpha \in \Pi^\theta$. We say that an object $L \in \operatorname{Irr}(\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)})$ is *α-singular* if

- $(\nu, \alpha) \geqslant 0$ and $[L] \notin \operatorname{im}[F_\alpha]$ or
- $(\nu, \alpha) \leqslant 0$ and $[L] \notin \operatorname{im}[E_\alpha]$.

Here and below we write $[L]$ for the class of L in $K_0(\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)})$. We write $[E_\alpha], [F_\alpha]$ for the maps between the K_0 spaces induced by the functors E_α, F_α.

There is an important alternative characterization of α-singular objects based on Theorem 5.10. Namely, suppose that θ' is a generic stability condition separated from θ by the single wall $\ker\alpha$ (meaning the chambers of θ, θ' share the common wall $\ker\alpha$).

Proposition 5.14. *Let $L \in \operatorname{Irr}(\mathcal{A}^\theta_\lambda\operatorname{-mod}_{\rho^{-1}(0)})$. The following conditions are equivalent:*

(1) *L is α-singular.*

(2) $H_0(\mathfrak{WC}_{\theta\to\theta'}L) \neq 0$.

Moreover, under these equivalent conditions there is a unique α-singular simple constituent of $H_*(\mathfrak{WC}_{\theta\to\theta'}L)$, say L', and it is a quotient of $H_0(\mathfrak{WC}_{\theta\to\theta'}L)$. The map $L \mapsto L'$ is a bijection between the sets of α-singular simples in $\mathcal{A}^\theta_\lambda\operatorname{-mod}_{\rho^{-1}(0)}, \mathcal{A}^{\theta'}_\lambda\operatorname{-mod}_{\rho^{-1}(0)}$.

This proposition will be proved in Section 9.2.

5.4.3. *Short wall-crossing through affine wall.* Thanks to the previous paragraph, we only need to show (modulo technicalities to be addressed later) that the wall-crossing functor through $\ker\delta$ cannot homologically shift a simple by more than $\dim\mathcal{M}^\theta(v)/2-1$.

Let us explain an idea of the proof of this claim, which is an extension of what was done in the proof of Proposition 5.4. By Proposition 5.2, the wall-crossing functor through $\ker\delta$ is given by $\mathcal{A}^0_{\lambda,\chi}(v)\otimes^L_{\mathcal{A}^0_\lambda(v)}\bullet$ (here we assume that $\lambda,\lambda+\chi\in\mathfrak{P}^{iso}$ and there are generic stability conditions θ,θ' separated by $\ker\delta$ such that $(\lambda,\theta),(\lambda+\chi,\theta')\in\mathfrak{AL}(v)$). Now let us set $\mathfrak{P}_0:=\lambda+\ker\delta,\mathfrak{P}_0':=\lambda+\chi+\ker\delta$. Then $\mathcal{A}^0_{\lambda,\chi}(v)$ is the specialization of $\mathcal{A}^0_{\mathfrak{P}_0,\chi}(v)$. We will show that "homological shift behavior" of the functors $\mathcal{A}^0_{\lambda_1,\chi}(v)\otimes_{\mathcal{A}^0_{\lambda_1}(v)}\bullet$ is the same for Zariski generic parameters $\lambda_1\in\mathfrak{P}_0$ (this is the most non-trivial part of the proof; the very first step here is results from 3.2.2). Then we need to show that for a *Weil generic* λ_1 the homological shifts under the functor $\mathcal{A}^0_{\lambda_1,\chi}(v)\otimes^L_{\mathcal{A}^0_{\lambda_1}(v)}\bullet$ are less than $\frac{1}{2}\dim\mathcal{M}^\theta(v)$ (recall that "Weil generic" means "lying outside of countably many proper closed algebraic subvarieties"). The point of considering Weil generic parameters is that here the functor $\mathcal{A}^0_{\lambda_1,\chi}(v)\otimes^L_{\mathcal{A}^0_{\lambda_1}(v)}\bullet$ becomes the long wall-crossing functor (indeed, all walls but possibly $\ker\delta$ are non-essential). Basically, we prove that the algebra $\mathcal{A}^0_{\lambda_1}(v)$ has no finite dimensional simples, and our claim about homological shifts follows from Proposition 5.13.

In fact, we show more than the bound for homological shifts, we prove that $\mathfrak{WC}_{\lambda\to\lambda+\chi}$ is *perverse* in the sense of Chuang and Rouquier, [R1, Section 2.6]. We characterize filtrations on the categories $\mathcal{A}^0_\lambda(v)\text{-mod},\mathcal{A}^0_{\lambda+\chi}(v)\text{-mod}$ that make $\mathfrak{WC}_{\lambda\to\lambda+\chi}$ perverse and deduce our claim about homological shifts from there. We use results on the representation theory of type A Rational Cherednik algebras to establish the perversity, which leads to restrictions on the framing.

Let us explain the most essential reason why we restrict to finite and affine quivers. In fact, as shown in the subsequent paper [L16], wall-crossing functors are perverse in a much more general situation (including all wall-crossings through hyperplanes for wild quivers). A difficulty of dealing with wild quivers is that one may need to cross many walls defined by imaginary roots and we do not know how to control the homological shifts of compositions in that case.

5.4.4. *Completion of the proof.* Let us define *extremal* objects.

Definition 5.15. We say that $L\in\operatorname{Irr}(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ is *extremal* if L does not lie in the category $\mathcal{C}$ from 5.4 and v is minimal with this property.

Of course, (II) is equivalent to the claim that no extremal objects exist.

Here are two important properties of extremal objects to be proved in Section 11.

Lemma 5.16. *An extremal object is singular for all α.*

The following is a crucial property of extremal objects.

Proposition 5.17. *The bijection $L\mapsto L'$ from Proposition 5.14 restricts to a bijection between the sets of extremal simples in $\mathcal{A}^\theta_\lambda\text{-mod}_{\rho^{-1}(0)},\mathcal{A}^{\theta'}_\lambda\text{-mod}_{\rho^{-1}(0)}$.*

We conclude that extremal objects are not homologically shifted by short wall-crossing functors through real walls (=walls defined by real roots). This finishes the proof of (II) in the case when Q is finite. To deal with the case of affine Q (under our restrictions on the framing) one uses results outlined in 5.4.3.

5.5. **Outline of proof: injectivity of CC.** Now we will explain how Proposition 5.12 implies (III).

Suppose first, that we know that

(*) the endomorphisms $[E_\alpha],[F_\alpha],\alpha\in\Pi^\theta$, define a representation of the Lie algebra $\mathfrak{a}$ in $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)})$

(so far we know that each pair $([E_\alpha],[F_\alpha])$ defines a representation of $\mathfrak{sl}_2$). So CC^λ becomes an epimorphism of $\mathfrak{a}$-modules. Using Proposition 5.12 we will show that it is an isomorphism.

It remains to establish (*). In fact, this reduces to the case when λ is rational (where we have a powerful tool – reduction to positive characteristic). The general case will be deduced from there and Proposition 5.12.

To prove (*) we will argue more or less as follows. We will show that the degeneration map, $[M]\mapsto[\operatorname{gr}M]$ defines an embedding $K_0(\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)})\hookrightarrow K_0(\operatorname{Coh}_{\rho^{-1}(0)}\mathcal{M}^\theta(v))$. We will see that $\mathfrak{a}$ naturally acts on $K_0(\operatorname{Coh}_{\rho^{-1}(0)}\mathcal{M}^\theta(v))$ (in fact, the whole algebra $\mathfrak{g}(Q)$ does) and our embedding (after some twist) intertwines $[E_\alpha]$ with e_α, $[F_\alpha]$ with f_α. The proofs here are based on K-theory version of results mentioned in Section 5.3. This proves (*) and finishes the proof of Theorem 1.2.

5.6. **Subsequent content.** Let us describe the content of the following sections. Section 6 is preparatory, there we discuss some relatively standard results based on the reduction to characteristic p. Then, in Section 7 we prove (I) (the inclusion $L^{\mathfrak{a}}_\omega[\nu]\subset\operatorname{im}\mathsf{CC}_\lambda$) as well as some stronger results needed in the proofs of (II) and (III) (concerning K-theory rather than middle homology).

In the subsequent three sections we study wall-crossing functors. In Section 8 we study the long wall-crossing functor and relate the homological shifts under this functor to codimension of support. In Section 9 we study short wall-crossing functors through walls defined by real roots and their interactions with singular simples. In Section 10 we study the short wall-crossing functor through the wall $\ker\delta$ and prove that it is a perverse equivalence.

Finally, in Section 11 we finish the proofs of (II) and (III) and hence of Theorem 1.2. We also discuss some generalizations of Conjecture 1.1.

6. Quantizations in positive characteristic and applications

In this section we deal with quantizations in positive characteristic and their applications to characteristic 0.

Let us explain two main applications first. They concern the existence of tilting generators on $\mathcal{M}^\theta(v)$ with some special properties and the injectivity of a natural map $K_0(\mathcal{A}_\lambda(v)\operatorname{-mod}_{fin})\to K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ that is the composition of the localization and degeneration (here we assume that the homological dimension of $\mathcal{A}_\lambda(v)$ is finite and $\lambda\in\mathbb{Q}^{Q_0}$).

Let us state a result about a tilting bundle. We say that a vector bundle $\mathcal{P}$ on a smooth algebraic variety X is a *tilting generator* if $\operatorname{Ext}^i(\mathcal{P},\mathcal{P})=0$ for $i>0$ and the homological dimension of the algebra $\operatorname{End}(\mathcal{P})$ is finite. When X is a Nakajima quiver variety (in fact, under some more general assumptions), the functor $R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{P},\bullet)$ is an equivalence $D^b(\operatorname{Coh}X)\xrightarrow{\sim}D^b(\operatorname{End}(\mathcal{P})^{opp}\operatorname{-mod})$ (see [BezKa2, Proposition 2.2]).

Here is our main result on the existence of compatible tilting generators on the quiver varieties $\mathcal{M}^\theta(v)$.

Proposition 6.1. *There is a $\mathbb{C}^\times$-equivariant (with respect to the contracting action) tilting generator $\mathcal{P}^\theta$ on $\mathcal{M}^\theta(v)$ such that the algebra* $\operatorname{End}(\mathcal{P}^\theta)$ *is independent of θ.*

Such a bundle $\mathcal{P}^\theta$ is constructed by Kaledin in [Ka1]. Our construction is quite similar to Kaledin's and is also inspired by an earlier construction in [BezKa2]. Namely, one fixes a suitable quantization of $\mathcal{M}^\theta(v)$ over an algebraically closed field of positive characteristic. It is an Azumaya algebra on the Frobenius twist $\mathcal{M}^\theta(v)^{(1)}$ which then can be shown to split on the formal neighborhood of the zero fiber. The splitting bundle then extends to a vector bundle that is shown to be tilting. Our proof of the splitting result is easier than Kaledin's. Besides, for our next main result of this section we need to use a particular choice of a quantization: one obtained by Hamiltonian reduction.

Now let us proceed to the second main result in this section: on the injectivity of the natural map $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \hookrightarrow K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$. Let λ be such that $\mathcal{A}_\lambda(v)$ is regular so that the localization functor $L\operatorname{Loc}^\theta_\lambda$ gives rise to an identification $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \xrightarrow{\sim} K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$. We have a well-defined map

$$[M] \mapsto [\operatorname{gr} M] : K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))).$$

Let us denote the composition $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \to K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ by γ^θ_λ.

Proposition 6.2. *Suppose, in addition, that $\lambda \in \mathbb{Q}^{Q_0}$. Then γ^θ_λ is injective.*

The proposition is true for any λ and, in fact, follows from the injectivity of the characteristic cycle map. But at this point we are only able to prove it for rational λ.

6.1. **Quiver varieties and quantizations in characteristic** p. We can define the GIT quotient $\mathcal{M}^\theta(v)$ in characteristic p for p large enough. More precisely, the moment map $\mu : T^*R \to \mathfrak{g}$ is defined over $\mathbb{Z}$. So we can reduce it modulo p and get $\mu_{\mathbb{F}} : T^*R_{\mathbb{F}} \to \mathfrak{g}_{\mathbb{F}}$, where $\mathbb{F}$ is an algebraically closed field of characteristic p. For p large enough, this is still a moment map and we can form the Hamiltonian reduction $\mathcal{M}^\theta(v)_{\mathbb{F}}$ that is a smooth symplectic algebraic variety over $\mathbb{F}$.

Lemma 6.3. *There is a finite localization S of $\mathbb{Z}$ and a smooth symplectic scheme $\mathcal{M}^\theta(v)_S$ over* $\operatorname{Spec}(S)$ *with the following properties:*
1. *$\mathcal{M}^\theta(v), \mathcal{M}^\theta(v)_{\mathbb{F}}$ are obtained from $\mathcal{M}^\theta(v)_S$ by base change (for every S-algebra $\mathbb{F}$ that is an algebraically closed field).*
2. *$\mathbb{C}[\mathcal{M}^\theta(v)], \mathbb{F}[\mathcal{M}^\theta(v)_{\mathbb{F}}]$ are obtained from $S[\mathcal{M}^\theta(v)_S]$ by base change.*
3. *$H^i(\mathcal{M}^\theta(v)_S, \mathcal{O}_{\mathcal{M}^\theta(v)_S}) = 0$, $H^i(\mathcal{M}^\theta(v)_{\mathbb{F}}, \mathcal{O}_{\mathcal{M}^\theta(v)_{\mathbb{F}}}) = 0$ for $i > 0$, where $\mathbb{F}$ is as in (1).*

Proof. We remark that $\mu^{-1}(0)^{\theta-ss} \to \mathcal{M}^\theta(v)$ is a principal G-bundle, in particular, it is locally trivial in the Zariski topology. It is defined over some finite localization S of $\mathbb{Z}$. After a finite localization, $\mu_S^{-1}(0)^{\theta-ss}$ – the stable locus of $\operatorname{Spec}(S[T^*R_S]/(\mu_S^*(\mathfrak{g}_S)))$ – becomes the total space of this principal bundle. (i) follows.

Fix an open affine cover of $\mathcal{M}^\theta(v)_S$. After a finite localization all cocycle groups in the Čech complex for $\mathcal{O}_{\mathcal{M}^\theta(v)_S}$ coincide with the corresponding coboundary groups and they are free over S. (2) and (3) follow. $\square$

This result can be generalized to $\mathcal{M}^\theta_{\mathfrak{p}}(v)$ in a straightforward way.

Quantizations of $\mathcal{M}^\theta(v)_{\mathbb{F}}$ were studied in [BFG], see Section 3,4,6 there. Take $\lambda \in \mathbb{F}_p^{Q_0}$. The algebra $D(R)_{\mathbb{F}}$ is Azumaya over the Frobenius twist $\mathbb{F}[T^*R_{\mathbb{F}}]^{(1)}$ so we can view $D_{R,\mathbb{F}}$

as a coherent sheaf on $(T^*R_{\mathbb{F}})^{(1)}$. According to [BFG, Section 3],

$$\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}} := [\mathcal{Q}_{\mathbb{F},\lambda}|_{(T^*R_{\mathbb{F}})^{(1)},\theta-ss}]^{G_{\mathbb{F}}}$$

is a sheaf of Azumaya algebras on $\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$. If we consider this sheaf in the conical topology, it becomes filtered, and the associated graded is $\mathrm{Fr}_*\,\mathcal{O}_{\mathcal{M}^\theta(v)_{\mathbb{F}}}$.

There is an extension of this construction to $\lambda \in \mathbb{F}^{Q_0}$. The difference is that $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ is now an Azumaya algebra over $\mathcal{M}^\theta_{\mathrm{AS}(\lambda)}(v)$, where AS is the Artin-Schreier map, see [BFG, Section 3.2]. We also have a version that works in families. We get a sheaf $\mathcal{A}^\theta_{\mathfrak{P}}(v)_{\mathbb{F}}$ of Azumaya algebra over $\mathfrak{p}_{\mathbb{F}} \times_{\mathfrak{p}^{(1)}_{\mathbb{F}}} \mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)}_{\mathbb{F}}$ that specializes to $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ for any $\lambda \in \mathbb{F}^{Q_0}$.

Let us write $\mathcal{A}_\lambda(v)_{\mathbb{F}}$ for the global sections of $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$.

Lemma 6.4. *Fix $\lambda^\circ \in \mathbb{Q}^{Q_0}$. There is a finite localization S of $\mathbb{Z}$ with the following property: for any $\lambda \in \lambda^\circ + \mathbb{Z}^{Q_0}$ there exists a filtered S-algebra $\mathcal{A}_\lambda(v)_S$ such that*

(1) $\mathrm{gr}\,\mathcal{A}_\lambda(v)_S = S[\mathcal{M}^\theta(v)_S]$.
(2) $\mathbb{C} \otimes_S \mathcal{A}_\lambda(v)_S = \mathcal{A}_\lambda(v)$.
(3) $\mathbb{F} \otimes_S \mathcal{A}_\lambda(v)_S = \mathcal{A}_\lambda(v)_{\mathbb{F}}$.

Proof. We may assume that Lemma 6.3 holds for S and moreover that $\lambda^\circ \in S^{Q_0}$ and that μ_S is flat. We can define the microlocal quantizations $\mathcal{A}^\theta_\lambda(v)_S$ of $\mathcal{M}^\theta(v)_S$ in the same way as was done for the complex numbers. Set $\mathcal{A}_\lambda(v)_S := \Gamma(\mathcal{A}^\theta_\lambda(v)_S)$.

(1) follows from (2) and (3) of Lemma 6.3. The microlocal quantization $\mathcal{A}^\theta_\lambda(v)$ is obtained from $\mathcal{A}^\theta_\lambda(v)_S$ by the base change to $\mathbb{C}$ followed by a suitable completion (needed to preserve the condition that the sheaf is still complete and separated with respect to the filtration). (2) follows. To prove (3), we notice that we have a natural homomorphism $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}} \to \mathrm{Fr}_*(\mathbb{F} \otimes_S \mathcal{A}^\theta_\lambda(v)_S)$. On the level of the associated graded sheaves, it is the identity automorphism of $\mathrm{Fr}_*\,\mathcal{O}_{\mathcal{M}^\theta(v)_{\mathbb{F}}}$. So it gives rise to an isomorphism $\mathcal{A}_\lambda(v)_{\mathbb{F}} = \Gamma(\mathcal{M}^\theta(v)_{\mathbb{F}}, \mathbb{F} \otimes_S \mathcal{A}^\theta_\lambda(v)_S) = \mathbb{F} \otimes_S \mathcal{A}_\lambda(v)_S$. □

6.2. **Splitting.** We write $\mathcal{M}(v)^{(1)}_{\mathbb{F}}$ for $\mathrm{Spec}(\mathbb{F}[\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}])$, $\mathcal{M}_{\mathfrak{p}}(v)^{(1)}_{\mathbb{F}}$ for $\mathfrak{p}_{\mathbb{F}} \times_{\mathfrak{p}^{(1)}_{\mathbb{F}}} \mathrm{Spec}(\mathbb{F}[\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)}_{\mathbb{F}}])$. Consider $\mathcal{M}^\theta(v)^{(1)\wedge_0}_{\mathbb{F}} = \rho^{-1}_{\mathbb{F}}(\mathcal{M}^\theta(v)^{\wedge_0}_{\mathbb{F}})$ and $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)\wedge_0}_{\mathbb{F}}$, the formal neighborhood of $\mathcal{M}^\theta(v)^{(1)\wedge_0}_{\mathbb{F}}$ in $\rho^{-1}_{\mathbb{F}}(\mathcal{M}_{\mathfrak{p}}(v)^{(1)\wedge_0})$.

Proposition 6.5. *The restrictions $\mathcal{A}^\theta_\lambda(v)^{\wedge_0}_{\mathbb{F}}$ of $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ to $\mathcal{M}^\theta(v)^{(1)\wedge_0}_{\mathbb{F}}$ and $\mathcal{A}^\theta_{\mathfrak{P}_{\mathbb{F}}}(v)^{\wedge_0}$ of $\mathcal{A}^\theta_{\mathfrak{P}_{\mathbb{F}}}(v)$ to $\mathcal{M}^\theta_{\mathfrak{p}_{\mathbb{F}}}(v)^{(1)\wedge_0}$ (where the fiber of $\mathcal{A}^\theta_{\mathfrak{P}_{\mathbb{F}}}(v)$ over $0 \in \mathfrak{p}$ is $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$) split.*

Proof. Let us prove the claim about $\mathcal{A}^\theta_\lambda(v)^{\wedge_0}_{\mathbb{F}}$ first.

Step 1. Consider the one-form β on $\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$ obtained by pairing of the symplectic form ω with the Euler vector field for the $\mathbb{F}^\times$-action (induced from the fiberwise dilation action on T^*R). We claim that the class of $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ in the Brauer group $\mathrm{Br}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}})$ comes from β (see [Mi, III.4] for a general discussion of Azumaya algebras coming from 1-forms). Let us prove this claim. Let π denote the quotient morphism $Z := (\mu^{(1)}_{\mathbb{F}})^{-1}(0)^{\theta-ss} \twoheadrightarrow \mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$ (by the $G^{(1)}_{\mathbb{F}}$-action). By [BFG, Remark 4.1.5], $\pi^*(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}})$ is Morita equivalent to $D_{R,\mathbb{F}}|_Z$. The class of $D_{R,\mathbb{F}}$ comes from the canonical 1-form $\tilde{\beta}$ on $(T^*R_{\mathbb{F}})^{(1)}$. The restriction of $\tilde{\beta}$ to Z coincides with $\pi^*\beta$. It follows that the class of $\pi^*\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ in the Brauer group coincides with the class defined by $\pi^*\beta$. Now recall that Z is a principal $G^{(1)}_{\mathbb{F}}$-bundle on $\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$ hence it is locally trivial in Zariski topology. It follows that the restriction of the class of $\mathcal{A}^\theta_\lambda(v)$ to an open subset $U\subset \mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$ (where the bundle trivializes) coincides with the restriction of the class of β. Since the restriction induces an embedding $\operatorname{Br}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}})\hookrightarrow \operatorname{Br}(U)$ ([Mi, III.2.22]), the claim in the beginning of the paragraph is proved.

Step 2. An Azumaya algebra defined by a one-form β' splits provided $\beta'=\alpha-\mathsf{C}(\alpha)$ for some 1-form α, where C stands for the Cartier map $\Omega^1_{cl}\to\Omega^1$ (here we write Ω^1 for the bundle of 1-forms and Ω^1_{cl} for the bundle of closed 1-forms). We claim that $\mathsf{C}:\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{cl})\to\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1)$ is surjective. This follows from the following exact sequences of sheaves:

$$0\to\Omega^1_{ex}\to\Omega^1_{cl}\xrightarrow{\mathsf{C}}\Omega^1\to 0, \tag{6.1}$$

$$0\to\mathcal{O}^p\to\mathcal{O}\to\Omega^1_{ex}\to 0. \tag{6.2}$$

Since $H^i(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\mathcal{O})=H^i(\mathcal{M}^\theta(v)_{\mathbb{F}},\mathcal{O})=0$ for $i=1,2$, (6.2) implies $H^1(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{ex})=0$. Hence, using (6.1), $\mathsf{C}:\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{cl})\twoheadrightarrow\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1)$.

Step 3. The global sections $\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{cl}),\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1)$ are graded with respect to the $\mathbb{F}^\times$-action (coming from the dilation action on T^*R), let $\Gamma(\ldots)_d$ denote the dth graded component. The map C sends $\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{cl})_d$ to $\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1)_{d/p}$ if d is divisible by p and to 0 else. Pick $\mathbb{F}$-linear sections $\mathsf{C}^{-1}:\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1)_d\to\Gamma(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\Omega^1_{cl})_{pd}$.

Let us point out that the degree of β is 2. It follows that $\alpha:=\sum_{i=0}^{+\infty}\mathsf{C}^{-i}(\beta)$ is a well-defined 1-form on $\mathcal{M}^\theta(v)^{(1)\wedge_0}_{\mathbb{F}}$. Indeed the ith summand has degree $2p^i$ and so $\mathsf{C}^{-i}(\beta)$ converges to zero in the topology defined by the maximal ideal of 0 in $\mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]$. Clearly, $\beta=\alpha-\mathsf{C}(\alpha)$.

This finishes the proof of the claim that $\mathcal{A}^\theta_\lambda(v)^{\wedge_0}_{\mathbb{F}}$ splits.

Let us proceed to the splitting of $\mathcal{A}^\theta_{\mathfrak{P}_{\mathbb{F}}}(v)^{\wedge_0}$.

Step 4. We will prove a stronger statement. Let $\tilde{\mathcal{A}}$ be an Azumaya algebra over $\rho_{\mathbb{F}}^{-1}(\mathcal{M}_{\mathfrak{p}}(v)^{(1)\wedge_0}_{\mathbb{F}})$ whose restriction to $\rho_{\mathbb{F}}^{-1}(\mathcal{M}(v)^{(1)\wedge_0}_{\mathbb{F}})$ splits. We will show that then the restriction of $\tilde{\mathcal{A}}$ to $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)\wedge_0}$ splits as well.

Let $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)k}_{\mathbb{F}}$ denote the kth infinitesimal neighborhood of $\rho_{\mathbb{F}}^{-1}(\mathcal{M}(v)^{(1)\wedge_0}_{\mathbb{F}})$ in $\rho_{\mathbb{F}}^{-1}(\mathcal{M}_{\mathfrak{p}_{\mathbb{F}}}(v)^{(1)\wedge_0})$, a scheme over $\operatorname{Spec}(\mathbb{F}[\mathfrak{p}]/\mathfrak{m}^{k+1})$, where $\mathfrak{m}$ is the maximal ideal. We remark that

$$H^i(\mathcal{M}^\theta(v)^{(1)\wedge_0}_{\mathbb{F}},\mathcal{O})=0,\ \text{for } i>0$$

(to simplify the notation we just write $\mathcal{O}$ for the structure sheaf). This follows from $H^i(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}},\mathcal{O})=0$ and the formal function theorem.

Step 5. We have a short exact sequence of sheaves on $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)\wedge_0}_{\mathbb{F}}$:

$$0\to S^k\mathcal{N}\to\mathcal{O}^\times_{\mathcal{M}^\theta_{\mathfrak{p}_{\mathbb{F}}}(v)^{(1)k+1}}\to\mathcal{O}^\times_{\mathcal{M}^\theta_{\mathfrak{p}_{\mathbb{F}}}(v)^{(1)k}}\to 0, \tag{6.3}$$

where $\mathcal{N}$ is the normal bundle to $\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}$ in $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{(1)}_{\mathbb{F}}$.

We claim that that $\mathcal{N}=\mathfrak{p}\otimes\mathcal{O}$. First, note that the conormal bundle to $\mu^{-1}(0)^{\theta-ss}$ in T^*R is the trivial bundle with the fiber $\mathfrak{g}$: if $\xi_1,\ldots,\xi_m$ is a basis in $\mathfrak{g}$, then $d\mu^*(\xi_1),\ldots,d\mu^*(\xi_m)$ is a basis in the conormal bundle. It follows the conormal bundle to $\mu^{-1}(0)^{\theta-ss}$ in $\mu^{-1}(\mathfrak{g}^{*G})^{\theta-ss}$ is trivial with fiber $\mathfrak{g}^G$. Since $\mathcal{N}$ is the equivariant descent of the latter bundle, we get $\mathcal{N}=\mathfrak{p}\otimes\mathcal{O}$.

This implies

$$H^i(\mathcal{M}^\theta(v)_{\mathbb{F}}^{(1)\wedge_0}, S^k\mathcal{N}) = 0 \text{ for } i > 0. \tag{6.4}$$

In particular, the Pickard groups of the schemes $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}$ are naturally identified.

Step 6. To check that $\tilde{\mathcal{A}}$ splits, it is enough to show that $\tilde{\mathcal{A}}|_{\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}}$ splits for each k. To see why this is enough, we notice that, since the splitting bundles are defined up to a twist with a line bundle, a splitting bundle lifts from $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}$ to $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k+1}$. This gives a splitting of the restriction of $\tilde{\mathcal{A}}$ to the required formal neighborhood.

We show that $\tilde{\mathcal{A}}|_{\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}}$ splits by using induction on k. The base, $k = 1$, has been established before in this proof. Let us establish the induction step. Recall that $\operatorname{Br}(\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}) \hookrightarrow H^2_{et}(\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}, \mathcal{O}^\times)$, see [Mi, Theorem 2.5]. From (6.3), (6.4), it follows that $H^2_{et}(\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}, \mathcal{O}^\times)$ and $H^2_{et}(\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k+1}, \mathcal{O}^\times)$ are naturally identified. In particular, if the restriction of $\tilde{\mathcal{A}}$ to $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k}$ splits, then so does the restriction to $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)k+1}$. $\square$

6.3. **Comparison for different resolutions.** Let $\hat{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ denote a splitting bundle for $\mathcal{A}^\theta_{\mathfrak{P}}(v)^{\wedge_0}_{\mathbb{F}}$. The bundle $\hat{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ has trivial higher self-extensions, compare with [BezKa2, Section 2.3], and hence has an $\mathbb{F}^\times$-equivariant structure, see [V].

We remark that since $\mathcal{M}^\theta(v)_{\mathbb{F}}$ is defined over $\mathbb{F}_p$, we have an isomorphism $\mathcal{M}^\theta(v)_{\mathbb{F}} \cong \mathcal{M}^\theta(v)_{\mathbb{F}}^{(1)}$ of $\mathbb{F}$-schemes. Therefore we can view $\hat{\mathcal{P}}^\theta_{\mathbb{F}}$ (the specialization of $\hat{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ to $0 \in \mathfrak{p}_{\mathbb{F}}$) as a bundle on $\mathcal{M}^\theta(v)^{\wedge_0}_{\mathbb{F}}$. Similarly, we can view $\hat{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ as a bundle on $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_0}_{\mathbb{F}}$. This is because the Artin-Schreier map is etale and so induces an isomorphism of $\mathbb{F}[\mathfrak{p}]^{\wedge_0}$ with itself. Since the bundle $\hat{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ has no higher self-extensions, we can extend it to a unique $\mathbb{F}^\times$-equivariant vector bundle on $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}$ to be denoted by $\mathcal{P}^\theta_{\mathfrak{p},\mathbb{F}}$.

One can lift $\mathcal{P}^\theta_{\mathfrak{p},\mathbb{F}}$ to characteristic 0 as explained in [BezKa2]. Let us recall how to do this. The bundle $\mathcal{P}^\theta_{\mathfrak{p},\mathbb{F}}$ is defined over some finite field $\mathbb{F}_q$. Let $\tilde{S}$ be an algebraic extension of the ring S from Section 6.1 that has $\mathbb{F}_q$ as a quotient field. Set $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\tilde{S}} := \operatorname{Spec}(\tilde{S}) \times_{\operatorname{Spec}(S)} \mathcal{M}^\theta_{\mathfrak{p}}(v)_S$. Since the bundle $\mathcal{P}^\theta_{\mathfrak{p},\mathbb{F}}$ has no higher Ext's it can be extended to a unique $\mathbb{G}_m$-equivariant bundle $\tilde{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ on the formal neighborhood $\mathcal{M}^\theta_{\mathfrak{p}}(v)^{\wedge_q}_{\tilde{S}}$ of $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}_q}$ in $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\tilde{S}}$ (the existence is guaranteed by vanishing of Ext^2, and the uniqueness by the vanishing of Ext^1). Since the bundle $\tilde{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ is $\mathbb{G}_m$-equivariant, and the action is contracting, this bundle is the completion of a unique $\mathbb{G}_m$-equivariant bundle $\mathcal{P}^\theta_{\mathfrak{p}_{\tilde{S}^{\wedge_q}}}$ on $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\tilde{S}^{\wedge_q}}$, where $\tilde{S}^{\wedge_q}$ stands for the q-adic completion of $\tilde{S}$. Since the quotient field of $\tilde{S}^{\wedge_q}$ embeds into $\mathbb{C}$, we get a bundle $\mathcal{P}^\theta_{\mathfrak{p}}$ on $\mathcal{M}^\theta_{\mathfrak{p}}(v)$. This bundle is $\mathbb{C}^\times$-equivariant and has no higher self-extensions. We remark that its restriction to $\mathcal{M}^\theta(v)$ has no higher self-extensions either because $\mathcal{P}^\theta_{\mathfrak{p}}$ is flat over $\mathbb{C}[\mathfrak{p}]$ and so is $\operatorname{End}(\mathcal{P}^\theta_{\mathfrak{p}})$.

Now we want to compare the endomorphism algebras of the bundles $\mathcal{P}^\theta_{\mathfrak{p}}$ for different θ.

Proposition 6.6. *For any (generic) θ, θ', we have $\operatorname{End}(\mathcal{P}^\theta_{\mathfrak{p}}) \cong \operatorname{End}(\mathcal{P}^{\theta'}_{\mathfrak{p}})$, an isomorphism of graded $\mathbb{C}[\mathfrak{p}]$-algebras.*

Proof. The proof is in several steps.

Step 1. Let $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)reg}$ denote the locus where the morphism $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)} \to \mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)}$ is an isomorphism (compare to the proof of Proposition 3.3), it is equal to the union of the open symplectic leaves in $\mathcal{M}_\lambda(v)^{(1)}, \lambda \in \mathfrak{p}_{\mathbb{F}}$. So $\mathcal{M}^\theta_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)reg} = \mathcal{M}^{\theta'}_{\mathfrak{p}}(v)_{\mathbb{F}}^{(1)reg}$. We claim that the restrictions of the bundles $\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}, \mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}}$ to this open subvariety differ by a twist with a line bundle. The latter will follow if we check that

$$\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}) \cong \operatorname{End}(\mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}}). \tag{6.5}$$

Indeed, the restriction of this endomorphism algebra to $\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{(1)reg}_{\mathbb{F}}$ is Azumaya and the restrictions of both $\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}, \mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}}$ are splitting bundles so differ by a twist with a line bundle.

To establish (6.5) we will first verify that $\operatorname{End}(\hat{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}}) = \operatorname{End}(\hat{\mathcal{P}}^{\theta'}_{\mathfrak{p},\mathbb{F}})$. The left hand side is $\Gamma(\mathcal{A}^{\theta}_{\mathfrak{P}_{\mathbb{F}}}(v)^{\wedge_0})$, while the right hand side is $\Gamma(\mathcal{A}^{\theta'}_{\mathfrak{P}_{\mathbb{F}}}(v)^{\wedge_0})$, both are isomorphic to $\mathcal{A}_{\mathfrak{P}_{\mathbb{F}}}(v)^{\wedge_0}$. Now by the formal function theorem, the completions $\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})^{\wedge_0}, \operatorname{End}(\mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}})^{\wedge_0}$ are isomorphic $\mathbb{F}[[\mathfrak{p}]]$-algebras. We want to deduce the isomorphism $\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}) \cong \operatorname{End}(\mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}})$ from here. This will follow if we show that the isomorphism $\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})^{\wedge_0} \cong \operatorname{End}(\mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}})^{\wedge_0}$ can be made $\mathbb{F}^{\times}$-equivariant by twisting the actions on the indecomposable summands of the vector bundles involved by characters of $\mathbb{F}^{\times}$.

To show that we first observe that the restrictions of the indecomposable summands of $\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}$ to $\rho_{\mathbb{F}}^{-1}(\mathcal{M}_{\mathfrak{p}}(v)^{(1)\wedge_0,reg}_{\mathbb{F}})$ are still indecomposable. This follows from the fact that the complement to $\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{(1)reg}_{\mathbb{F}}$ has codimension bigger than 2. Any two $\mathbb{F}^{\times}$-equivariant structures on an indecomposable bundle differ, up to an isomorphism, by a twist with a character. This is because the automorphism group of such a vector bundle is pro-unipotent. This implies the claim in the end of the previous paragraph.

Step 2. So now we can assume that

$$\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}|_{\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}}} \cong \mathcal{P}^{\theta'}_{\mathfrak{p},\mathbb{F}}|_{\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}}},$$

where we consider $\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}$ as a bundle on $\mathcal{M}^{\theta}_{\mathfrak{p}}(v)_{\mathbb{F}}$. We claim that the first self-Ext of these isomorphic bundles vanishes. The variety $\mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}}$ is Cohen-Macaulay by (4) of Corollary 2.4. Since $H^i(\mathcal{M}^{\theta}_{\mathfrak{p}}(v)_{\mathbb{F}}, \mathcal{E}nd(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})) = 0$ for $i > 0$, we see that $\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})$ is a Cohen-Macaulay $\mathbb{F}[\mathcal{M}^{\theta}_{\mathfrak{p},\mathbb{F}}]$-module. Therefore, for a subvariety $Y \subset \mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}}$ of codimension i, we have $H^j_Y(\mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}}, \operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})) = 0$ for $j < i$. Since $\mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}} \setminus \mathcal{M}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}}$ has codimension 3, we use a standard exact sequence for the cohomology with support to see that $H^i_{\mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}} \setminus \mathcal{M}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}}}(\mathcal{M}_{\mathfrak{p}}(v)_{\mathbb{F}}, \operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})) = 0$ for $i < 2$. Therefore, $H^1(\mathcal{M}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}}, \mathcal{E}nd(\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}})) = 0$ and we are done.

Step 3. We have a closed subscheme $\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg}_{\mathbb{F}_q} \subset \mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg}_{\tilde{S}}$. Consider its formal neighborhood $\mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg,\wedge_q}_{\tilde{S}}$. There is a natural morphism $\iota : \mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{reg,\wedge_q}_{\tilde{S}} \to \mathcal{M}^{\theta}_{\mathfrak{p}}(v)^{\wedge_q}_{\tilde{S}}$ of formal schemes. The bundles $\iota^*\tilde{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}}, \iota^*\tilde{\mathcal{P}}^{\theta'}_{\mathfrak{p},\mathbb{F}}$ are isomorphic by the previous step, because both deform $\mathcal{P}^{\theta}_{\mathfrak{p},\mathbb{F}}|_{\mathcal{M}^{\theta}_{\mathfrak{p},\mathbb{F}}(v)^{reg}}$. The induced homomorphism $\operatorname{End}(\tilde{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}}) \to \operatorname{End}(\iota^*\tilde{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}})$ is an isomorphism because both algebras are flat over $R^{\wedge_q}$, complete in the q-adic topology, and modulo the maximal ideal of $R^{\wedge_q}$ this homomorphism coincides with the isomorphism $\operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p}_{\mathbb{F}}}) \to \operatorname{End}(\mathcal{P}^{\theta}_{\mathfrak{p}_{\mathbb{F}}}|_{\mathcal{M}^{\theta}_{\mathfrak{p}_{\mathbb{F}}}(v)^{reg}})$.

By Step 1, $\iota^*\tilde{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}}$ is independent of θ. So $\operatorname{End}(\tilde{\mathcal{P}}^{\theta}_{\mathfrak{p},\mathbb{F}})$ is $\mathbb{G}_m$-equivariantly isomorphic to $\operatorname{End}(\tilde{\mathcal{P}}^{\theta'}_{\mathfrak{p},\mathbb{F}})$. This yields an isomorphism required in this proposition. □

Remark 6.7. The argument in the above proof implies that the bundle $\mathcal{P}^{\theta}_{\mathfrak{p}}|_{\mathcal{M}(v)^{reg}}$ is independent of θ and has zero first self-extensions.

6.4. **Proof of Proposition 6.1.** Let us prove Proposition 6.1. For $\mathcal{P}^{\theta}$ we take the restriction of $\mathcal{P}^{\theta}_{\mathfrak{p}}$ to $\mathcal{M}^{\theta}(v)$. This bundle has no higher self-extensions, this has already been mentioned in Section 6.3. Because of that $\mathrm{End}(\mathcal{P}^\theta) = \mathrm{End}(\mathcal{P}^\theta_{\mathfrak{p}})/(\mathfrak{p})$. So the algebras $\mathrm{End}(\mathcal{P}^\theta)$ for different θ are $\mathbb{C}^\times$-equivariantly identified.

It remains to show that $\mathrm{End}(\mathcal{P}^\theta)$ has finite homological dimension. This is what we do in the remainder of this section. For this algebra to have finite homological dimension we will need to make special choices of λ and of p.

First of all, let us notice that there is $\lambda \in \mathbb{Q}^{Q_0}$ such that the homological dimension of $\mathcal{A}_\lambda(v) \otimes \mathcal{A}_\lambda(v)^{opp}$ is finite. Indeed, for any λ there is $k \in \mathbb{Z}$ such that the abelian localization theorem holds for $\mathcal{A}_{\lambda+k\theta}(v) \otimes \mathcal{A}_{\lambda+k\theta}(v)^{opp}$ on $\mathcal{M}^\theta(v) \times \mathcal{M}^{-\theta}(v)$. It follows that the homological dimension of $\mathcal{A}_{\lambda+k\theta}(v) \otimes \mathcal{A}_{\lambda+k\theta}(v)^{opp}$ is finite.

We claim that for $p \gg 0$, the homological dimension of $\mathcal{A}_\lambda(v)_{\mathbb{F}}$ is finite. This follows from a more general result.

Lemma 6.8. *Let S be a finite localization of $\mathbb{Z}$. Let A be an S-algebra such that $A \otimes_S A^{op}$ is Noetherian. Suppose that for $A_{\mathbb{C}} = \mathbb{C} \otimes_S A$, the homological dimension of $A_{\mathbb{C}} \otimes_{\mathbb{C}} A_{\mathbb{C}}^{opp}$ is finite. Then the algebra $A_{\mathbb{F}} := \mathbb{F} \otimes_S A$ has finite homological dimension for all $p \gg 0$. Moreover, the projective dimension of the regular $A_{\mathbb{F}}$-bimodule is finite.*

Proof. For an algebra $\mathcal{A}$ over a field, the homological dimension is finite provided the projective dimension of the regular bimodule $\mathcal{A}$ is.

Let F be a finitely generated free A-bimodule, M a finitely generated A-bimodule and φ a homomorphism $F \to M$ such that the homomorphism $\varphi_{\mathbb{C}}$ is surjective. Because all bimodules involved are finitely generated, $\varphi_{S'}$ is an epimorphism for a finite localization S' of S. So, using induction, we reduce to showing that if M is a finitely generated A-bimodule such that $M_{\mathrm{Frac}(S)}$ is a projective $A_{\mathrm{Frac}(S)}$-bimodule, then $M_{S'}$ is a projective $A_{S'}$-bimodule for a finite localization S' of S. Fix an epimorphism $\varphi : F \twoheadrightarrow M$ of A-bimodules. Then $\varphi_{\mathrm{Frac}(S)}$ admits a left inverse, ι. Clearly, ι is defined over a finite localization S' of S. It follows that $M_{S'}$ is projective.

Therefore the projective dimension of the regular $A_{S'}$-bimodule is finite. So the projective dimension of $A_{\overline{\mathbb{F}}_p}$ is finite for $p \gg 0$ proving the lemma. $\square$

Since the projective dimension of the regular $\mathcal{A}_\lambda(v)_{\mathbb{F}}$-bimodule is finite, so is the projective dimension of the regular $\mathcal{A}_\lambda(v)_{\mathbb{F}}^{\wedge 0}$-bimodule, because $\mathcal{A}_\lambda(v)_{\mathbb{F}}^{\wedge 0}$ is a flat (left and right) $\mathcal{A}_\lambda(v)_{\mathbb{F}}$-module. It follows that $\mathcal{A}_\lambda(v)_{\mathbb{F}}^{\wedge 0}$ has finite homological dimension. In other words, the homological dimension of $\mathrm{End}(\hat{\mathcal{P}}^\theta_{\mathbb{F}})$ is finite. Because of the contracting $\mathbb{F}^\times$-action, the homological dimension of $\mathrm{End}(\mathcal{P}^\theta_{\mathbb{F}})$ is also finite. The same holds for the deformation $\mathrm{End}(\tilde{\mathcal{P}}^\theta_{\mathbb{F}})$ of $\mathrm{End}(\mathcal{P}^\theta_{\mathbb{F}_q})$ over $\tilde{S}^{\wedge_q}$ (here $\tilde{\mathcal{P}}^\theta_{\mathbb{F}}$ is the specialization of the bundle $\tilde{\mathcal{P}}^\theta_{\mathfrak{p},\mathbb{F}}$ constructed in the beginning of Section 6.3 to the zero parameter). Again, thanks to the contracting action of $\mathbb{G}_m$, the homological dimension of $\mathrm{End}(\mathcal{P}^\theta_{\tilde{S}^{\wedge_q}})$ is finite, and we are done.

6.5. **Proof of Proposition 6.2.** In this section we prove Proposition 6.2 using reduction to characteristic p. We start with reducing finite dimensional representations mod p.

6.5.1. *Reduction of representations mod p.* Let $\lambda \in \mathbb{Q}^{Q_0}$ be such that $\mathcal{A}_\lambda(v) \otimes \mathcal{A}_\lambda(v)^{opp}$ has finite homological dimension. Here we will discuss the reduction of the finite dimensional representations of $\mathcal{A}_\lambda(v)$ modulo p for $p \gg 0$. Let $\mathbb{F}$ still be an algebraically closed field of characteristic p. Note that by Section 6.4, the algebra $\mathcal{A}_\lambda(v)_{\mathbb{F}}$ has finite homological dimension. Let S have the same meaning as in Lemma 6.4.

Let $M \in \mathcal{A}_\lambda(v)\text{-mod}_{fin}$. Pick an S-lattice $M_S \subset M$. If p is invertible in S, then $M_{\mathbb{F}} := \mathbb{F} \otimes_S M_S$ makes sense. It is a standard result that $[M_{\mathbb{F}}] \in K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_{fin})$ depends only on $[M]$ and the map $[M] \mapsto [M_{\mathbb{F}}]$ is linear.

Inside $\mathcal{A}_\lambda(v)_{\mathbb{F}}$ we have a central subalgebra $\mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]$ known as the p-center. Consider the category $\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0$ consisting of all finite dimensional $\mathcal{A}_\lambda(v)_{\mathbb{F}}$-modules supported at $0 \in \mathcal{M}(v)^{(1)}_{\mathbb{F}}$.

Here is the main result of this section.

Proposition 6.9. *The following is true provided $p \gg 0$:*

1. $M_{\mathbb{F}} \in \mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0$.
2. *If M is irreducible, then $M_{\mathbb{F}}$ is irreducible.*
3. *If M^1, M^2 are two non-isomorphic finite dimensional irreducible modules, then $M^1_{\mathbb{F}}$ and $M^2_{\mathbb{F}}$ are non-isomorphic.*

It follows from Proposition 6.9 that we get an injective map $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \hookrightarrow K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0)$. In 6.5.2 we will see that there is an isomorphism $K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ intertwining the injective map above with γ^θ_λ.

Proof of Proposition 6.9. Let $M \in \mathcal{A}_\lambda(v)\text{-mod}_{fin}$ be irreducible and let I be its annihilator in $A := \mathcal{A}_\lambda(v)$ so that $A/I = \mathrm{End}_{\mathbb{C}}(M)$. Then the annihilator of M_S in A_S is $I_S := A_S \cap I$.

We may assume that M is a free finite rank module over S. Then I_S and A_S/I_S are flat over S. Replacing S with a finite algebraic extension we achieve that that $A_S/I_S \xrightarrow{\sim} \mathrm{End}_S(M_S)$. For p sufficiently large, $\mathbb{F}$ is an S-algebra. So, for $I_{\mathbb{F}} := \mathbb{F} \otimes_S I_S$, we have $A_{\mathbb{F}}/I_{\mathbb{F}} \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}}(M_{\mathbb{F}})$. It follows that $M_{\mathbb{F}}$ is irreducible and $I_{\mathbb{F}}$ is the annihilator of $M_{\mathbb{F}}$ in $A_{\mathbb{F}}$. (2) is proved.

Let M^1, M^2 be two non-isomorphic simples. Let I be the intersection of their annihilators in A. Then we can form the ideals $I_S \subset A_S$ and $I_{\mathbb{F}} \subset A_{\mathbb{F}}$ similarly to the above. We will get $A_{\mathbb{F}}/I_{\mathbb{F}} \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}}(M^1_{\mathbb{F}}) \oplus \mathrm{End}_{\mathbb{F}}(M^2_{\mathbb{F}})$. This implies (3).

Finally, let us prove (1). Note that $\dim \mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]/(I_{\mathbb{F}} \cap \mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]) \leqslant (\dim M)^2$ so is bounded with respect to p. On the other hand, it is easy to see that $I_{\mathbb{F}} \cap \mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]$ is a Poisson ideal in $\mathbb{F}[\mathcal{M}(v)^{(1)}_{\mathbb{F}}]$. Since the codimension is bounded with respect to p, we see that the subvariety of $\mathcal{M}(v)^{(1)}_{\mathbb{F}}$ defined by this ideal is the union of points that are symplectic leaves. Since we have a contracting $\mathbb{F}^\times$-action on $\mathcal{M}(v)^{(1)}_{\mathbb{F}}$ we see that the subvariety is actually $\{0\}$. This finishes the proof of (1). $\square$

6.5.2. *Isomorphism* $K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$. Note that $R\Gamma(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}) = \mathcal{A}_\lambda(v)_{\mathbb{F}}$. By the construction, the algebra $\mathcal{A}_\lambda(v)_{\mathbb{F}}$ has finite homological dimension. By [BezKa2, Section 2.2], the functor $R\Gamma$ gives an equivalence $D^b(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod})$. Consider the subcategory $D^b_0(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}) \subset D^b(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod})$ of all complexes whose homology are finite dimensional with generalized p-character 0 and the similarly defined subcategory $D^b_{(\rho^{(1)})^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}) \subset D^b(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod})$. The equivalence $R\Gamma$ restricts to $D^b_{(\rho^{(1)})^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}) \xrightarrow{\sim} D^b_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod})$.

Since $\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}$ splits on $\mathcal{M}^\theta(v)^{\wedge_0}_{\mathbb{F}}$, we further get an equivalence

$$\mathcal{F} : D^b_{(\rho^{(1)})^{-1}(0)}(\mathrm{Coh}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}})) \xrightarrow{\sim} D^b_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod})$$

given by $N \mapsto R\Gamma(\mathcal{P}^\theta_{\mathbb{F}} \otimes N)$. So we get an isomorphism

$$[\mathcal{F}] : K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} K_0(\mathrm{Coh}_{(\rho^{(1)})^{-1}(0)}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}})).$$

The Frobenius push-forward induces the isomorphism

$$[\mathrm{Fr}_*] : K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)_{\mathbb{F}})) \xrightarrow{\sim} K_0(\mathrm{Coh}_{(\rho^{(1)})^{-1}(0)}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}}))$$

(recall that all K_0's we consider are over $\mathbb{C}$, the map $[\mathrm{Fr}_*]$ is not invertible over $\mathbb{Z}$). But $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ is naturally identified with $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)_{\mathbb{F}}))$ since $p \gg 0$.

We get an isomorphism $K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ given by $[\mathrm{Fr}_*]^{-1} \circ [\mathcal{F}]^{-1}$ to be denoted by ι. Set $\iota' = [\mathcal{P}^\theta]\iota$. Our K_0 is a $\mathbb{C}$-vector space, so the multiplication by $[\mathcal{P}^\theta]$ (the class of a vector bundle) is an invertible transformation.

6.5.3. *Completion of proof.* It remains to prove the following lemma.

Lemma 6.10. *We have $\gamma^\theta_\lambda([M]) = \iota'([M_{\mathbb{F}}])$.*

Proof. Note that we still have the degeneration map

$$K_0(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}_{\rho^{-1}(0)}) \to K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)_{\mathbb{F}})), [N] \mapsto [\mathrm{gr}\, N].$$

So we get the map $\gamma^\theta_{\lambda,\mathbb{F}} : K_0(\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}}\text{-mod}_{fin}) \to K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)_{\mathbb{F}}))$. Let us check that $\gamma^\theta_\lambda([M_{\mathbb{C}}]) = \gamma^\theta_{\lambda,\mathbb{F}}([M_{\mathbb{F}}])$. Let $M^i_S := H_i(\mathcal{A}^\theta_\lambda(v)_S \otimes^L_{\mathcal{A}_\lambda(v)_S} M_S)$. By localizing S further, we can achieve that each $\mathrm{gr}\, M^i_S$ is flat over S. Form the base changes $M^i_{\mathbb{C}} = \mathbb{C} \otimes_S M^i_S$ and $M^i_{\mathbb{F}} = \mathbb{F} \otimes_S M^i_S$. We get $[\mathrm{gr}\, M^i] = [\mathrm{gr}\, M^i_{\mathbb{F}}]$. It follows that $\gamma^\theta_\lambda([M_{\mathbb{C}}]) = \gamma^\theta_{\lambda,\mathbb{F}}([M_{\mathbb{F}}])$.

Now take $N \in \mathrm{Coh}_{(\rho^{(1)})^{-1}(0)}(\mathcal{M}^\theta(v)^{(1)}_{\mathbb{F}})$. The corresponding object $M \in D^b_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod})$ is $R\Gamma(\mathcal{P}^\theta_{\mathbb{F}} \otimes N)$. But

$$\mathcal{A}^\theta_\lambda(v)_{\mathbb{F}} \otimes^L_{\mathcal{A}_\lambda(v)_{\mathbb{F}}} M = \mathcal{P}^\theta_{\mathbb{F}} \otimes N. \tag{6.6}$$

The class of the right hand side of (6.6) is $[\mathcal{P}^\theta_{\mathbb{F}}][N]$. To finish the proof we observe that the associated graded of this class is $[\mathcal{P}^\theta_{\mathbb{F}}][\mathrm{gr}\, N]$. $\square$

7. Proof of the lower bound

In this section we prove (I) from the beginning of Section 5 and various related statements.

In Section 7.1 we prove Theorem 5.10 relating the wall-crossing functor to a Rickard functor. In Section 7.2 we study the K-theory of quiver varieties, in particular, we use results of Section 6 to identify the K_0-groups $K_0(\mathrm{Coh}_{\rho^{-1}(0)}\, \mathcal{M}^\theta(v))$ for different generic θ. We show that the Chern character maps are isomorphisms that intertwine these identifications with the identification of homology explained in Section 2.1. In Section 7.3 we prove that the identifications $K_0(\mathrm{Coh}_{\rho^{-1}(0)}\, \mathcal{M}^\theta(v)) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}\, \mathcal{M}^{\theta'}(v))$ intertwine the maps from $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$. This shows, in particular, that $\mathsf{CC} : K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \to L_\omega[\nu]$ is independent of the choice of θ. Finally, in Section 7.4 we equip $\bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}\, \mathcal{M}^\theta(v))$ with an $\mathfrak{a}$-action and modify the degeneration map

$$\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to \bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}\, \mathcal{M}^\theta(v))$$

so that it intertwines $[E_\alpha]$ with e_α and $[F_\alpha]$ with f_α. We deduce Proposition 5.9 (and hence (I)) from here.

7.1. **Wall-crossing vs Rickard complexes.** In this subsection we prove Theorem 5.10. Our proof follows the scheme of construction of E_i, F_i: we use reduction in stages to reduce to what essentially is the case of a quiver with a single vertex and no loops.

We will use the notation of Section 5.2 and of 2.1.3. Recall that we assume that $\theta_k > 0$ for all k and $\lambda_i \in \mathbb{Z}_{\geqslant 0}$. In the proof we will need to deal with various functors that we will now describe.

7.1.1. *Quotient functors.* Consider the quotient functors

$$\pi^{\theta_i}(v) : D_R\text{-mod}^{G,\lambda} \twoheadrightarrow D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}^{\underline{G},\underline{\lambda}},$$
$$\underline{\pi}^{\theta}(v) : D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}^{\underline{G},\underline{\lambda}} \twoheadrightarrow \mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}.$$

so that $\pi^{\theta}_{\lambda}(v) = \underline{\pi}^{\theta}(v) \circ \pi^{\theta_i}(v)$ (below we will omit the subscript). Recall, 2.5.4, that the functor $\pi^{\theta}(v)$ extends to a quotient functor $D^b_{G,\lambda}(D_R\text{-mod}) \to D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$ still denoted by $\pi^{\theta}(v)$. For completely similar reasons, we get quotient functors

$$\pi^{\theta_i}(v) : D^b_{G,\lambda}(D_R\text{-mod}) \twoheadrightarrow D^b_{\underline{G},\underline{\lambda}}(D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}),$$
$$\underline{\pi}^{\theta}(v) : D^b_{\underline{G},\underline{\lambda}}(D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}) \twoheadrightarrow \mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}.$$

such that $\pi^{\theta}(v)$ still decomposes as $\underline{\pi}^{\theta}(v) \circ \pi^{\theta_i}(v)$. Assuming that $\lambda \in \mathfrak{P}^{ISO}$ and $(\lambda, \theta) \in \mathfrak{AL}(v)$, the functor $\pi^{\theta}(v)$ admits a left adjoint functor $L\pi^{\theta}(v)^!$. Under the same assumptions, $\underline{\pi}^{\theta}(v)$ admits a derived left adjoint functor $L\underline{\pi}^{\theta}(v)^!$. Further, possibly after replacing λ with $\lambda + k\theta$ for $k > 0$ we may assume, in addition, that $\pi^{\theta_i}(v)$ admits a derived left adjoint $L\pi^{\theta_i}(v)^!$. So we have $L\pi^{\theta}(v)^! = L\pi^{\theta_i}(v)^! \circ L\pi^{\underline{\theta}}(v)^!$.

7.1.2. *Wall-crossing functors.* Recall that λ' stands for $s_i \bullet^{s_i \bullet v} \lambda$. Consider the functor

$$\mathfrak{WC}^i_{\lambda\to\lambda'} : D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}) \to D^b_{\underline{G},\underline{\lambda'}}(\mathcal{A}^{-\theta_i}_{\lambda'_i}(v)\text{-mod})$$

that is the composition of

$$\mathfrak{WC}_{\lambda_i\to\lambda'_i} : D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}) \to D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{-\theta_i}_{\lambda'_i}(v)\text{-mod})$$

and the equivalence

$$D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{-\theta_i}_{\lambda'_i}(v)\text{-mod}) \xrightarrow{\sim} D^b_{\underline{G},\underline{\lambda'}}(\mathcal{A}^{-\theta_i}_{\lambda'_i}(v)\text{-mod})$$

(an integral change of the twisted equivariance condition).

We have $(\lambda, \theta) \in \mathfrak{AL}(v)$ by our assumptions. Also $(\lambda_i, \theta_i) \in \mathfrak{AL}(v_i)$ because $\lambda_i, \theta_i \geqslant 0$. These two observations imply $(\lambda', s_i\theta) \in \mathfrak{AL}(s_i \bullet v)$ and $(\lambda'_i, -\theta_i) \in \mathfrak{AL}(\tilde{w}_i - v_i)$.

Lemma 7.1.

$$\mathfrak{WC}_{\lambda\to\lambda'} = \underline{\pi}^{s_i\bullet\theta}(v) \circ \mathfrak{WC}^i_{\lambda\to\lambda'} \circ L\underline{\pi}^{\theta}(v)^!. \tag{7.1}$$

Proof. Note that we have the next four equalities

$$\mathfrak{WC}^i_{\lambda\to\lambda'} = \pi^{-\theta_i}(v) \circ (\mathbb{C}_{\lambda'-\lambda} \otimes \bullet) \circ L\pi^{\theta_i}(v)^!,$$
$$\mathfrak{WC}_{\lambda\to\lambda'} = \pi^{s_i\theta}(v) \circ (\mathbb{C}_{\lambda'-\lambda} \otimes \bullet) \circ L\pi^{\theta}(v)^!,$$
$$\pi^{s_i\theta}(v) = \underline{\pi}^{s_i\theta}(v) \circ \pi^{-\theta_i}(v),$$
$$L\pi^{\theta}(v)^! = L\pi^{\theta_i}(v)^! \circ L\underline{\pi}^{\theta}(v)^!.$$

The last two equalities were discussed in 7.1.1. To prove the second one we use Lemma 2.17 and isomorphisms of functors $\pi^0_\lambda(v) \cong \pi^\theta_\lambda(v), \pi^0_{\lambda'}(v) \cong \pi^{s_i\theta}_{\lambda'}(v)$. The first equality is analogous.

These four equalities imply (7.1). $\square$

7.1.3. *LMN isomorphisms.* Tracking the construction of the LMN isomorphisms, see Sections 2.1.3 and 2.2.4, we see that

$$s_{i*} \circ \underline{\pi}^{\sigma_i\theta}(v) = \underline{\pi}^\theta(s_i \bullet v) \circ \tilde{s}_{i*}, \tag{7.2}$$

where we write $\tilde{s}_{i*}$ for the equivalence

$$D^b_{\underline{G},\underline{\lambda}'}(\mathcal{A}^{-\theta_i}_{\lambda_i'}(v)\text{-mod}) \xrightarrow{\sim} D^b_{\underline{G},\underline{\lambda}}(\mathcal{A}^{\theta_i}_{\lambda_i}(s_i \bullet v)\text{-mod})$$

that comes from the quantum LMN isomorphism $\mathcal{A}^{-\theta_i}_{\lambda_i'}(v) \xrightarrow{\sim} \mathcal{A}^{\theta_i}_{\lambda_i}(v)$. Combining (7.1) with (7.2), we get

$$s_{i*} \circ \mathfrak{WC}_{\lambda\to\lambda'} = \underline{\pi}^\theta(s_i \bullet v) \circ (\tilde{s}_{i*} \circ \mathfrak{WC}^i_{\lambda\to\lambda'}) \circ L\underline{\pi}^\theta(v)^!. \tag{7.3}$$

7.1.4. *Rickard complexes.* We consider Rickard complexes that (in a somewhat different framework) were suggested by Chuang and Rouquier, [CR, Section 6]. We will use the version of [CDK, Section 8].

Set $k = v_i, N = \tilde{w}_i$. We want to define an object Θ in the homotopy category of 1-morphisms in $\mathcal{U}(\mathfrak{sl}_2)$ (going from the object $N-2k$ to the object $2k-N$). This will be the complex

$$\Theta^m[-m] \to \Theta^{m-1}[1-m] \to \ldots \to \Theta^1[-1] \to \Theta^0,$$

where $m = \min(k, N-k)$. Here

$$\Theta^i = \mathcal{F}^{(N-k-i)}\mathcal{E}^{(k-i)},$$

and [?] denotes the grading shift in $\mathcal{U}(\mathfrak{sl}_2)$. The differentials in the complex come from adjunctions between $\mathcal{E}, \mathcal{F}$. We note that what is denoted by Θ in [CDK, Section 8] is the cone of $\psi(\Theta)$ (where ψ was introduced in 5.2.2).

Recall that Webster's functors E, F give rise to an action, say α, of the 2-category $\mathcal{U}(\mathfrak{sl}_2)$ on the category

$$\bigoplus_{v_i=0}^{\tilde{w}_i} D^b_{\underline{G},\underline{\lambda}}(\mathcal{D}^\lambda_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}). \tag{7.4}$$

We get an endofunctor $\alpha(\Theta)$ of (7.4). It follows from [CDK, Theorem 8.1] that it is an equivalence.

7.1.5. *Comparison.* Now let us compare the wall-crossing functors and the functors coming from Rickard complexes.

Proposition 7.2. *The functor*

$$\alpha(\Theta) : D^b_{\underline{G},\underline{\lambda}}(D^{\lambda_i}_{\mathrm{Gr}(v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod}) \to D^b_{\underline{G},\underline{\lambda}}(D^{\lambda_i}_{\mathrm{Gr}(\tilde{w}_i - v_i,\tilde{w}_i)} \otimes D_{\underline{R}}\text{-mod})$$

coincides with $\tilde{s}_{i*} \circ \mathfrak{WC}^i_{\lambda\to\lambda'}$.

Proof. The statement will be deduced from a result of [CDK] which provides an isomorphism $I \cong \alpha'(\Theta)$ between two equivalences

$$I,\, \alpha'(\Theta) : D^b(D_{\mathrm{Gr}(k,N)}\text{-mod}) \xrightarrow{\sim} D^b(D_{\mathrm{Gr}(N-k,N)}\text{-mod})$$

(also for the GL_n-equivariant categories). Here I is the Radon transform functor given by the convolution with $j_*(\mathcal{O}_U)[k(N-k)] \in D^b(D_{\mathrm{Gr}(k,N)\times \mathrm{Gr}(N-k,N)}\text{-mod})$, where $U \subset \mathrm{Gr}(k,N) \times \mathrm{Gr}(N-k,N)$ is the open GL_n-orbit and $j : U \hookrightarrow \mathrm{Gr}(k,N) \times \mathrm{Gr}(N-k,N)$ is the open embedding.

Consider the homomorphism

$$\psi' : \mathcal{U}(\mathfrak{sl}_2) \to D^b(D_{\mathrm{Gr}(k,N)\times \mathrm{Gr}(N-k,N)}\text{-mod})$$

that is completely analogous to ψ considered in Section 5.2. The object

$$\psi'(\Theta) \in D^b(D_{\mathrm{Gr}(k,N)\times \mathrm{Gr}(N-k,N)}\text{-mod})$$

is shown in [CDK, Corollary 8.6] to be isomorphic to the complex $j_*(\mathcal{O}_U)[k(N-k)]$ of D-modules, where j is the embedding $U \to \mathrm{Gr}(k,N) \times \mathrm{Gr}(N-k,N)$. The same is true for the GL_n-equivariant derived category.

It remains to show how this statement implies the proposition. Clearly, $\psi_i(\Theta)$ is identified with $\psi(\Theta) \boxtimes \delta_{\overline{R}*}(\mathcal{O})$, where we used the obvious identification $\big(\mathrm{Gr}(v_i, \tilde{w}_i) \times \overline{R}\big) \times \big(\mathrm{Gr}(\tilde{w}_i - v_i, \tilde{w}_i) \times \overline{R}\big) = \mathrm{Gr}(v_i, \tilde{w}_i) \times \mathrm{Gr}(\tilde{w}_i - v_i, \tilde{w}_i) \times \overline{R}^2$. Here $\delta_{\overline{R}} : \overline{R} \to \overline{R}^2$ is the diagonal embedding. Since $\psi(\Theta) \cong j_*(\mathcal{O}_U[k(N-k)]) \boxtimes \delta_{\overline{R}*}(\mathcal{O})$, to complete the proof it is enough to notice that $\tilde{s}_{i*} \circ \mathfrak{WC}^i_{\lambda\to\lambda'}$ is given by the convolution with $j_*(\mathcal{O}_U) \boxtimes \delta_{\overline{R}}(\mathcal{O})$ (in the $\underline{G}$-equivaraint derived category). This is proved similarly to [BB2, Theorem 12]. $\square$

7.1.6. *Completion of proof.* Let us complete the proof of Theorem 5.10. We have the endofunctor Θ_i of $\bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ induced by $\alpha(\Theta)$, so that

$$\Theta_i \circ \pi^{\underline{\theta}}(v) \cong \pi^{\underline{\theta}}(s_i \bullet v) \circ \alpha(\Theta).$$

This implies

$$\Theta_i = \pi^{\underline{\theta}}(s_i \bullet v) \circ \alpha(\Theta) \circ L\pi^{\underline{\theta}}(v)^!. \tag{7.5}$$

Thanks to (7.5) and (7.3), we see that Theorem 5.10 follows from Proposition 7.2.

7.2. **K-theory and cohomology of quiver varieties.** Recall that the homology groups $H_*(\mathcal{M}^\theta(v))$ and the cohomology groups $H^*(\mathcal{M}^\theta(v))$ are independent of θ, see 2.1.8. Note that we have Chern character maps

$$K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \to H_*(\mathcal{M}^\theta(v)), K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \to H^*(\mathcal{M}^\theta(v))$$

The main result of this section is as follows.

Proposition 7.3. *Let θ, θ' be generic stability conditions. Then the Chern character maps $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^?(v))) \to H_*(\mathcal{M}^?(v)), K_0(\mathrm{Coh}(\mathcal{M}^?(v))) \to H^*(\mathcal{M}^?(v))$ are isomorphisms.*

The proof of this proposition occupies the remainder of the section. It is quite indirect but we will prove other useful facts along the way. In particular, we will produce an independent identification of K_0-groups for different θ and then show that the Chern character map intertwines this identification with that of the (co)homology groups.

7.2.1. *Identifications of K_0's.* First, let us produce isomorphisms

$K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta'}(v))), K_0(\operatorname{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\operatorname{Coh}(\mathcal{M}^{\theta'}(v)))$.

Recall that we have tilting generators $\mathcal{P}^?$ on $\mathcal{M}^?(v)$ ($?$ is θ, θ') with naturally identified endomorphism algebras, Proposition 6.1. Set $\tilde{A} := \operatorname{End}(\mathcal{P}^?)^{opp}$. This algebra has a central subalgebra $\mathbb{C}[\mathcal{M}(v)]$ so we can consider the category $\tilde{A}\text{-mod}_0$ of all finite dimensional $\tilde{A}$-modules supported at $0 \in \mathcal{M}(v)$.

By [BezKa2, Section 2.2], the functor $R\operatorname{Hom}(\mathcal{P}^?, \bullet)$ is an equivalence $D^b(\operatorname{Coh}(\mathcal{M}^?(v))) \xrightarrow{\sim} D^b(\tilde{A}\text{-mod})$. This gives an identification $K_0(\operatorname{Coh}(\mathcal{M}^?(v))) \xrightarrow{\sim} K_0(\tilde{A}\text{-mod})$. Moreover, the functor $R\operatorname{Hom}(\mathcal{P}^?, \bullet)$ restricts to an equivalence $D^b_{\rho^{-1}(0)}(\operatorname{Coh}(\mathcal{M}^?(v))) \xrightarrow{\sim} D^b_0(\tilde{A}\text{-mod})$. This gives an identification $K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^?(v))) \xrightarrow{\sim} K_0(\tilde{A}\text{-mod}_0)$. We will use the composite identification $K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta'}(v)))$.

7.2.2. *Equivariant vs non-equivariant K_0-groups.* One application of the derived equivalences $D^b(\operatorname{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} D^b(\tilde{A}\text{-mod}), D^b_{\rho^{-1}(0)}(\operatorname{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} D^b_0(\tilde{A}\text{-mod})$ is a comparison of equivariant and non-equivariant K-theories of $\mathcal{M}^\theta(v)$. Namely, let T be a torus acting on $\mathcal{M}^\theta(v)$. Let $R(T)$ denote the representation ring of T (over $\mathbb{C}$).

Lemma 7.4. *We have isomorphisms* $K_0(\operatorname{Coh}(\mathcal{M}^\theta(v))) \otimes R(T) \cong K_0(\operatorname{Coh}^T(\mathcal{M}^\theta(v)))$ *and* $K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \otimes R(T) \cong K_0(\operatorname{Coh}^T_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$.

Proof. The bundle $\mathcal{P}^\theta$ is T-equivariant because it is tilting, compare with the beginning of Section 6.3. So $R\operatorname{Hom}(\mathcal{P}^\theta, \bullet)$ gives equivalences $D^b(\operatorname{Coh}^T(\mathcal{M}^\theta(v))) \xrightarrow{\sim} D^b(\tilde{A}\text{-mod}^T)$ and $D^b_{\rho^{-1}(0)}(\operatorname{Coh}^T(\mathcal{M}^\theta(v))) \xrightarrow{\sim} D^b_0(\tilde{A}\text{-mod}^T)$. So we need to check that

$$K_0(\tilde{A}\text{-mod}^T) \cong K_0(\tilde{A}\text{-mod}) \otimes R(T), \tag{7.6}$$

$$K_0(\tilde{A}\text{-mod}^T_0) \cong K_0(\tilde{A}\text{-mod}_0) \otimes R(T) \tag{7.7}$$

Let us prove (7.6). A basis in $K_0(\tilde{A}\text{-mod})$ is given by the classes of the indecomposable projective modules (there are finitely many of those, they correspond to the number of isomorphism classes of the indecomposable direct summands of $\mathcal{P}^\theta$). So a basis in $K_0(\tilde{A}\text{-mod}^T)$ is given by the classes of T-equivariant lifts of the indecomposable projectives. Each indecomposable projective admits a graded lift, unique up to a twist with a character of T. (7.6) follows.

(7.7) follows similarly. Here we consider the simple modules instead of the projectives. There are finitely many of those because the algebra $\tilde{A}$ is finite over $\mathbb{C}[\mathcal{M}(v)]$. $\square$

7.2.3. *Dimensions of* $K_0(\operatorname{Coh}(\mathcal{M}^\theta(v))), K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ *coincide.* We will use Lemma 7.4 to prove the following result.

Lemma 7.5. *We have* $\dim K_0(\operatorname{Coh}(\mathcal{M}^\theta(v))) = \dim K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$.

Proof. Let T be the contracting torus acting on $\mathcal{M}^\theta(v)$, the action is induced from the dilation action on T^*R. Consider the localized K_0-groups $K_0(\operatorname{Coh}^T(\mathcal{M}^\theta(v)))_{loc} := \operatorname{Frac}(R(T)) \otimes_{R(T)} K_0(\operatorname{Coh}^T(\mathcal{M}^\theta(v))), K_0(\operatorname{Coh}^T_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))_{loc}$. Thanks to Lemma 7.4, it is enough to check that

$$\dim K_0(\operatorname{Coh}^T(\mathcal{M}^\theta(v)))_{loc} = \dim K_0(\operatorname{Coh}^T_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))_{loc}.$$

Note that $\mathcal{M}^\theta(v)^T \subset \rho^{-1}(0)$. By the equivariant localization in K-theory, we have $K_0(\operatorname{Coh}^T(\mathcal{M}^\theta(v)))_{loc} \cong K_0(\operatorname{Coh}(\mathcal{M}^\theta(v)^T)) \otimes \operatorname{Frac}(R(T)) \cong K_0(\operatorname{Coh}^T_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))_{loc}$. $\square$

7.2.4. *Non-degeneracy of pairing* $K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \times K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \to \mathbb{C}$. The assignment

$$(M,N) \mapsto \sum_{i=0}^{\infty}(-1)^i \dim \mathrm{Ext}^i(M,N), M \in \mathrm{Coh}(\mathcal{M}^\theta(v)), N \in \mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))$$

descends to a pairing

$$\langle\cdot,\cdot\rangle : K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \times K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \to \mathbb{C}. \tag{7.8}$$

Proposition 7.6. *The pairing $\langle\cdot,\cdot\rangle$ is non-degenerate.*

Proof. By Lemma 7.5, it is enough to show that the left radical of the pairing is trivial. The isomorphisms $K_0(\tilde{A}\text{-mod}) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))), K_0(\tilde{A}\text{-mod}_0) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ come from the derived equivalence $D^b(\tilde{A}\text{-mod}) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{M}^\theta(v)))$. Therefore the pairing (7.8) gets intertwined with the similarly defined pairing

$$K_0(\tilde{A}\text{-mod}) \times K_0(\tilde{A}\text{-mod}_0) \to \mathbb{C}$$

also to be denoted by $\langle\cdot,\cdot\rangle$. So we need to show that the right radical of the latter is zero.

Recall that the algebra $\tilde{A}$ is realized as the (opposite of the) endomorphism algebra of the $\mathbb{C}^\times$-equivariant (with respect to the contracting action) vector bundle $\mathcal{P}^\theta$ on $\mathcal{M}^\theta(v)$. We can assume that every indecomposable occurs in $\mathcal{P}^\theta$ with multiplicity 1. Let I be the labeling set of the indecomposables, let $\mathcal{P}_i, i \in I$, be the indecomposable corresponding to i, and let $e_i, i \in I$, be the primitive idempotent in $\tilde{A}$ corresponding to i.

Let $\mathfrak{m}$ denote the maximal ideal of 0 in $\mathbb{C}[\mathcal{M}(v)]$. We claim that $e_i, i \in I$, span a maximal commutative subalgebra in the quotient B of $\tilde{A}/\tilde{A}\mathfrak{m}$ modulo the radical. Note that B (the direct sum of several matrix algebras) is acted on $\mathbb{C}^\times$ and all elements e_i are invariant. If they do not span a maximal commutative subalgebra in B, then there is an indecomposable $\mathbb{C}^\times$-invariant idempotent e in B different from any of e_i's and commuting with the e_i's. This idempotent lifts to a $\mathbb{C}^\times$-invariant idempotent the completion $\tilde{A}^{\wedge_0}$ again commuting with the e_i's. But the $\mathbb{C}^\times$-invariant part of this completion is contained in $\tilde{A}$. However, from the construction of the e_i's, there is no idempotent in $\tilde{A}$ commuting with all of them.

We conclude that we have a partition of I labelled by the irreducible B-modules (that are precisely the irreducible modules in $\tilde{A}\text{-mod}_0$) with the following property: i lies in the part labelled by N if and only if $\langle[\tilde{A}e_i], N\rangle > 0$ (in which case it is 1). This shows that the right radical of $\langle\cdot,\cdot\rangle$ is zero. $\square$

7.2.5. *Bijectivity of Chern character maps.* Now we are ready to prove Proposition 7.3.

Proposition 7.7. *The Chern character maps*

$$\mathsf{ch} : K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \to H^*(\mathcal{M}^\theta(v)), K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \to H_*(\mathcal{M}^\theta(v))$$

are isomorphisms.

Proof. It is a standard consequence of the Riemann-Roch theorem that $\langle[M],[N]\rangle = (\mathsf{Td}\cdot\mathsf{ch}([M]), \mathsf{ch}([N]))$, where Td is the Todd class, an invertible element in $H^*(\mathcal{M}^\theta(v))$, and in the right hand side we have the standard pairing between the homology and the cohomology. By [Ka1, Corollary 1.10], $\mathsf{ch} : K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \to H^*(\mathcal{M}^\theta(v))$ is surjective. The pairing $\langle\cdot,\cdot\rangle$ is nondegenerate by Proposition 7.6. We deduce that $\mathsf{ch} : K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} H^*(\mathcal{M}^\theta(v))$. Now since $(\cdot,\cdot)$ is also nondegenerate, we deduce that $\mathsf{ch} : K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} H_*(\mathcal{M}^\theta(v))$. $\square$

7.2.6. *Alternative characterization of* $K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^{\theta'}(v)))$. We are going to give an alternative characterization of the isomorphism and use it to show that the Chern character maps intertwine the identification of the K_0-groups with that of cohomology groups.

Note that $\tilde{A}$ is a graded (with respect to the contracting $\mathbb{C}^\times$-action) algebra of finite homological dimension. Then we have the following classical result.

Lemma 7.8. *Let $\tilde{\mathcal{A}}$ be a filtered deformation of $\tilde{A}$. Then the degeneration map $K_0(\tilde{\mathcal{A}}\text{-mod}) \to K_0(\tilde{A}\text{-mod}), [M] \mapsto [\mathrm{gr}\, M]$, is an isomorphism. The inverse sends the class $[\tilde{A}e_i]$ of an indecomposable projective module to the class of its unique deformation.*

Now let us consider the $\mathbb{C}[\mathfrak{p}]$-algebra $\tilde{\mathcal{A}}_\mathfrak{p} = \mathrm{End}(\mathcal{P}^\theta_\mathfrak{p})^{opp}$ whose specialization at $0 \in \mathfrak{p}$ coincides with $\tilde{A}$. Let $\tilde{\mathcal{A}}_p$ denote the specialization of $\tilde{\mathcal{A}}_\mathfrak{p}$ to $p \in \mathfrak{p}$ so that $\mathrm{gr}\, \tilde{\mathcal{A}}_p = \tilde{A}$. For p generic, we have an equivalence $R\,\mathrm{Hom}(\mathcal{P}^\theta_p, \bullet) : D^b(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}_p\text{-mod})$ that is independent of θ because $\mathcal{P}^\theta_p$ is. We also have the degeneration map $K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \to K_0(\mathrm{Coh}(\mathcal{M}^\theta(v)))$.

Lemma 7.9. *The following diagram is commutative.*

$$\begin{array}{ccc} K_0(\tilde{\mathcal{A}}_p\text{-mod}) & \longrightarrow & K_0(\tilde{A}\text{-mod}) \\ \downarrow & & \downarrow \\ K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) & \longrightarrow & K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \end{array}$$

In particular, $K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^\theta(v)))$.

Proof. Note that $\mathcal{P}^\theta = \mathrm{gr}\,\mathcal{P}^\theta_p$ (here we view $\mathcal{P}^\theta_p$ as a filtered sheaf on $\mathcal{M}^\theta(v)$). Now pick $M \in \mathbb{C}[\mathcal{M}_p(v)]\text{-mod}$ and equip it with a good filtration so that we can view it as a sheaf on $\mathcal{M}^\theta(v)$ with coherent associated graded. It follows that we have the following equality in $K_0(\tilde{A}\text{-mod})$:

$$\sum_i (-1)^i [\mathrm{Ext}^i(\mathcal{P}^\theta, \mathrm{gr}\, M)] = \sum_i (-1)^i [\mathrm{gr}\,\mathrm{Ext}^i(\mathcal{P}^\theta_p, M)]$$

The left hand side is the image of $[M]$ under

$$K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \to K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \to K_0(\tilde{A}\text{-mod}),$$

while the right hand side is the image of $[M]$ under

$$K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \to K_0(\tilde{\mathcal{A}}_p\text{-mod}) \to K_0(\tilde{A}\text{-mod}).$$

This finishes the proof. □

Corollary 7.10. *The following is true:*

1. *The identification* $K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^{\theta'}(v)))$ *from 7.2.1 is the composition* $K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^{\theta'}(v)))$.
2. *The Chern character maps intertwine the identifications* $K_0(\mathrm{Coh}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^{\theta'}(v)))$ *and* $H^*(\mathcal{M}^\theta(v)) \xrightarrow{\sim} H^*(\mathcal{M}^{\theta'}(v))$.

Proof. (1) is a direct corollary of Lemma 7.9. (2) reduces to checking that the Chern character maps intertwine the degeneration maps $K_0(\mathbb{C}[\mathcal{M}_p(v)]\text{-mod}) \xrightarrow{\sim} K_0(\mathrm{Coh}(\mathcal{M}^\theta(v)))$

and $H^*(\mathcal{M}_p(v)) \xrightarrow{\sim} H^*(\mathcal{M}^\theta(v))$, which is a standard property of the Chern character maps (Chern characters commute with specialization). □

This finishes the proof of Proposition 7.3.

7.3. **$\mathfrak{WC}$ vs degeneration to $K_0(\mathrm{Coh})$.** In the previous section we have produced the identification $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta'}(v)))$ for different generic θ, θ'.

Let λ be such that $\mathcal{A}_\lambda(v)$ has finite homological dimension. Then we have an identification

$$K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin}) \xrightarrow{\sim} K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$$

and the degeneration map

$$K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))).$$

Proposition 7.11. *The identification $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v))) \xrightarrow{\sim} K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta'}(v)))$ intertwines the maps from $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$.*

Note that Propositions 7.7, 7.11 imply Proposition 5.11.

The proof of Proposition 7.11 occupies the rest of the section.

7.3.1. *Algebras $\tilde{\mathcal{A}}_\lambda$.* Consider the vector bundle $\mathcal{P}^\theta_\mathfrak{p}$ on $\mathcal{M}^\theta_\mathfrak{p}(v)$. It has trivial self-extensions and therefore quantizes to a left $\mathcal{A}^\theta_\mathfrak{P}(v)$-module to be denoted by $\mathcal{P}^\theta_\mathfrak{P}$. Set $\tilde{\mathcal{A}}_\mathfrak{P}(v) := \mathrm{End}(\mathcal{P}^\theta_\mathfrak{P})^{opp}$. This is a filtered $\mathbb{C}[\mathfrak{P}]$-algebra with $\operatorname{gr}\tilde{\mathcal{A}}_\mathfrak{P}(v) = \tilde{A}_\mathfrak{p}$.

Lemma 7.12. *$\tilde{\mathcal{A}}_\mathfrak{P}(v)$ does not depend on θ.*

Proof. Indeed, $\mathcal{P}^\theta_\mathfrak{P}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}}$ is a filtered deformation of $\mathcal{P}^\theta_\mathfrak{p}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}}$, unique because the latter bundle has zero 1st self-extensions (see Remark 6.7). By the same remark, $\mathcal{P}^\theta_\mathfrak{p}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}} = \mathcal{P}^{\theta'}_\mathfrak{p}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}}$. So $\mathcal{P}^\theta_\mathfrak{P}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}} = \mathcal{P}^{\theta'}_\mathfrak{P}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}}$. By the same arguments as in Step 1 of Proposition 6.6 and in the proof of Proposition 3.3, we see that $\mathrm{End}(\mathcal{P}^\theta_\mathfrak{P}) = \mathrm{End}(\mathcal{P}^\theta_\mathfrak{P}|_{\mathcal{M}_\mathfrak{p}(v)^{reg}})$. This finishes the proof. □

7.3.2. *Equivalence $D^b(\mathcal{A}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}_\lambda(v)\text{-mod})$.* Let $\mathcal{P}^\theta_\lambda, \tilde{\mathcal{A}}_\lambda(v)$ be the specializations of $\mathcal{P}^\theta_\mathfrak{P}, \tilde{\mathcal{A}}_\mathfrak{P}(v)$ at $\lambda \in \mathfrak{P}$. So we have a functor $R\operatorname{Hom}(\mathcal{P}^\theta_\lambda, \bullet) : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to D^b(\tilde{\mathcal{A}}_\lambda(v)\text{-mod})$. Arguing as in [GL, Section 5], we use that $\operatorname{gr}\mathcal{P}^\theta_\lambda = \mathcal{P}^\theta$ and the claim that $R\operatorname{Hom}(\mathcal{P}^\theta, \bullet)$ is an equivalence to deduce that $R\operatorname{Hom}(\mathcal{P}^\theta_\lambda, \bullet)$ is an equivalence. By the construction, the functor $R\operatorname{Hom}(\mathcal{P}^\theta_\lambda, \bullet)$ restricts to an equivalence $D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to D^b_{fin}(\tilde{\mathcal{A}}_\lambda(v)\text{-mod})$.

The composition $R\operatorname{Hom}(\mathcal{P}^\theta_\lambda, L\operatorname{Loc}^\theta_\lambda(\bullet))$ is an equivalence $D^b(\mathcal{A}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b(\tilde{\mathcal{A}}_\lambda(v)\text{-mod})$ to be denoted by κ. This equivalence κ restricts to $D^b_{fin}(\mathcal{A}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b_{fin}(\tilde{\mathcal{A}}_\lambda(v)\text{-mod})$. Note that the equivalence κ is given by $R\operatorname{Hom}_{\mathcal{A}_\lambda(v)}(\mathcal{B}^{(\theta)}_\lambda, \bullet)$, where $\mathcal{B}^{(\theta)}_\lambda$ is an object in $D^b(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)\text{-bimod})$ defined as $R\Gamma((\mathcal{P}^\theta_\lambda)^*)$.

Below we will prove that, for $M \in \mathcal{A}_\lambda(v)\text{-mod}_{fin}$, the class $[\kappa(M)] \in K_0(\tilde{\mathcal{A}}_\lambda(v)\text{-mod}_{fin})$ is independent of θ and use this independence to prove Proposition 7.11. The first step is to introduce HC bimodules over $\mathcal{A}_\lambda(v)$, $\tilde{\mathcal{A}}_\lambda(v)$.

7.3.3. *Harish-Chandra $\mathcal{A}_\lambda(v)$-$\tilde{\mathcal{A}}_\lambda(v)$-bimodules.* In the proof below we will need the notions of Harish-Chandra $\mathcal{A}_\lambda(v)$-$\tilde{\mathcal{A}}_\lambda(v)$ and $\mathcal{A}_{\mathfrak{P}}(v)$-$\tilde{\mathcal{A}}_{\mathfrak{P}}(v)$-bimodules.

Recall that $\mathbb{C}[\mathcal{M}(v)]$ sits as a central subalgebra in $\tilde{A}$. We say that a $\mathcal{A}_\lambda(v)$-$\tilde{\mathcal{A}}_\lambda(v)$-bimodule $\mathcal{B}$ is HC if it admits a *good filtration*, i.e., a bimodule filtration such that $\operatorname{gr}\mathcal{B}$ is a finitely generated $\mathbb{C}[\mathcal{M}(v)]$-module (meaning, in particular, that the left action of $\mathbb{C}[\mathcal{M}(v)]$ coincides with the right action). In particular, $H^j(\mathcal{B}^{(\theta)}_\lambda)$ is HC, compare with [BPW, Theorem 6.5].

The argument of [L12, Section 4.3] generalizes to HC $\mathcal{A}_\lambda(v)$-$\tilde{\mathcal{A}}_\lambda(v)$-bimodules so we see that any HC bimodule still has finite length.

Now consider $\mathcal{A}_{\mathfrak{P}}(v)$-$\tilde{\mathcal{A}}_{\mathfrak{P}}(v)$-bimodules. We filter the algebras $\mathcal{A}_{\mathfrak{P}}(v), \tilde{\mathcal{A}}_{\mathfrak{P}}(v)$ as in the proof of Lemma 3.5 so that $\mathbb{C}[\mathfrak{P}]$ is in degree 0. Then we can define HC bimodules the same way as in the previous paragraph. We can define HC bimodules over $\mathcal{A}_S(v) := S\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{A}_{\mathfrak{P}}(v), \tilde{\mathcal{A}}_S(v)$ for any Noetherian $\mathbb{C}[\mathfrak{P}]$-algebra S. We will be mostly interested in $S=\mathbb{C}(\mathfrak{p})$ for the reasons explained below.

Let us give an example of a HC $\mathcal{A}_{\mathfrak{P}}(v)$-$\tilde{\mathcal{A}}_{\mathfrak{P}}(v)$-bimodule. Set $\mathcal{B}^{(\theta)}_{\mathfrak{P}} := R\Gamma((\mathcal{P}^\theta_{\mathfrak{P}})^*)$. So we get $\mathcal{A}_{\mathfrak{P}}(v)$-$\tilde{\mathcal{A}}_{\mathfrak{P}}(v)$-bimodules $H^j(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$.

Lemma 7.13. *All $H^j(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$ are HC bimodules.*

Proof. Note that the sheaf $\mathcal{P}^\theta_{\mathfrak{P}}$ acquires a filtration similar to those on $\tilde{\mathcal{A}}_{\mathfrak{P}}(v), \mathcal{A}_{\mathfrak{P}}(v)$ we currently consider. We have $\operatorname{gr}\mathcal{P}^\theta_{\mathfrak{P}}=\mathbb{C}[\mathfrak{P}]\otimes\mathcal{P}^\theta$. Now we argue as in the proof of [BPW, Theorem 6.5] and complete the proof of the lemma. □

We will be interested in the K_0 groups of categories of HC bimodules. First of all, note that for any $\mathcal{B}\in D^b_{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)\text{-bimod})$, the functor $R\operatorname{Hom}(\mathcal{B},\bullet)$ restricts to $D^b_{fin}(\mathcal{A}_\lambda(v)\text{-mod})\to D^b_{fin}(\tilde{\mathcal{A}}_\lambda(v))$. This gives rise to a bilinear map

$$K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)))\otimes K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})\to K_0(\tilde{\mathcal{A}}_\lambda(v)\text{-mod}_{fin}).$$

Now let us construct the specialization map

$$K_0(\mathrm{HC}(\mathcal{A}_{\mathbb{C}(\mathfrak{P})}(v)\text{-}\tilde{\mathcal{A}}_{\mathbb{C}(\mathfrak{P})}(v)))\to K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v))).$$

Pick $\mathcal{B}_{\mathbb{C}(\mathfrak{P})}\in \mathrm{HC}(\mathcal{A}_{\mathbb{C}(\mathfrak{P})}(v)\text{-}\tilde{\mathcal{A}}_{\mathbb{C}(\mathfrak{P})}(v))$. We can find a $\mathbb{C}[\mathfrak{P}]$-lattice $\mathcal{B}_{\mathbb{C}[\mathfrak{P}]}$ that is a HC bimodule. It is a standard fact that the K_0-class $[\mathbb{C}_\lambda\otimes^L_{\mathbb{C}[\mathfrak{P}]}\mathcal{B}_{\mathfrak{P}}]\in K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)))$ depends only on $[\mathcal{B}_{\mathbb{C}(\mathfrak{P})}]$. This gives a required map.

For later applications let us note that we also have a well-defined map

$$K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)))\otimes K_0(\mathcal{A}_\lambda(v)\text{-mod})\to K_0(\tilde{\mathcal{A}}_\lambda(v)\text{-mod}).$$

7.3.4. *Independence of $[\mathcal{B}^{(\theta)}_\lambda]$ of θ.* Here our goal is to check that the class of $\mathcal{B}^{(\theta)}_\lambda$ in $K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)))$ is independent of θ. Thanks to the discussion in 7.3.3, this will show that the class $[R\operatorname{Hom}(\mathcal{B}^{(\theta)}_\lambda, M)]$ is independent of θ.

Consider $\mathcal{B}^{(\theta)}_{\mathfrak{P}}\in D^b(\mathcal{A}_{\mathfrak{P}}(v)\text{-}\tilde{\mathcal{A}}_{\mathfrak{P}}(v)\text{-bimod})$, by Lemma 7.13, this is an object with Harish-Chandra cohomology. Note that $\mathcal{B}^{(\theta)}_\lambda=\mathbb{C}_\lambda\otimes^L_{\mathbb{C}[\mathfrak{P}]}\mathcal{B}^{(\theta)}_{\mathfrak{P}}$. So $[\mathcal{B}^{(\theta)}_\lambda]$ is the image of $[\mathbb{C}(\mathfrak{P})\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{B}^{(\theta)}_{\mathfrak{P}}]$ in $K_0(\mathrm{HC}(\mathcal{A}_\lambda(v)\text{-}\tilde{\mathcal{A}}_\lambda(v)))$. The following lemma shows that the bimodule $\mathbb{C}(\mathfrak{P})\otimes_{\mathbb{C}[\mathfrak{P}]}\mathcal{B}^{(\theta)}_{\mathfrak{P}}$ itself is independent of θ.

Lemma 7.14. *The $\mathcal{A}_{\mathfrak{P}}(v)$-$\tilde{\mathcal{A}}_{\mathfrak{P}}(v)$-bimodule $H^0(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$ is independent of θ. All higher cohomology are torsion over $\mathfrak{P}$.*

Proof. The claim about H^0 follows from the fact that $\mathcal{P}^{\theta}_{\mathfrak{P}}|_{\mathcal{M}_{\mathfrak{p}}(v)^{reg}}$ is independent of θ, see the proof of Lemma 7.12. Let us prove that the higher cohomology are torsion. Note that $H^i(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$ is a finitely generated $\mathcal{A}_{\mathfrak{P}}(v)$-module. By Lemma 3.5, $\mathrm{Supp}_{\mathfrak{P}}(H^i(\mathcal{B}^{(\theta)}_{\mathfrak{P}}))$ is a constructibe subset of $\mathfrak{P}$. If $H^i(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$ is not torsion, then $\mathrm{Supp}_{\mathfrak{P}}(H^i(\mathcal{B}^{(\theta)}_{\mathfrak{P}}))$ contains a principal Zariski open subset $\mathfrak{P}^0$. We may assume that all $\mathbb{C}[\mathfrak{P}^0]\otimes_{\mathbb{C}[\mathfrak{P}]} H^j(\mathcal{B}^{(\theta)}_{\mathfrak{P}})$ are free over $\mathbb{C}[\mathfrak{P}^0]$. It follows that for $\lambda\in\mathfrak{P}^0$, we have $H^j(\mathcal{B}^{(\theta)}_{\lambda})=H^j(\mathcal{B}^{(\theta)}_{\mathfrak{P}})_{\lambda}$. On the other hand, $H^j(\mathcal{B}^{(\theta)}_{\lambda})=0$ for $j>0$ if $(\lambda,\theta)\in\mathfrak{AL}(v)$. By Proposition 4.2, the set of $\lambda\in\mathfrak{P}$ such that $(\lambda,\theta)\in\mathfrak{AL}(v)$ is Zariski dense. We arrive at a contradiction with $\mathfrak{P}^0\subset\mathrm{Supp}_{\mathfrak{P}}(H^i(\mathcal{B}^{(\theta)}_{\mathfrak{P}}))$ that finishes the proof. □

7.3.5. *Completion of the proof.* Now we know that the class of $[\kappa(M)]\in K_0(\tilde{\mathcal{A}}_{\lambda}(v)\text{-mod}_{fin})$ depends only on $[M]\in K_0(\mathcal{A}_{\lambda}(v)\text{-mod}_{fin})$.

Similarly to the proof of Lemma 7.9, we see that the following diagram commutes. The horizontal arrows are the degeneration maps, and the vertical ones come from derived equivalences.

$$\begin{tikzcd}
K_0(\tilde{\mathcal{A}}_{\lambda}(v)\text{-mod}_{fin}) \arrow[r] \arrow[d] & K_0(\tilde{A}\text{-mod}_0) \arrow[d] \\
K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)}) \arrow[r] & K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))
\end{tikzcd}$$

So the class of $[L\,\mathrm{Loc}^{\theta}_{\lambda}(M)]$ is also independent of θ. This completes the proof of Proposition 7.11.

Remark 7.15. For similar reasons, $[L\,\mathrm{Loc}^{\theta}_{\lambda}(M)]\in K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$ is independent of θ for $M\in\mathcal{A}_{\lambda}(v)\text{-mod}$.

7.4. **Actions on K_0.** In this section we will produce an action of the Lie algebra $\mathfrak{a}$ on $\bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))$ and show that, after a suitable modification, degeneration maps $\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)}\to K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))$ intertwine $[E_\alpha]$ with e_α and $[F_\alpha]$ with f_α. This will imply Proposition 5.9.

Let us produce an identification $K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})\xrightarrow{\sim}K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))$. Note that the element $[O(\chi)]\in K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))$ is unipotent for any $\chi\in\mathbb{Z}^{Q_0}$. So, as the class in K_0, $[\mathcal{O}(\lambda)]$ makes sense for $\underline{\text{any}}$ $\lambda\in\mathbb{C}^{Q_0}$. We identify $K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$ with $K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))$ by $[M]\mapsto[\mathcal{O}(\varrho(v)-\lambda)\otimes\mathrm{gr}\,M]$. We note that this identification is independent of the choice of the orientation on Q. Also the classes of shift functors $\mathcal{T}_{\lambda,\chi}$ are sent to the identity. We modify the degeneration maps $K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)})\to K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))$ in a similar fashion so that our identifications intertwine the natural maps $K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod}_{\rho^{-1}(0)})\xrightarrow{\sim}K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})^*$ and $K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))\xrightarrow{\sim}K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))^*$.

Recall that the algebra $\mathfrak{g}(Q)$ acts on $\bigoplus_v D^b(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$ for $\lambda\in\mathbb{Z}^{Q_0}$ and $\theta_i>0$ for all $i>0$, see [We1]. Now we define an action of $\mathfrak{g}(Q)$ on $K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))$ from the identification $\bigoplus_v K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))\cong\bigoplus_v K_0(\mathcal{A}^{\theta}_{\lambda}(v)\text{-mod})$. And then we define a $\mathfrak{g}(Q)$-action on $\bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^{\theta}(v)))=\bigoplus_v K_0(\mathrm{Coh}(\mathcal{M}^{\theta}(v)))^*$ so that e_i,f_i act by f_i^*,e_i^*.

Proposition 7.16. *There is a choice of Serre generators $e_\alpha, f_\alpha \in \mathfrak{a}, \alpha \in \Pi^\theta$, such that the modified degeneration map $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to \bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ intertwines $[E_\alpha]$ with e_α and $[F_\alpha]$ with f_α.*

Proof. First note that $[E_i], [F_i]$ are independent of λ, this can be deduced from Remark 5.5 and our identification of K_0's. From Proposition 7.11 and our identifications of K_0's it follows that the wall-crossing functors are the identity on the K_0 level. By Theorem 5.10, we have $[\Theta_i] = [s_{i*}]$. Note that $[\Theta_i]$ on $K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ equals s_i, where s_i stands for the action of image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ on $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})$. Now, for $\alpha \in \Pi^\theta$, set $e_\alpha = \sigma(e_i), f_\alpha = \sigma(f_i)$ if $\alpha = \sigma\alpha^i$. By the definition of the functors E_α, F_α, we see that $[F_\alpha] = f_\alpha, [E_\alpha] = e_\alpha$ on $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})$. But we have an isomorphism $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})^* \xrightarrow{\sim} \bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ that intertwines the natural map $K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})^*$ with the modified degeneration map $K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to \bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$. The isomorphism intertwines e_α with f^*_α and f_α with e^*_α. On the other hand, the map $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to \bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod})^*$ intertwines $[E_\alpha]$ with $[F_\alpha]^*$ and $[F_\alpha]$ with $[E_\alpha]^*$ because of the adjointness properties of the functors E_α, F_α. So it follows that the modified degeneration map $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to \bigoplus_v K_0(\mathrm{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$ intertwines $[E_\alpha]$ with e_α and $[F_\alpha]$ with f_α. This finishes the proof. □

Proof of Proposition 5.9. The equality $[\Theta_i] = [s_{i*}]$ from the proof of Proposition 7.16 implies $\mathsf{CC}(\Theta_i) = s_i$. Using this and (2) of Lemma 5.7 we deduce $\mathsf{CC}(E_\alpha) = e_\alpha, \mathsf{CC}(F_\alpha) = f_\alpha$ similarly to the proof of Proposition 7.16. □

8. Long wall-crossing and dimension of support

In this section we prove Proposition 5.13. A key ingredient is a comparison of the long wall-crossing functor to the homological duality functor. Then we mention some further properties of long wall-crossing bimodules and finish the proof of (2) of Proposition 4.3.

8.1. **Homological duality.** By homological duality functors we mean the functors

$$D : D^b(\mathcal{A}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda(v)^{opp}\text{-mod})^{opp},$$
$$D^{-\theta} : D^b(\mathcal{A}^{-\theta}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}^{-\theta}_\lambda(v)^{opp}\text{-mod})^{opp}$$

given by

$$\mathrm{RHom}_{\mathcal{A}_\lambda(v)}(\bullet, \mathcal{A}_\lambda(v))[-N], \mathrm{R}\,\mathcal{H}om_{\mathcal{A}^{-\theta}_\lambda(v)}(\bullet, \mathcal{A}^{-\theta}_\lambda(v))[-N],$$

where $N := \frac{1}{2}\dim \mathcal{M}^\theta(v)$. Since $R\Gamma^{-\theta}_\lambda$ is a derived equivalence mapping $\mathcal{A}^\theta_\lambda(v)$ to $\mathcal{A}_\lambda(v)$, the following diagram is commutative.

$$\begin{tikzcd}
D^b(\mathcal{A}^{-\theta}_\lambda(v)\text{-mod}) \arrow[r, "D^{-\theta}"] \arrow[d, "R\Gamma^{-\theta}_\lambda"] & D^b(\mathcal{A}^{-\theta}_\lambda(v)^{opp}\text{-mod})^{opp} \arrow[d, "R\Gamma^{-\theta}_{\lambda,opp}"] \\
D^b(\mathcal{A}_\lambda(v)\text{-mod}) \arrow[r, "D"] & D^b(\mathcal{A}_\lambda(v)^{opp}\text{-mod})^{opp}
\end{tikzcd}$$

Here $R\Gamma^{-\theta}_{\lambda,opp}$ stands for the derived global section functor for right modules.

Lemma 8.1. *The functor $D^{-\theta}$ gives a contravariant abelian equivalence between the categories of holonomic $\mathcal{A}_\lambda^{-\theta}(v)$- and $\mathcal{A}_\lambda^{-\theta}(v)^{opp}$-modules.*

Proof. The claim boils down to checking that if $\mathcal{N}$ is a holonomic $\mathcal{A}_\lambda^{-\theta}(v)$-module, then $\mathcal{E}xt^i(\mathcal{N}, \mathcal{A}_\lambda^{-\theta}(v)) = 0$ whenever $i \neq N$. By the standard commutative algebra, see, e.g., [Ei, Proposition 18.4], we see that $\mathcal{E}xt^i(\operatorname{gr}\mathcal{N}, \mathcal{O}_{\mathcal{M}^{-\theta}(v)}) \neq 0$ implies $i \geqslant N$. Moreover, if $i > N$, then the support of $\mathcal{E}xt^i(\operatorname{gr}\mathcal{N}, \mathcal{O}_{\mathcal{M}^{-\theta}(v)})$ has dimension $< N$. The space $\mathcal{E}xt^i(\mathcal{N}, \mathcal{A}_\lambda^{-\theta}(v))$ has a natural filtration with $\operatorname{gr}\mathcal{E}xt^i(\mathcal{N}, \mathcal{A}_\lambda^{-\theta}(v)) \hookrightarrow \mathcal{E}xt^i(\operatorname{gr}\mathcal{N}, \mathcal{O}_{\mathcal{M}^{-\theta}(v)})$. Since the filtration is separated, we see that $\mathcal{E}xt^i(\mathcal{N}, \mathcal{A}_\lambda^{-\theta}(v)) = 0$ for $i < N$ and

$$\dim \operatorname{Supp} \mathcal{E}xt^i(\mathcal{N}, \mathcal{A}_\lambda^{-\theta}(v)) < N$$

for $i > N$. Since the support of any $\mathcal{A}_\lambda^{-\theta}(v)^{opp}$-module is coisotropic, see Section 2.4, it cannot have dimension less than N and we are done. $\square$

Now consider the functor D for the categories of $\mathcal{A}_\lambda(v)$-modules.

Lemma 8.2. *Let $\mathcal{N}$ be a simple holonomic $\mathcal{A}_\lambda(v)$-module. Then the following is true*

(1) $H^i(D\mathcal{N}) = 0$ *for* $i < N - \dim \operatorname{Supp}\mathcal{N}$ *or* $i > N$.
(2) $H^i(D\mathcal{N})$ *is a nonzero module with support of dimension* $\dim \operatorname{Supp}\mathcal{N}$ *when* $i = N - \dim \operatorname{Supp}\mathcal{N}$.

Proof. The algebra $\mathbb{C}[\mathcal{M}(v)]$ is Cohen-Macaulay, see Corollary 2.4. Then [Ei, Proposition 18.4] implies that, for a finitely generated $\mathbb{C}[\mathcal{M}(v)]$-module M, the minimal number r such that $\operatorname{Ext}^r(M, \mathbb{C}[\mathcal{M}(v)]) \neq 0$ equals $\dim \mathcal{M}(v) - \dim \operatorname{Supp} M$. Moreover, we have $\dim \operatorname{Supp} \operatorname{Ext}^r(M, \mathbb{C}[\mathcal{M}(v)]) = \dim \operatorname{Supp} M$ and $\dim \operatorname{Supp} \operatorname{Ext}^i(M, \mathbb{C}[\mathcal{M}(v)]) < \dim \operatorname{Supp} M$ for $i > r$.

The case $i < N - \dim \operatorname{Supp}\mathcal{N}$ is done similarly to the proof of Lemma 8.1 using the facts quoted in the previous paragraph. To deal with the case of $i > N$ we notice that the homological dimension of $\mathcal{A}_\lambda(v)$ coincides with that of $\mathcal{A}_\lambda^\theta(v)$ because Γ_λ^θ is an abelian equivalence. The homological dimension of $\mathcal{A}_\lambda^\theta(v)$ does not exceed that of $\operatorname{Coh}\mathcal{M}^\theta(v)$ that equals $2N = \dim \mathcal{M}^\theta(v)$. This completes the $i > N$ case.

Let us prove (2). As in the proof of Lemma 8.1, we see that $\dim \operatorname{Supp} H^i(D\mathcal{N}) < N - \dim \operatorname{Supp}\mathcal{N}$ for $i > N - \dim \operatorname{Supp}\mathcal{N}$. Since $\mathcal{D}^2 = \mathrm{id}$, the inequality $H^i(D\mathcal{N}) \neq 0$ for $i = N - \dim \operatorname{Supp}\mathcal{N}$ follows. $\square$

Now we note that we have an isomorphism $\mathcal{A}_\lambda^{-\theta}(v)^{opp} = \mathcal{A}_{\lambda^*}^{-\theta}(v)$ (the equality of quantizations of $\mathcal{M}^{-\theta}(v)$). Here $\lambda^* := 2\varrho(v) - \lambda$. This follows, for example, from [L6, Proposition 5.4.4]. So in the above constructions, we can replace $\mathcal{A}_\lambda^{-\theta}(v)^{opp}$ with $\mathcal{A}_{\lambda^*}^{-\theta}(v)$ and $\mathcal{A}_\lambda(v)^{opp}$ with $\mathcal{A}_{\lambda^*}(v)$.

8.2. **Proof of Proposition 5.13.** Thanks to Lemma 4.3, replacing λ with $\lambda + k\theta$ for $k \gg 0$, we may assume that $(\lambda^*, -\theta) \in \mathfrak{AL}(v)$ (and still $(\lambda, \theta) \in \mathfrak{AL}(v)$). Now we have the following commutative diagram.

$$\begin{tikzcd}
D^b(\mathcal{A}_{\lambda^-}^{-\theta}(v)) \arrow[d, "R\Gamma_{\lambda^-}^{-\theta}"] & D^b(\mathcal{A}_\lambda^{-\theta}(v)) \arrow[l, "\mathcal{T}_{\lambda,\lambda^- - \lambda}"'] \arrow[r, "D^{-\theta}"] \arrow[d, "R\Gamma_\lambda^{-\theta}"] & D^b(\mathcal{A}_{\lambda^*}^{-\theta}(v))^{opp} \arrow[d, "R\Gamma_{\lambda^*}^{-\theta}"] \\
D^b(\mathcal{A}_{\lambda^-}(v)) & D^b(\mathcal{A}_\lambda(v)) \arrow[l, "\mathfrak{WC}_{\lambda\to\lambda^-}"'] \arrow[r, "D"] & D^b(\mathcal{A}_{\lambda^*}(v))^{opp}
\end{tikzcd}$$

Here we write $D^b(\mathcal{A}_\lambda(v))$ for $D^b(\mathcal{A}_\lambda(v)\text{-mod})$, etc.

The functor $R\Gamma^{-\theta}_{\lambda^-}\circ\mathcal{T}_{\lambda,\lambda^- -\lambda}$ is an abelian equivalence $\mathcal{A}^{-\theta}_\lambda(v)\text{-mod}\xrightarrow{\sim}\mathcal{A}_{\lambda^-}(v)\text{-mod}$. The functor $R\Gamma^{-\theta}_{\lambda^*}\circ D^{-\theta}$ intertwines the standard t-structures on $D^b_{hol}(\mathcal{A}^{-\theta}_\lambda(v)), D^b_{hol}(\mathcal{A}_{\lambda^*}(v))$. So we see that the pull-backs of the t-structures on $D^b_{hol}(\mathcal{A}_{\lambda^*}(v))$ and on $D^b_{hol}(\mathcal{A}^{-\theta}_{\lambda^-}(v))$ to $D^b_{hol}(\mathcal{A}_\lambda(v))$ coincide (with the push-forward of the t-structure on $D^b_{hol}(\mathcal{A}^{-\theta}_\lambda(v))$).

Let us prove (1). Thanks to Lemmas 8.1,8.2, the functor D homologically shifts a simple M by $N-\dim\operatorname{Supp} M$. Part (1) now follows from the coincidence of the t-structures on $D^b_{hol}(\mathcal{A}_\lambda(v))$ established in the previous paragraph.

Let us prove part (2). The functor $\mathfrak{WC}_{\lambda\to\lambda^-}$ restricts to a derived equivalence $D^b_{fin}(\mathcal{A}_\lambda(v)\text{-mod})\to D^b_{fin}(\mathcal{A}_{\lambda^-}(v)\text{-mod})$. By Lemma 8.2, for a finite dimensional module M, the only nonzero homology of $\mathfrak{WC}_{\lambda\to\lambda^-}M$ is H_N. We are done.

Remark 8.3. Equip $\mathcal{A}_\lambda(v)\text{-mod}_{hol}$ with a filtration by the dimension of support: let $\mathcal{A}_\lambda(v)\text{-mod}^{\leqslant i}_{hol}$ consist of all modules whose dimension of support does not exceed i. The functor $\mathfrak{WC}_{\lambda\to\lambda^-}$ sends an object of $\mathcal{A}_\lambda(v)\text{-mod}^{\leqslant i}_{hol}$ to a complex whose homology are in $\mathcal{A}_{\lambda^-}(v)\text{-mod}^{\leqslant i}_{hol}$. The arguments of the proofs of Lemma 8.2 and Proposition 5.13 implies that the functor $H_i(\mathfrak{WC}_{\lambda\to\lambda^-}\bullet)$ gives rise to an equivalence $\mathcal{A}_\lambda(v)\text{-mod}^{\leqslant i}_{hol}/\mathcal{A}_\lambda(v)\text{-mod}^{\leqslant i-1}_{hol}\xrightarrow{\sim}\mathcal{A}_{\lambda^-}(v)\text{-mod}^{\leqslant i}_{hol}/\mathcal{A}_{\lambda^-}(v)\text{-mod}^{\leqslant i-1}_{hol}$. In particular, $\mathfrak{WC}_{\lambda\to\lambda^-}$ is a perverse equivalence

$$D^b_{hol}(\mathcal{A}_\lambda(v)\text{-mod})\to D^b_{hol}(\mathcal{A}_{\lambda^-}(v)\text{-mod})$$

in the sense of Chuang and Rouquier. See Section 10.1 below for a precise definition of a perverse equivalence in the case of derived categories (the general case of triangulated categories is completely analogous, see, e.g., [ABM]). We do not need this result in the rest of the paper so we do not provide details.

8.3. **Further results.** We will need some further results on long wall-crossing functors. Let λ,λ^-,θ have the same meaning as above.

Lemma 8.4. *A long wall-crossing $\mathcal{A}_{\lambda^-}(v)$-$\mathcal{A}_\lambda(v)$-bimodule $\mathcal{A}^{(-\theta)}_{\lambda\to\lambda^-}(v)$ is simple.*

Proof. The HC $\mathcal{A}^{-\theta}_{\lambda^-}(v)$-$\mathcal{A}^{-\theta}_\lambda(v)$ bimodule $\mathcal{A}^{-\theta}_{\lambda\to\lambda^-}(v)$ is simple because its rank equals 1. The categories $\operatorname{HC}(\mathcal{A}^{-\theta}_{\lambda^-}(v)\text{-}\mathcal{A}^{-\theta}_\lambda(v))$ (see [BPW, Section 6.1] for the definition of this category) and $\operatorname{HC}(\mathcal{A}_{\lambda^-}(v)\text{-}\mathcal{A}_\lambda(v))$ of Harish-Chandra bimodules are equivalent, see [BPW, Corollary 6.6]. □

Remark 8.5. We can consider $\mathcal{A}^{(-\theta)}_{\lambda\to\lambda^-}$ as an $\mathcal{A}^{opp}_\lambda$-$\mathcal{A}^{opp}_{\lambda^-}$-bimodule. It is straightforward to see that it is still a long wall-crossing bimodule.

8.4. **Corollaries.** Now we are ready to prove (2) of Proposition 4.3. To start with, let us prove a stronger version of Proposition 3.14.

Proposition 8.6. *For each indecomposable root $\alpha\leqslant v$, there is a finite subset $\Sigma_\alpha\subset\mathbb{C}$ such that the algebra $\mathcal{A}_\lambda(v)$ is simple whenever $\langle\alpha,\lambda\rangle\not\in\Sigma_\alpha+\mathbb{Z}$ for all $\alpha\leqslant v$.*

Proof. As in the proof of Proposition 3.14, we will first show that, for each root $\alpha\leqslant v$, there is a finite subset $\Sigma_\alpha(v)$ such that the algebras $\mathcal{A}^0_\lambda(v),\mathcal{A}_\lambda(v)$ have no finite dimensional representations provided $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha(v)+\mathbb{Z}$ for all $\alpha\leqslant v$.

Step 1. Let us construct $\Sigma_\alpha(v)$. Pick a Zariski generic point $p\in\ker\alpha$. The variety $\mathcal{M}_p(v)$ has a minimal symplectic leaf, compare with Step 2 of the proof of Proposition 5.4. It corresponds to a semisimple representation of the form $r_0+r_1^{\oplus k}$, where $\dim r_1=\alpha$

and k is maximal such that $(v-k\alpha,1)$ is a root of the quiver Q^w. So we can form the slice algebras $\hat{\mathcal{A}}^0_{\hat{r}(\lambda)}(\hat{v}), \hat{\mathcal{A}}_{\hat{r}(\lambda)}(\hat{v})$, compare to Step 2 of the proof of Proposition 5.4. Pick a sufficiently big positive integer m. The set of $\langle\lambda,\alpha\rangle$ such that the translation bimodules $\hat{\mathcal{A}}^0_{\hat{r}(\lambda),m}(\hat{v}), \hat{\mathcal{A}}^0_{\hat{r}(\lambda)+m,-m}(\hat{v})$ are not mutually inverse Morita equivalences between $\hat{\mathcal{A}}_{\hat{r}(\lambda),m}(\hat{v})$ and $\mathcal{A}_{\hat{r}(\lambda)+m,-m}(\hat{v}))$ is finite by Proposition 4.5. We take this set for $\Sigma_\alpha(v)$.

Step 2. Let θ,θ' be two stability conditions from chambers opposite with respect to $\ker\alpha$. Similarly to the proof of Proposition 5.4 we see that $\mathfrak{WC}_{\theta\to\theta'}$ is an abelian equivalence provided $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha(v)+\mathbb{Z}$.

Step 3. Now suppose that $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha(v)$ for all indecomposable $\alpha\leqslant v$. By Theorem 5.3, the long wall-crossing functor $\mathfrak{WC}_{\theta\to-\theta}$ is an abelian equivalence. By Theorem 5.13, the algebra $\mathcal{A}_\lambda(v)$ has no finite dimensional representations if abelian localization holds for (λ,θ). In general, note that if L is a finite dimensional representation of $\mathcal{A}_\lambda(v)$, then $L\operatorname{Loc}^\theta_\lambda(L)$ is a nonzero object supported on $\rho^{-1}(0)$ (indeed, the functor $R\Gamma$ is a left inverse to $L\operatorname{Loc}$, both functors are considered between bounded from the right derived categories). It follows that for $n\gg 0$, the algebra $\mathcal{A}_{\lambda+n\theta}(v)$ has a finite dimensional representation. This completes the proof of the claim that $\mathcal{A}_\lambda(v),\mathcal{A}^0_\lambda(v)$ have no finite dimensional representations provided $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha(v)+\mathbb{Z}$ for all indecomposable $\alpha\leqslant v$.

Step 4. Similarly to the proof of Proposition 3.14, if all proper slice algebras $\hat{\mathcal{A}}_{\hat{r}(\lambda)}(\hat{v})$ have no finite dimensional representations, then the algebra $\mathcal{A}_\lambda(v)$ is simple. Now recall that $r^{-1}(\hat{\mathfrak{p}}^{sing})\subset\mathfrak{p}^{sing}$, see 2.1.6. Using this we get subsets $\Sigma_\alpha\in\mathbb{C}$ (for each indecomposable root α) such that the proper slice algebras $\hat{\mathcal{A}}_{\hat{r}(\lambda)}(\hat{v})$ have no finite dimensional representations when $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha+\mathbb{Z}$. So for such λ, the algebra $\mathcal{A}_\lambda(v)$ is simple. □

Proof of (2) of Proposition 4.3. Let χ be inside of the chamber of θ and such that $H^1(\mathcal{M}^\theta(v),\mathcal{O}(\chi))=H^1(\mathcal{M}^{-\theta}(v),\mathcal{O}(-\chi))=0$. If $\langle\lambda,\alpha\rangle\not\in\Sigma_\alpha+\mathbb{Z}$ for any indecomposable root α, then the algebras $\mathcal{A}_\lambda(v),\mathcal{A}_{\lambda+\chi}(v)$ are simple. Consider the bimodule homomorphisms

$$\mathcal{A}^0_{\lambda+\chi,-\chi}(v)\otimes_{\mathcal{A}_{\lambda+\chi}(v)}\mathcal{A}^0_{\lambda,\chi}(v)\to\mathcal{A}_\lambda(v),$$
$$\mathcal{A}^0_{\lambda,\chi}(v)\otimes_{\mathcal{A}_\lambda(v)}\mathcal{A}^0_{\lambda+\chi,-\chi}(v)\to\mathcal{A}_{\lambda+\chi}(v).$$

It is enough to show that are iso. Note that the generic rank of $\operatorname{gr}\mathcal{A}^0_{\lambda,\chi}(v)$ on $\mathcal{M}(v)$ is 1. Indeed, we have $\mathcal{A}^0_{\lambda,\chi}(v)=\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ by Proposition 4.6. The generic rank of $\operatorname{gr}\mathcal{A}^{(\theta)}_{\lambda,\chi}(v)$ on $\mathcal{M}(v)^{reg}$ is 1 by the construction. For similar reasons, the generic rank of $\operatorname{gr}\mathcal{A}^0_{\lambda+\chi,-\chi}(v)$ equals 1. For similar reasons, the bimodule homomorphisms above become iso after microlocalizing to $\mathcal{M}(v)^{reg}$. Since the algebras $\mathcal{A}_\lambda(v),\mathcal{A}_{\lambda+\chi}(v)$ are simple, we deduce that the bimodule homomorphisms are indeed iso. □

9. Finite short wall-crossing

In this section we investigate various questions related to categorification functors E_α,F_α, the wall-crossing functor through $\ker\alpha$ and connections between them. In Section 9.1 we study the category $\mathcal{C}$ introduced in the beginning of Section 5.4. In particular, we show that every simple in $\mathcal{C}$ is *regular holonomic* in a suitable sense. We use this to show that Proposition 5.12 implies (II), while (II) and (III) imply Proposition 5.12. In Section 9.2 we study singular objects from 5.4.2 and prove Proposition 5.14.

9.1. **Category $\mathcal{C}$.** Recall that $\mathcal{C}\subset\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}$ is the Serre subcategory spanned by the homology of the objects of the form $\mathcal{F}L_0$, where $\mathcal{F}$ is some monomial in the functors E_α,F_α and $L_0\in\mathcal{A}^\theta_\lambda(\sigma\bullet w)$ for $\sigma\in W(Q)$ such that $\sigma\omega$ is dominant for $\mathfrak{a}$.

9.1.1. *Regular holonomic modules.* Let us define *regular holonomic* simples in $\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}_{\rho^{-1}(0)}$. An object in $\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}$ is called *regular holonomic* if it is obtained from a regular holonomic (G,λ)-equivariant $D(R)$-module by applying $\pi^\theta_\lambda(v)$. Actually, we are not interested in all regular holonomic modules. Recall the torus $T=(\mathbb{C}^\times)^{Q_1}\times(\mathbb{C}^\times)^{Q_0}$ from 2.1.3 acting on R. We will consider only weakly T-equivariant modules. Note that a unique indecomposable $\mathcal{A}^\theta_\lambda(\sigma\bullet w)$-module is weakly T-equivariant.

It is a standard fact that the category of regular holonomic weakly T-equivariant D-modules stays the same under changing the orientation of R (partial Fourier transforms preserve weakly T-equivariant regular holonomic modules, [Br]) so the notion of a weakly T-equivariant regular holonomic $\mathcal{A}^\theta_\lambda(v)$-module is well-defined.

Lemma 9.1. *Let $\lambda'=\sigma\bullet^v\lambda$. Under the isomorphism $\mathcal{A}^\theta_\lambda(v)\cong\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma\bullet v)$, a weakly T-equivariant regular holonomic module remains weakly T-equivariant regular holonomic.*

Proof. The part concerning the T-action follows from the observation, see 2.2.4, that $\sigma_*:\mathcal{A}^\theta_\lambda(v)\operatorname{-mod}\xrightarrow{\sim}\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma\bullet v)\operatorname{-mod}$ is T-equivariant. It is enough to prove the claim that the resulting module is still regular holonomic for a simple reflection s_i. Recall that in this case the isomorphism $\mathcal{A}^\theta_\lambda(v)\cong\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma\bullet v)$ is induced by the isomorphism

$$D_R/\!\!/\!\!/^{\theta_i}_{\lambda_i}\operatorname{GL}(v_i)\cong D_R/\!\!/\!\!/^{-\theta_i}_{\lambda'_i}\operatorname{GL}((s_i\bullet v)_i). \tag{9.1}$$

The former reduction is just $D^{\lambda_i}_{\operatorname{Gr}(v_i,\tilde{w}_i)}\otimes D_{\underline{R}}$ and so there is an internal notion of a regular holonomic module.

We claim that a simple $D^{\lambda_i}_{\operatorname{Gr}(v_i,\tilde{w}_i)}\otimes D_{\underline{R}}$-module is regular holonomic if and only if it is obtained from a simple regular holonomic D_R-module under the quotient functor. Indeed, let L be a simple $(\operatorname{GL}(v_i),\lambda_i)$-equivariant D-module on R whose support intersects R^{θ_i-ss}. Then L is regular holonomic if and only if the induced twisted D-module on the quotient $R/\!/^{\theta_i}\operatorname{GL}(v_i)$ is regular holonomic. This follows from the classification of simple regular holonomic D-modules: those are precisely the intermediate extensions of regular holonomic local systems on smooth locally closed subvarieties, see [Bo, Theorem 7.10.6, 7.12]. Our claim in the beginning of the paragraph is proved.

Now, under the identifications of $D_R/\!\!/\!\!/^{\theta_i}_{\lambda_i}\operatorname{GL}(v_i)\operatorname{-mod}$ and $D_R/\!\!/\!\!/^{-\theta_i}_{\lambda'_i}\operatorname{GL}((s_i\bullet v)_i)\operatorname{-mod}$ with the category of D-modules on $\operatorname{Gr}(v_i,\tilde{w}_i)\times\underline{R}$, the equivalence induced by (9.1) becomes the identity, this follows from the construction of an isomorphism. We deduce that the equivalence induced by the isomorphism $\mathcal{A}^\theta_\lambda(v)\cong\mathcal{A}^{\sigma\theta}_{\lambda'}(\sigma\bullet v)$ maps regular holonomic modules to regular holonomic ones. ☐

Corollary 9.2. *The simples in $\mathcal{C}$ are weakly T-equivariant regular holonomic.*

Proof. Let us show first that Webster's functors E_i,F_i preserve the category of direct sums of semisimple weakly T-equivariant regular holonomic $\mathcal{A}^\theta_\lambda(v)$-modules with homological shifts. The functors E_i,F_i on $\bigoplus_{v_i}D^b(D^{\lambda_i}_{\operatorname{Gr}(v_i,\tilde{w}_i)}\otimes D_{\underline{R}}\operatorname{-mod}^{\underline{G},\underline{\lambda}})$ have this property by the construction. By the proof of Lemma 9.1, a simple regular holonomic $\mathcal{A}^\theta_\lambda(v)$-module is an image of such a module from $D^{\lambda_i}_{\operatorname{Gr}(v_i,\tilde{w}_i)}\otimes D_{\underline{R}}\operatorname{-mod}^{\underline{G},\underline{\lambda}}$ and our claim follows.

Lemma 9.1 and the claim that the equivalences σ_* are T-equivariant show that the functors E_α,F_α preserve semisimple complexes of weakly T-equivariant regular holonomic modules when $\alpha\in\Pi^\theta$. So it is enough to check that any object in $\mathcal{A}^\theta_\lambda(\sigma\bullet w)\operatorname{-mod}$ is regular holonomic. This is definitely true for $\sigma=1$ (the space R is zero). For arbitrary σ, the claim again follows from Lemma 9.1. ☐

9.1.2. *Crystal.* Consider the full subcategory $\mathcal{C}' \subset D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ consisting of all objects M such that $M \cong H_*(M)$ and $H_*(M)$ is a semisimple object of $\mathcal{C}$. For $L \in \operatorname{Irr}(\mathcal{C})$, we write $d_\alpha(L)$ for the minimal dimension of an irreducible $\mathfrak{sl}_2$-module in $U(\mathfrak{sl}_2)[L]$ (where we consider the action corresponding to the operators $[E_\alpha], [F_\alpha]$)

Lemma 9.3. *The functors E_α, F_α for all $\alpha \in \Pi^\theta$ preserve the subcategory $\mathcal{C}' \subset D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$. Furthermore, we have*

$$\begin{aligned} F_\alpha L &= \bigoplus_{i=0}^{k} \tilde{f}_\alpha L[m+2i] \oplus \bigoplus_{L', d(L')>d(L)} L'[?], \\ E_\alpha L &= \bigoplus_{i=0}^{\ell} \tilde{e}_\alpha L[n+2i] \oplus \bigoplus_{L'', d(L'')>d(L)} L''[?] \end{aligned} \tag{9.2}$$

Here $\tilde{f}_\alpha, \tilde{e}_\alpha$ are maps $\operatorname{Irr}(\mathcal{C}) \to \operatorname{Irr}(\mathcal{C}) \sqcup \{0\}$ forming a crystal for $\mathfrak{sl}_2$, and k, m, ℓ, n are some numbers whose precise values are not important for us.

Proof. The claim that the functors E_α, F_α preserve the category $\mathcal{C}'$ follows from the proof of Corollary 9.2. So we get a categorical $\mathfrak{sl}_2$-action on the additive category $\mathcal{C}'$. [R2, Theorem 5.8] applies to this action. It follows that the basis $[L], L \in \operatorname{Irr}(\mathcal{C})$, is a dual perfect basis for the $\mathfrak{sl}_2$-action on $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ (meaning that the dual basis in $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})^*$ is perfect in the sense of Berenstein and Kazhdan, see [BerKa, Section 5]). This gives rise to crystal operators $\tilde{e}_\alpha, \tilde{f}_\alpha$ on $\bigsqcup_v \operatorname{Irr}(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ (9.2) follows. $\square$

Corollary 9.4. *We have $\mathsf{CC}(K_0(\mathcal{C})) = L^{\mathfrak{a}}_\omega$.*

Proof. It is clear from the construction of $K_0(\mathcal{C})$ and Proposition 5.9 that $L^{\mathfrak{a}}_\omega \subset \mathsf{CC}(K_0(\mathcal{C}))$. So let us prove the opposite inclusion. Let v be minimal such that $\mathsf{CC}(K_0(\mathcal{C}_v)) \supsetneq L^{\mathfrak{a}}_\omega[\nu]$. Then ν is dominant (otherwise $\mathsf{CC}(K_0(\mathcal{C}_v)) \subset \sum_\alpha \operatorname{im} F_\alpha$). Pick $L \in \operatorname{Irr}(K_0(\mathcal{C}_v))$ with $\mathsf{CC}(L) \notin L^{\mathfrak{a}}_\omega[\nu]$ and maximal number $\max_\alpha d_\alpha(L)$ among all such objects L so that $d_\alpha(L) > 0$. Let $\alpha \in \Pi^\theta$ be such that the maximum is achieved for α. So if $L' \in \operatorname{Irr}(\mathcal{C})$ satisfies $d_\alpha(L') > d_\alpha(L)$, then $\mathsf{CC}(L') \in L^{\mathfrak{a}}_\omega$. Applying (9.2) for the functor F_α and the simple $\tilde{e}_\alpha L$ we see that, by Proposition 5.9, $\mathsf{CC}(L) \in L^{\mathfrak{a}}_\omega$. $\square$

So, indeed, Proposition 5.12 implies (II) (and is equivalent to (II) modulo (III)).

9.2. **Singular simples.** Let us start with an easy alternative characterization of a singular object.

Lemma 9.5. *Let $\alpha \in \Pi^\theta$ and $L \in \operatorname{Irr}(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$. Then the following are equivalent.*

1. *L is α-singular.*
2. *$L \notin \operatorname{im} \tilde{f}_\alpha$ if $\langle \nu, \alpha_i^\vee \rangle \geqslant 0$ or $L \notin \operatorname{im} \tilde{e}_\alpha$ if $\langle \nu, \alpha_i^\vee \rangle \leqslant 0$.*

Proof. This follows from (9.2). $\square$

Proof of Proposition 5.14. The proof is in several steps. Thanks to the construction of the functors E_α, F_α, we may assume that $\alpha = \alpha^i$ and $\theta = \theta^+$. Recall the quotient functor $\underline{\pi} : \mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod} \twoheadrightarrow \mathcal{A}^\theta_\lambda(v)\text{-mod}$. Let $\tilde{L}$ denote the simple in $\mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}$ with $\underline{\pi}(\tilde{L}) = L$. Note that L is α-singular if and only if $\tilde{L}$ is singular (for the Webster functors E_i, F_i).

Step 1. Let us prove that (2) implies (1). Assume the contrary: $H_0(\mathfrak{WC}_{\theta\to\theta'}L)\neq 0$ but L is not α-singular. Then $\tilde{L}$ is not singular: if L_1 is such that, say, $\tilde{f}_\alpha L_1=L$ and $\tilde{L}_1$ is the simple in $\mathcal{A}^{\theta_i}_{\lambda_i}(v)$-mod with $\underline{\pi}(\tilde{L}_1)=L_1$, then $\tilde{f}\tilde{L}_1=\tilde{L}$. On the other hand, $H_0(\mathfrak{WC}^i_{\theta_i\to-\theta_i}\tilde{L})\neq 0$, this follows from 7.1.2. Since $\tilde{L}$ is holonomic, a direct analog of Proposition 5.13 applies. Thanks to that, we see that the singular support of $\Gamma^{\theta_i}_{\lambda_i}(\tilde{L})$ intersects an open symplectic leaf in the affinization of $T^*\operatorname{Gr}(v_i,\tilde{w}_i)\times T^*\underline{R}$. Equivalently, the singular support of $\tilde{L}$ intersects $\mathbb{O}\times T^*\underline{R}$, where $\mathbb{O}$ is the open $\operatorname{GL}(\tilde{w}_i)$-orbit in $T^*\operatorname{Gr}(v_i,\tilde{w}_i)$. We claim that this contradicts the condition that $\tilde{L}$ is not singular. Indeed, in the sake of being definite, assume $2v_i\leqslant\tilde{w}_i$ so that $\tilde{L}\in\operatorname{im}\tilde{f}_i$. From the construction of the functor F_i, the singular support of $\tilde{L}$ lies in the image of $Y\times T^*\underline{R}$ in $T^*\operatorname{Gr}(v_i,\tilde{w}_i)$, where we write Y for the conormal bundle to $\operatorname{Fl}(v_i-1,v_i;\tilde{w}_i)\subset\operatorname{Gr}(v_i-1,\tilde{w}_i)\times\operatorname{Gr}(v_i,\tilde{w}_i)$. But that image of Y does not intersect $\mathbb{O}$. Contradiction. This finishes the proof of the implication (2)$\Rightarrow$(1).

Step 2. Let us prove that (1) implies (2). Here we will use Theorem 5.10 that says that $s_{i*}\circ\mathfrak{WC}_{\theta\to\theta'}=\Theta_i$. So (2) is equivalent to $H_0(\Theta_iL)\neq 0$. Below, to simplify the notation, we write E,F,Θ for E_i,F_i,Θ_i.

In the proof we will assume that $2v_i\leqslant\tilde{w}_i$, the other case is similar. Recall that ΘL is the iterated cone of

$$F^{(\ell)}L[-m]\to F^{(\ell+1)}EL[1-m]\to\ldots\to F^{(\ell+m)}E^{(m)}L,$$

where $m=v_i,\ell=\tilde{w}_i-2v_i$. Then Θ^2L is the cone of the double complex with the term in the slot $(i-v_i,j-\tilde{w}_i+v_i)$ of the form $F^{(i)}E^{(\ell+i)}F^{(\ell+j)}E^{(j)}L[i+j-\tilde{w}_i]$. The terms with $i>0$ do not contain L in their homology because L is singular. If $i=0,j>0$, then we can commute $E^{(\ell)}$ and $F^{(\ell+j)}$ using the categorical $\mathfrak{sl}_2$-relations, see, e.g., (iii) in [CDK, Section 2.2]. We get that these terms do not contain L either. For the same reason, the $(0,0)$ term splits into a direct sum of L and some object that does not contain L in the homology. The summand L will contribute to H_0 of the iterated cone of Θ^2L. Now recall that $L\mapsto\Theta L$ is right t-exact by Theorem 5.10. Since $H_0(\Theta^2L)\neq 0$, we deduce that $H_0(\Theta L)\neq 0$. This finishes the proof of the implication (1)$\Rightarrow$(2).

Step 3. Let us prove the claim about the singular simples in $H_*(\mathfrak{WC}_{\theta\to\theta'}L)$: only one occurs as a composition factor and it is a quotient of H_0. It is equivalent to an analogous claim for $H_*(\mathfrak{WC}^i_{\theta_i\to-\theta_i}\tilde{L})$. By Step 1, the singular support of $\tilde{L}$ contains a point that is stable for $-\theta_i$. Let π_+,π_- be the quotient functors from D_R-mod$^{G,\lambda}$ to the quotient categories for the stability conditions $\theta_i,-\theta_i$. Then $\mathfrak{WC}^i_{\theta_i\to-\theta_i}=\pi_-L\pi_+^!$, see (5.3). Let $\widehat{L}$ be the simple in $D(R)$-mod$^{G,\lambda}$ such that $\pi_+(\widehat{L})=\tilde{L}$. We see that $\pi_-(\widehat{L})$ occurs in $H_0(\mathfrak{WC}_{\theta_i\to-\theta_i}(\tilde{L}))$ (as a quotient, in fact, because $\widehat{L}$ is a quotient of $\pi_+^!(\tilde{L})$). Clearly, the multiplicity is 1. The other composition factors of $H_j(\mathfrak{WC}_{\theta_i\to-\theta_i}(\tilde{L}))$ are the images of simples in $\ker\pi_+$. So the singular supports do not intersect $\mathbb{O}\times T^*\underline{R}$. Reversing the argument of Step 1, we see that these simples are shifted by $\mathfrak{WC}$ and hence by Θ. By Step 2, they cannot be singular. So we get simples $\tilde{L}':=\pi_-(\widehat{L})\in\mathcal{A}^{-\theta_i}_{\lambda_i'}(v)$-mod and the corresponding simple $L'\in\mathcal{A}^{\theta'}_{\lambda'}(v)$-mod$_{\rho^{-1}(0)}$. Note that $\tilde{L}'$ singular by the argument above in this step. So L' is singular.

Step 4. It remains to prove that the map $L\mapsto L'$ is a bijection between the sets of α-singular objects. By Step 3, $s_*\tilde{L}'$ is the only singular simple constituent of $\bigoplus_i H_i(\Theta\tilde{L})$, it is a quotient of $H_0(\Theta\tilde{L})$. The proof of (2)$\Rightarrow$(1) implies that $\tilde{L}$ is the only singular constituent of $\bigoplus_i H_i(\Theta(s_*\tilde{L}'))$. This gives rise to a map from the set of α-singular simples in $\mathcal{A}_\lambda^{s_i\theta}(v)\text{-mod}_{\rho^{-1}(0)}$ to the set of singular simples in $\mathcal{A}_\lambda^\theta(v)\text{-mod}_{\rho^{-1}(0)}$ that is the inverse to $L\mapsto L'$. □

10. Affine short wall-crossing

Here we consider the situation when Q is an affine quiver, $v=n\delta, w=\epsilon_0$, and $\mathcal{A}_\lambda(v)=eH_{\kappa,c}(n)e$. In this section we study the wall-crossing functor through the wall $\ker\delta$ proving in particular that the homological shifts of modules under this functor are less than n and that the functor is a perverse equivalence.

10.1. **Results.** Let us introduce some conventions and notation.

Pick a generic stability condition θ in a classical chamber C' that has $\ker\delta$ as a wall. Let C' denote the classical chamber sharing the wall $\ker\delta$ with C.

We consider a parameter λ° such that $(\lambda,\theta)\in\mathfrak{AL}(v)$ for any $\lambda\in\lambda^\circ+(C\cap\mathbb{Z}^{Q_0})$, such a parameter exists by (2) of Proposition 4.3. The parameter λ° is represented in the form (κ,c°), where $\kappa=\langle\lambda^\circ,\delta\rangle$ so that $\mathcal{A}_{\lambda^\circ}(v)=eH_{\kappa,c^\circ}(n)e$. We view c° as an element of $\ker\delta$ (this includes some renormalization of the usual parameters for the SRA's). Similarly, choose a parameter $\lambda'^\circ=(\kappa',c'^\circ)\in\lambda+\mathbb{Z}^{Q_0}$ such that $(\lambda',\theta')\in\mathfrak{AL}(v)$ for any $\lambda'\in\lambda'^\circ+(C'\cap\mathbb{Z}^{Q_0})$. Set $\chi:=\lambda'^\circ-\lambda^\circ$. Note that $\kappa'-\kappa=\langle\chi,\delta\rangle\in\mathbb{Z}$.

For $\lambda=(\kappa,c)$, we set $\mathcal{A}^c:=\mathcal{A}_\lambda(v),\mathcal{A}'^c:=\mathcal{A}_{\lambda+\chi}(v),\mathcal{B}^c:=\mathcal{A}^0_{\lambda,\chi}(v)$. We also consider the universal versions: we write $\mathfrak{p}_0$ for $\ker\delta$ and consider the objects $\mathcal{A}^{\mathfrak{p}_0}:=\mathcal{A}_{\lambda+\mathfrak{p}_0}(v),\mathcal{A}'^{\mathfrak{p}_0},\mathcal{B}^{\mathfrak{p}_0}:=\mathcal{A}^0_{\lambda+\mathfrak{p}_0,\chi}(v)$ so that $\mathcal{A}^c,\mathcal{A}'^c,\mathcal{B}^c$ are the specializations of $\mathcal{A}^{\mathfrak{p}_0},\mathcal{A}'^{\mathfrak{p}_0},\mathcal{B}^{\mathfrak{p}_0}$.

Let m denote the denominator of κ (we set $m=\infty$ if κ is irrational) if $\kappa\notin\mathbb{Z}$. For $\kappa\in\mathbb{Z}$ we assume that $m=\infty$.

We will define chains of ideals $\{0\}=\mathcal{J}^{\mathfrak{p}_0}_{q+1}\subsetneq\mathcal{J}^{\mathfrak{p}_0}_q\subsetneq\ldots\subsetneq\mathcal{J}^{\mathfrak{p}_0}_1\subsetneq\mathcal{J}^{\mathfrak{p}_0}_0=\mathcal{A}^{\mathfrak{p}_0},\{0\}=\mathcal{J}'^{\mathfrak{p}_0}_{q+1}\subsetneq\mathcal{J}'^{\mathfrak{p}_0}_q\subsetneq\ldots\subsetneq\mathcal{J}'^{\mathfrak{p}_0}_1\subsetneq\mathcal{J}'^{\mathfrak{p}_0}_0=\mathcal{A}'^{\mathfrak{p}_0}$, where $q=\lfloor n/m\rfloor$, and consider the corresponding specializations $\mathcal{J}_i^c,\mathcal{J}_i'^c$.

One more piece of notation: $d_i:=(q+1-i)(m-1)$.

Here is our main technical result.

Theorem 10.1. *There is a principal open subset $\mathfrak{p}_0^0\subset\mathfrak{p}_0$ such that the HC bimodules $\mathcal{J}_i^{\mathfrak{p}_0},\mathcal{A}^{\mathfrak{p}_0}/\mathcal{J}_i^{\mathfrak{p}_0},\mathcal{J}'^{\mathfrak{p}_0}_i,\mathcal{A}'^{\mathfrak{p}_0}/\mathcal{J}'^{\mathfrak{p}_0},\mathcal{B}^{\mathfrak{p}_0},\operatorname{Tor}_j^{\mathcal{A}^{\mathfrak{p}_0}}(\mathcal{B}_{\mathfrak{p}_0},\mathcal{A}^{\mathfrak{p}_0}/\mathcal{J}_i^{\mathfrak{p}_0}),\operatorname{Tor}_j^{\mathcal{A}'^{\mathfrak{p}_0}}(\mathcal{A}'^{\mathfrak{p}_0}/\mathcal{J}'^{\mathfrak{p}_0}_i,\mathcal{B}_{\mathfrak{p}_0}))$ localized to $\mathfrak{p}_0^0$ are free both as left and as right modules over $\mathbb{C}[\mathfrak{p}_0^0]$ and moreover, for any $c\in\mathfrak{p}_0^0$, the following holds:*

1. $\mathcal{J}_i^c\mathcal{J}_j^c=\mathcal{J}^c_{\max(i,j)},\mathcal{J}_i'^c\mathcal{J}_j'^c=\mathcal{J}'^c_{\max(i,j)}$.
2. *For all i,j, we have $\mathcal{J}_i'^c\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B}^c,\mathcal{A}^c/\mathcal{J}_i^c)=\operatorname{Tor}_j^{\mathcal{A}'^c}(\mathcal{A}'^c/\mathcal{J}_i'^c,\mathcal{B}^c)\mathcal{J}_i^c=0$.*
3. *We have $\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B}_c,\mathcal{A}^c/\mathcal{J}_i^c)=0$ for $j<d_i$.*
4. *We have $\mathcal{J}'^c_{i-1}\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B}^c,\mathcal{A}^c/\mathcal{J}_i^c)=0$ for $j>d_i$.*
5. *Set $\mathcal{B}_i^c:=\operatorname{Tor}_{d_i}^{\mathcal{A}^c}(\mathcal{B}^c,\mathcal{A}^c/\mathcal{J}_i^c)$. Then $\mathcal{J}'^c_{i-1}\mathcal{B}_i^c=\mathcal{B}_i^c$.*
6. *The kernel and the cokernel of the natural homomorphism*
$$\mathcal{B}_i^c\otimes_{\mathcal{A}^c}\operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}_i^c,\mathcal{A}'^c/\mathcal{J}_i'^c)\to\mathcal{A}'^c/\mathcal{J}_i'^c$$
are annihilated by $\mathcal{J}'^c_{i-1}$ on the left and on the right. Similarly, the kernel and the cokernel of the natural homomorphism
$$\operatorname{Hom}_{\mathcal{A}^c}(\mathcal{B}_i^c,\mathcal{A}^c/\mathcal{J}_i^c)\otimes_{\mathcal{A}'^c}\mathcal{B}_i^c\to\mathcal{A}^c/\mathcal{J}_i^c$$
are annihilated on the left and on the right by $\mathcal{J}^c_{i-1}$.

We remark that under the freeness condition we have imposed on $\mathfrak{p}_0^0$, the bimodules with superscript c are the specializations of those with superscript $\mathfrak{p}_0$ provided $c \in \mathfrak{p}_0^0$. For example, $\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B}_c, \mathcal{A}^c/\mathcal{J}_i^c) = \operatorname{Tor}_j^{\mathcal{A}^{\mathfrak{p}_0}}(\mathcal{B}_{\mathfrak{p}_0}, \mathcal{A}^{\mathfrak{p}_0}/\mathcal{J}_i^{\mathfrak{p}_0})_c$. The existence of an open subset satisfying the freeness condition follows from (2) of Corollary 3.6.

The scheme of the proof of Theorem 10.1 is as follows. We first prove the theorem for the algebras $\bar{A}_\kappa(n), \bar{A}_{\kappa'}(n)$, where $m = n$, in Section 10.3. In this case we just have one proper ideal in either of these two algebras. Then, in Section 10.4, we construct the ideals $\mathcal{J}_i^{\mathfrak{p}_0}, \mathcal{J}_i'^{\mathfrak{p}_0}$ in general. After that we prove (2)-(6) of Theorem 10.1, first, for a Weil generic parameter c and then for a Zariski generic parameter, Section 10.5.

Of course, $c^\circ + (C_\theta \cap \ker\delta \cap \mathbb{Z}^{Q_0})$ intersects $\mathfrak{p}_0^0$. We remark that for $c \in [c^\circ + (C_\theta \cap \ker\delta \cap \mathbb{Z}^{Q_0})] \cap \mathfrak{p}_0^0$, thanks to Lemma 5.2, the functor $\mathcal{B}^c \otimes^L_{\mathcal{A}^c} \bullet$ is just $\mathfrak{WC}_{\theta\to\theta'}$.

Theorem 10.1 is used to prove that $\mathfrak{WC}_{\theta\to\theta'} : D^b(\mathcal{A}_{\lambda^\circ}(v)\text{-mod}) \to D^b(\mathcal{A}_{\lambda'^\circ}(v)\text{-mod})$ is a perverse equivalence.

Let us recall the general definition. Let $\mathcal{C}, \mathcal{C}'$ be two abelian categories equipped with filtrations $\{0\} = \mathcal{C}_{N+1} \subsetneq \mathcal{C}_N \subsetneq \mathcal{C}_{N-1} \subsetneq \ldots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0 = \mathcal{C}, \{0\} = \mathcal{C}'_{N+1} \subsetneq \mathcal{C}'_N \subsetneq \ldots \subsetneq \mathcal{C}'_1 \subsetneq \mathcal{C}'_0 = \mathcal{C}'$ by Serre subcategories. Following Chuang and Rouquier, [R1, Section 2.6], we say that a derived equivalence $\varphi : D^b(\mathcal{C}) \to D^b(\mathcal{C}')$ is perverse with respect to the filtrations above if

(i) φ restricts to an equivalence $D^b_{\mathcal{C}_i}(\mathcal{C}) \to D^b_{\mathcal{C}'_i}(\mathcal{C}')$, where we write $D^b_{\mathcal{C}_i}(\mathcal{C})$ for the full subcategory of $D^b(\mathcal{C})$ consisting of all complexes with homology in $\mathcal{C}_i$.
(ii) $H_j(\varphi M) = 0$ for $M \in \mathcal{C}_i$ and $j < i$.
(iii) The functor $M \mapsto H_i(\varphi M)$ induces an equivalence $\mathcal{C}_i/\mathcal{C}_{i+1} \xrightarrow{\sim} \mathcal{C}'_i/\mathcal{C}'_{i+1}$. Moreover, $H_j(\varphi M) \in \mathcal{C}'_{i+1}$ for $j > i$ and $M \in \mathcal{C}_i$.

We remark that, thanks to (iii), a perverse equivalence induces a natural bijection between the simple objects in $\mathcal{C}$ and $\mathcal{C}'$. We will write $S \mapsto S'$ for this bijection.

Theorem 10.2. *Set* $\mathcal{C} := \mathcal{A}_{\lambda^\circ}(v)\text{-mod}, \mathcal{C}' := \mathcal{A}_{\lambda'^\circ}(v)\text{-mod}$. *Define* $\mathcal{C}_i$ *to be the subcategory of all modules in* $\mathcal{C}$ *annihilated by* $\mathcal{J}_{q+1-\lfloor i/(m-1)\rfloor}$ *(this is a Serre subcategory by (1) of Theorem 10.1) and* $\mathcal{C}'_i \subset \mathcal{C}_i$ *analogously. Then, perhaps after replacing* λ° *with* $\lambda^\circ + \psi$ *for* $\psi \in C \cap \ker\delta \cap \mathbb{Z}^{Q_0}$, *the following holds.*

(1) $\mathfrak{WC}_{\theta\to\theta'}$ *is a perverse equivalences with respect to these filtrations.*
(2) *The induced equivalence* $\mathcal{C}_{j(m-1)}/\mathcal{C}_{j(m-1)+1} \to \mathcal{C}'_{j(m-1)}/\mathcal{C}'_{j(m-1)+1}$ *is given by* $\mathcal{B}^c_{q+1-j} \otimes_{\mathcal{A}^c} \bullet$. *Moreover, for a simple* $S \in \mathcal{C}_{j(m-1)} \setminus \mathcal{C}_{j(m-1)+1}$, *the head of* $\mathcal{B}^c_{q+1-j} \otimes_{\mathcal{A}^c} S$ *coincides with* S'.
(3) *The bijection* $S \mapsto S'$ *preserves the associated varieties of the annihilators.*

Remark 10.3. A direct analog of Theorem 10.2 holds for all wall-crossing functors through faces of classical chambers. This is proved in a subsequent paper, [L16], by the second named author. In particular, the short wall-crossing functors through real walls studied in Section 9 and the bijection between singular objects $L \mapsto L'$ we considered from Proposition 5.14 is a restriction of the bijection from (3) of generalized Theorem 10.2. Still, Proposition 5.14 cannot be entirely replaced by a generalization of Theorem 10.2 as the former relates the wall-crossing functor to the categorification functors E_α, F_α.

10.2. **HC bimodules for Symplectic reflection algebras.** Let $V, \Gamma, \mathcal{H}_c$ etc. have the same meaning as in 2.2.6. In this section we recall some known facts about HC bimodules over the algebras $\mathcal{H}_c$ and $e\mathcal{H}_c e$. The most important cases are $\Gamma = \Gamma_n, S_n$.

Let us recall a description of the symplectic leaves in V/Γ. The leaves are parameterized by conjugacy classes of stabilizers for the Γ-action on V. Namely, to a stabilizer Γ' we assign the image of $\{v \in V | \Gamma_v = \Gamma'\}$ in V/Γ.

Below we will need a property of restriction functors in the SRA setting. A similar property was obtained in [L5, Proposition 3.7.2] for a related, "upgraded", restriction functor.

Take a symplectic leaf $\mathcal{L} \subset V/\Gamma_n$ and consider the full subcategory $\mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\mathfrak{P}}(v)) \subset \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}(v))$ consisting of all HC bimodules $\mathcal{M}$ such that $\mathrm{V}(\mathcal{M}) \cap \mathcal{M}_0(v) \subset \overline{\mathcal{L}}$. Similarly, define the subcategory $\mathrm{HC}_{fin}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$.

Proposition 10.4. *For $x \in \mathcal{L}$, the functor $\bullet_{\dagger,x} : \mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\mathfrak{P}}(v)) \to \mathrm{HC}_{fin}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v}))$ admits a right adjoint $\bullet^{\dagger,x} : \mathrm{HC}_{fin}(\hat{\mathcal{A}}_{\mathfrak{P}}(\hat{v})) \to \mathrm{HC}_{\overline{\mathcal{L}}}(\mathcal{A}_{\mathfrak{P}}(v))$.*

Proof. As in the proof of [L5, Proposition 3.7.2], we reduce the proof to showing the following claim: a Poisson $\mathbb{C}[\overline{\mathcal{L}}]$-submodule of $\mathbb{C}[\mathcal{L}]^{\wedge_x}$ that is finitely generated over $\mathbb{C}[\overline{\mathcal{L}}]$ and is weakly equivariant under the action of $\mathbb{C}^\times$ on $\mathcal{L}$ is contained in $\mathbb{C}[\mathcal{L}] \subset \mathbb{C}[\mathcal{L}]^{\wedge_x}$. This is a special case of [L12, Lemma 3.9]. □

Here is an application of the restriction functors obtained in [L5, Section 5]. Consider the case when $\Gamma = \mathfrak{S}_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^{n-1}$ is the reflection representation of $\mathfrak{S}_n$. The resulting algebra $\mathcal{H}_\kappa(n)$ is known as the Rational Cherednik algebra of type A, in our previous notation $\mathcal{H}_{\kappa,\varnothing}(n) = D(\mathbb{C}) \otimes \mathcal{H}_\kappa(n), e\mathcal{H}_\kappa(n)e = \bar{\mathcal{A}}_\kappa(n)$. One can describe all two-sided ideals in $\mathcal{H}_\kappa(n)$, see [L5, Section 5.8].

Proposition 10.5. *If κ is irrational or $\kappa = \frac{r}{m}$, where $\mathrm{GCD}(r,m) = 1$ and $m > n$, then the algebra $\mathcal{H}_\kappa(n)$ is simple. Otherwise, there are $q := \lfloor n/m \rfloor$ proper ideals that form a chain: $\{0\} = \mathcal{J}_{q+1} \subsetneq \mathcal{J}_q \subsetneq \mathcal{J}_{q-1} \subsetneq \ldots \subsetneq \mathcal{J}_1 \subsetneq \mathcal{J}_0 := \mathcal{H}_\kappa(n)$. The associated variety of $\mathcal{H}_\kappa/\mathcal{J}_i$ is the closure of the symplectic leaf associated to the parabolic subgroup $\mathfrak{S}_m^{q+1-i} \subset \mathfrak{S}_n$. Moreover, we have $\mathcal{J}_i\mathcal{J}_j = \mathcal{J}_{\max(i,j)}$.*

We will also need the following lemma that is a consequence of Proposition 10.5 and Lemma 3.2. Consider a point x in the symplectic leaf of $(\mathfrak{h} \oplus \mathfrak{h}^*)/\mathfrak{S}_n$ corresponding to the parabolic subgroup $\mathfrak{S}_m^q \subset \mathfrak{S}_n$, where $q = \lfloor n/m \rfloor$.

Lemma 10.6. *The functor $\bullet_{\dagger,x}$ is faithful.*

Let us use the notation from Proposition 10.5. Set $\hat{\mathcal{H}} = H_\kappa(m)$ and let $\hat{\mathcal{J}}$ be the only proper two-sided ideal in $\hat{\mathcal{H}}$. Let $\hat{\mathcal{J}}_i$ denote the two-sided ideal in $\hat{\mathcal{H}}^{\otimes q}$ that is obtained as the sum of all q-fold tensor products of i copies of $\hat{\mathcal{J}}$ and $q-i$ copies of $\hat{\mathcal{H}}$ (in all possible orders).

Lemma 10.7. *Let $\mathcal{J}$ be a two-sided ideal in $\hat{\mathcal{H}}^{\otimes q}$. If $\mathrm{V}(\hat{\mathcal{H}}^{\otimes q}/\mathcal{J}) = \mathrm{V}(\hat{\mathcal{H}}^{\otimes q}/\hat{\mathcal{J}}_i)$, then $\mathcal{J} = \hat{\mathcal{J}}_i$.*

Proof. Let x be a generic point in an irreducible component in $\mathrm{V}(\hat{\mathcal{H}}^{\otimes q}/\hat{\mathcal{J}}_i)$. Recall that these components are labelled by i-element subsets of $\{1,\ldots,q\}$. Then $\mathcal{J}_{\dagger,x}$ is a proper ideal in $\hat{\mathcal{H}}^{\otimes i}$. It follows that $\mathcal{J}$ lies in the kernel of the projection of $\hat{\mathcal{H}}^{\otimes q}$ to the product of $q-i$ copies of $\hat{\mathcal{H}}$ and i copies of $\hat{\mathcal{H}}/\hat{\mathcal{J}}_i$ (in all possible orders). But by Step 3 in the proof of [L5, Theorem 5.8.1], the intersection of these kernels is $\hat{\mathcal{J}}_i$. So $\mathcal{J} \subset \hat{\mathcal{J}}_i$. Moreover, by *loc.cit.*, the kernels are precisely the minimal prime ideals containing $\mathcal{J}$. So $\hat{\mathcal{J}}_i$ is the radical of $\mathcal{J}$. By *loc.cit.*, $\hat{\mathcal{J}}_i^2 = \hat{\mathcal{J}}_i$. Therefore $\hat{\mathcal{J}}_i = \mathcal{J}$. □

10.3. **Type A: case $\kappa=\frac{r}{n}$.** Here we assume that $n=m,\kappa=\frac{r}{n}$ with $\mathrm{GCD}(r,n)=1$ and $\kappa\not\in(-1,0)$. In this case we have just one proper ideal $\mathcal{J}\subset\mathcal{H}:=\mathcal{H}_\kappa(n)$ and the quotient $\mathcal{H}/\mathcal{J}$ is finite dimensional, this is a special case of Proposition 10.5. The algebra $\bar{\mathcal{A}}_\kappa(m)$ is Morita equivalent to $\mathcal{H}$.

Lemma 10.8. *Claims (2)-(6) of Theorem 10.1 hold for the algebras $\bar{\mathcal{A}}_\kappa(m)$.*

Proof. These claims amount to the following two claims ((*) implies those claims for $i=1$, while (**) implies them for $i=0$):

- (*) We have $\operatorname{Tor}_j^{\mathcal{H}}(\mathcal{B},\mathcal{H}/\mathcal{J})=\operatorname{Tor}_j^{\mathcal{H}'}(\mathcal{H}'/\mathcal{J}',\mathcal{B})=0$ if $j\neq n-1$. Furthermore, $\operatorname{Tor}_{n-1}^{\mathcal{H}}(\mathcal{B},\mathcal{H}/\mathcal{J})=\operatorname{Tor}_{n-1}^{\mathcal{H}'}(\mathcal{H}'/\mathcal{J}',\mathcal{B})=\operatorname{Hom}_{\mathbb{C}}(L,L')$, where we write L (resp., L') for the simple finite dimensional $\mathcal{H}$-module (resp., $\mathcal{H}'$-module).
- (**) The kernels and cokernels of the natural homomorphisms $\mathcal{B}\otimes_{\mathcal{H}}\operatorname{Hom}_{\mathcal{H}'}(\mathcal{B},\mathcal{H}')\to\mathcal{H}',\operatorname{Hom}_{\mathcal{H}}(\mathcal{B},\mathcal{H})\otimes_{\mathcal{H}'}\mathcal{B}\to\mathcal{H}$ are finite dimensional.

The Tor vanishing statement in (*) is a consequence of (1) of Proposition 5.13 (by Remark 8.5, $\mathcal{B}\otimes^L_{\mathcal{H}'}\bullet$ is still the wall-crossing functor for the categories of right modules). The category of finite dimensional $\mathcal{H}$-modules (resp., of finite dimensional $\mathcal{H}'$-modules) is a semisimple category with a single indecomposable object L (resp., L'). By (2) of Proposition 5.13, $\mathcal{B}\otimes^L_{\mathcal{H}}L=L'[n-1]$ and $L'\otimes^L_{\mathcal{H}'}\mathcal{B}=L[n-1]$. So the equality for the Tor_{n-1}'s follows from the previous sentence and isomorphisms $\mathcal{H}/\mathcal{J}=\operatorname{Hom}_{\mathbb{C}}(L,L),\mathcal{H}'/\mathcal{J}'=\operatorname{Hom}_{\mathbb{C}}(L',L')$.

Let us proceed to (**). Apply the functor $\bullet_{\dagger,x}$ to the homomorphisms of interest, where $x\in\mathcal{M}_0(v)$ is generic. For the homomorphism

$$\mathcal{B}\otimes_{\mathcal{H}}\operatorname{Hom}_{\mathcal{H}'}(\mathcal{B},\mathcal{H}')\to\mathcal{H}'$$

we get a natural homomorphism

$$\mathcal{B}_{\dagger,x}\otimes_{\mathcal{H}_{\dagger,x}}\operatorname{Hom}_{\mathcal{H}'_{\dagger,x}}(\mathcal{B}_{\dagger,x},\mathcal{H}'_{\dagger,x})\to\mathcal{H}_{\dagger,x}.$$

But the algebras and bimodules involved are all just $\mathbb{C}$. So we see that the latter homomorphism is an isomorphism. It follows that the kernel and the cokernel of $\mathcal{B}\otimes_{\mathcal{H}}\operatorname{Hom}_{\mathcal{H}'}(\mathcal{B},\mathcal{H}')\to\mathcal{H}'$ are killed by $\bullet_{\dagger,x}$. So they have proper associated varieties and hence are finite dimensional. $\square$

10.4. **Chain of ideals.** For a partition $\mu=(\mu_1,\ldots,\mu_k)$ with $|\mu|\leqslant n$ set $\bar{\mathcal{A}}(\mu)=\bigotimes_{i=1}^k\bar{\mathcal{A}}_\kappa(\mu_i)$ and define $\bar{\mathcal{A}}'(\mu)$ similarly. Consider the restriction functors $\bullet_{\dagger,\mu}:\mathrm{HC}(\mathcal{A}^{\mathfrak{p}_0})\to\mathrm{HC}(\mathbb{C}[\mathfrak{p}_0]\otimes\bar{\mathcal{A}}(\mu)),\mathrm{HC}(\mathcal{A}'^{\mathfrak{p}_0}\text{-}\mathcal{A}^{\mathfrak{p}_0})\to\mathrm{HC}(\mathbb{C}[\mathfrak{p}_0]\otimes\bar{\mathcal{A}}'(\mu)\text{-}\mathbb{C}[\mathfrak{p}_0]\otimes\bar{\mathcal{A}}(\mu))$ etc. Let $\bullet^{\dagger,\mu}$ denote the right adjoint functor (defined on bimodules that are finitely generated over $\mathbb{C}[\mathfrak{p}_0]$ by Proposition 10.4). Recall, Proposition 10.5, that the ideals in the algebra $\bar{\mathcal{A}}(qm)$ form a chain: $\bar{\mathcal{A}}(qm)=\bar{\mathcal{J}}_0(qm)\supsetneq\bar{\mathcal{J}}_1(qm)\supsetneq\ldots\supsetneq\bar{\mathcal{J}}_q(qm)\supsetneq\bar{\mathcal{J}}_{q+1}(qm)=\{0\}$. We set $\bar{\mathcal{J}}_i(m^q):=\bar{\mathcal{J}}_i(qm)_{\dagger,m^q}$. These are precisely the ideals appearing before Lemma 10.7.

We set $\mathcal{J}_i^{\mathfrak{p}_0}$ to be the kernel of the natural map

$$\mathcal{A}^{\mathfrak{p}_0}\to(\mathbb{C}[\mathfrak{p}_0]\otimes[\bar{\mathcal{A}}(m)/\bar{\mathcal{J}}_1(m)]^{\otimes q+1-i})^{\dagger,(m^{q+1-i})},$$

and define $\mathcal{J}_i'^{\mathfrak{p}_0}$ similarly.

Remark 10.9. Note that, by the definition of $\mathcal{J}_i^{\mathfrak{p}_0}$ the following is true. If $\mathcal{J}\subset\mathcal{A}^{\mathfrak{p}_0}$ is such that $\mathcal{J}_{\dagger,(m^{q+1-i})}$ is in the kernel of $\mathbb{C}[\mathfrak{p}_0]\otimes\bar{\mathcal{A}}(m^{q+1-i})\twoheadrightarrow\mathbb{C}[\mathfrak{p}_0]\otimes[\bar{\mathcal{A}}(m)/\bar{\mathcal{J}}_1(m)]^{\otimes q+1-i}$, then $\mathcal{J}\subset\mathcal{J}_i^{\mathfrak{p}_0}$.

We are going to establish some properties of these ideals. First, let us describe properties that hold for all parameters c.

Lemma 10.10. *The following is true.*

(a) $(\mathcal{J}_i^c)_{\dagger,(m^{q+1-i})}$ *coincides with the maximal ideal of* $\bar{\mathcal{A}}(m^{q+1-i})$.

(b) $\mathrm{V}(\mathcal{A}^c/\mathcal{J}_i^c)$ *coincides with* $\overline{\mathcal{L}}_{(m^{q+1-i})}$, *where the latter is the closure of the symplectic leaf corresponding to the subgroup* $\mathfrak{S}_m^{q+1-i}\subset\Gamma_n$.

(c) $(\mathcal{J}_i^c)_{\dagger,(m^q)}=\bar{\mathcal{J}}_i(m^q)$. *Moreover,* $(\mathcal{J}_i^c)_{\dagger,(qm)}=\bar{\mathcal{J}}_i(qm)$.

(d) $\mathcal{J}_q^{\mathfrak{p}_0}\subset\mathcal{J}_{q-1}^{\mathfrak{p}_0}\subset\ldots\subset\mathcal{J}_1^{\mathfrak{p}_0}$.

Similar claims hold for $\mathcal{J}_i'^c,\mathcal{J}_i'^{\mathfrak{p}_0}$.

Proof. The ideal $(\mathcal{J}_i^c)_{\dagger,(m^{q+1-i})}\subset\bar{\mathcal{A}}(m^{q+1-i})$ is contained in the maximal ideal as the latter is the kernel of $\bar{\mathcal{A}}(m^{q+1-i})\twoheadrightarrow(\bar{\mathcal{A}}(m)/\bar{\mathcal{J}}_1(m))^{\otimes(q+1-i)}$. The inclusion $\mathrm{V}(\mathcal{A}^c/\mathcal{J}_i^c)\subset\overline{\mathcal{L}}_{(m^{q+1-i})}$ follows from Proposition 10.4. By Lemma 3.9, $\mathrm{V}(\bar{\mathcal{A}}^{\otimes q+1-i}/\mathcal{J}^c_{i,\dagger,(m^{q+1-i})})$ is a point. The equality in (a) follows from Lemma 10.7. In its turn, the equality in (a) implies the equality in (b).

(b) implies that $\mathrm{V}(\bar{\mathcal{A}}(m^q)/(\mathcal{J}_i^c)_{\dagger,(m^q)})=\mathrm{V}(\bar{\mathcal{A}}(m^q)/\bar{\mathcal{J}}_i(m^q))$. Lemma 10.7 yields $(\mathcal{J}_i^c)_{\dagger,(m^q)}=\bar{\mathcal{J}}_i(m^q)$. The equality $(\mathcal{J}_i^c)_{\dagger,(qm)}=\bar{\mathcal{J}}_i(qm)$ is proved similarly using Proposition 10.5.

Let us prove (d). We remark that $(\mathcal{J}_i^c)_{\dagger,(m^{q+2-i})}$ is a proper ideal because its associated variety (computed using Lemma 3.9) is proper. Hence $(\mathcal{J}_i^c)_{\dagger,(m^{q+2-i})}$ is contained in the maximal ideal of $\bar{\mathcal{A}}(m^{q+2-i})$. It follows that $(\mathcal{J}_i^{\mathfrak{p}_0})_{\dagger,(m^{q+2-i})}$ lies in the kernel of the epimorphism $\mathbb{C}[\mathfrak{p}_0]\otimes\bar{\mathcal{A}}(m^{q+2-i})\twoheadrightarrow\mathbb{C}[\mathfrak{p}_0]\otimes(\bar{\mathcal{A}}(m)/\bar{\mathcal{J}}_1(m))^{\otimes(q+2-i)}$. The inclusion $\mathcal{J}_i^{\mathfrak{p}_0}\subset\mathcal{J}_{i-1}^{\mathfrak{p}_0}$ follows from Remark 10.9. □

Now let us analyze what happens when c is Weil generic.

Lemma 10.11. *Let c be Weil generic. Then the following is true:*

(1) *The functor* $\bullet_{\dagger,(m^q)}$ *is faithful.*

(2) *The ideals* $\mathcal{J}_i^c, i=1,\ldots,q$, *exhaust all proper ideals in* $\mathcal{A}^c$.

(3) $\mathcal{J}_i^c\mathcal{J}_j^c=\mathcal{J}_{\max(i,j)}^c$.

Proof. Let us show that for a Weil generic c, the algebra $\mathcal{A}^c$ has no finite dimensional representations. Similarly to the proof of Proposition 3.14, we see that otherwise there is a two-sided ideal $\mathbf{J}\subset\mathcal{A}^{\mathfrak{p}_0}$ such that $\mathcal{A}^{\mathfrak{p}_0}/\mathbf{J}$ is generically flat and finite over $\mathbb{C}[\mathfrak{p}_0]$ and $\mathrm{Supp}^r_{\mathfrak{P}}(\mathcal{A}^{\mathfrak{p}_0}/\mathbf{J})=\mathfrak{p}_0$. So, by Proposition 3.13, for the Poisson ideal $\operatorname{gr}\mathbf{J}\subset\mathbb{C}[\mathcal{M}_{\mathfrak{p}_0}(v)]$ we have $\mathrm{Supp}_{\mathfrak{p}}(\mathbb{C}[\mathcal{M}_{\mathfrak{p}_0}(v)]/\operatorname{gr}\mathbf{J})=\mathfrak{p}_0$. It follows that, for every $p\in\mathfrak{p}_0$, the variety $\mathcal{M}_p(v)$ contains a point that is a symplectic leaf. We remark that $\mathcal{M}_{\mathfrak{p}_0}(v)=\mathcal{M}_{\mathfrak{p}_0}(\delta)^n/\mathfrak{S}_n$ ((the power is taken over $\mathfrak{p}_0$)), this follows from the description of $\mathcal{M}_{\mathfrak{p}_0}(v)$ as the generalized Calogero-Moser space, see [EG, Section 11]. For p generic, $\mathcal{M}_p(\delta)$ is smooth and symplectic and so the minimal dimension of a symplectic leaf in $\mathcal{M}_p(v)$ is 2. We arrive at a contradiction that shows that $\mathcal{A}^c$ has no finite dimensional representations provided c is Weil generic.

Now we are in position to prove (1). This boils down to checking that the associated variety of any HC $\mathcal{A}^c$-bimodule (or HC $\mathcal{A}'^c$-$\mathcal{A}^c$-bimodule, etc.) contains $\mathcal{L}_{(m^q)}$. First of all, let us show that the associated variety contains $\mathcal{L}_n$, the symplectic leaf corresponding to $\mathfrak{S}_n\subset\Gamma_n$. Indeed, the slice algebra $\bar{\mathcal{A}}_\lambda(\hat{v})$ for any leaf not containing $\mathcal{L}_n$ has a tensor factor isomorphic to $e\mathcal{H}_{\kappa,c}(n')e$ with nonzero $n'\leqslant n$. But that algebra has no finite dimensional irreducible representations by the first paragraph of the proof, a contradiction. So we see that the associated variety is contained in $\mathcal{L}_n$. Hence it is $\mathcal{L}_\mu$ for some partition μ. The slice algebra $\bar{\mathcal{A}}_\lambda(\hat{v})$ is the product $\bigotimes_{i=1}^k e\mathcal{H}_\kappa(\mu_i)e$.

That the associated variety contains $\mathcal{L}_{(m^q)}$ follows from the fact that the algebra $\mathcal{H}_\kappa(n')$ has a finite dimensional irreducible representation if and only if κ has denominator precisely n'. The proof of faithfulness of $\bullet_{\dagger,(m^q)}$ is now complete.

Let us proceed to the proof of (2) and (3). By (b) of Lemma 10.10, $\mathrm{V}(\mathcal{A}^c/\mathcal{J}_i^c) = \overline{\mathcal{L}}_{(m^{q+1-i})}$. The functor $\bullet_{\dagger,(qm)}$ is faithful, this follows from the argument in the previous paragraph. So the map $\mathcal{J}\mapsto\mathcal{J}_{\dagger,(qm)}$ embeds the poset of two-sided ideals in $\mathcal{A}^c$ into that for $\bar{\mathcal{A}}(qm)$. (2) follows from here and Proposition 10.5. To check (3) note that $\bullet_{\dagger,(qm)}$ respects the products of ideals and again use Proposition 10.5. □

Now let us transfer some of the properties in Lemma 10.11 to the case when c is only Zariski generic.

Lemma 10.12. *We have $\mathcal{J}_i^c\mathcal{J}_j^c = \mathcal{J}_{\max(i,j)}^c$ for c in some non-empty Zariski open subset of $\mathfrak{p}_0$.*

Proof. Consider the quotient $\mathcal{J}^{\mathfrak{p}_0}_{\max(i,j)}/\mathcal{J}_i^{\mathfrak{p}_0}\mathcal{J}_j^{\mathfrak{p}_0}$. By (3) of Lemma 10.11, its specialization to a Weil generic c is zero. It follows from (2) of Corollary 3.6 that the specialization of this HC bimodule to a Zariski generic $c\in\mathfrak{p}_0$ is zero. This implies our claim. □

10.5. **Proof of Theorem 10.1.** We write $\bar{\mathcal{B}}$ for the wall-crossing $\bar{\mathcal{A}}_{\kappa'}(m)$-$\bar{\mathcal{A}}_\kappa(m)$-bimodule. Without restrictions on c, we know that

$$\begin{aligned}
&\mathcal{B}^c_{\dagger,(m^{q+1-i})} = \bar{\mathcal{B}}^{\otimes q+1-i},\\
&\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B},\mathcal{A}^c/\mathcal{J}_i^c)_{\dagger,(m^{q+1-i})} = \operatorname{Tor}_j^{\bar{\mathcal{A}}(m^{q+1-i})}(\bar{\mathcal{B}}^{\otimes q+1-i},(\bar{\mathcal{A}}(m)/\mathcal{J}(m))^{\otimes q+1-i}),\\
&\operatorname{Tor}_j^{\mathcal{A}'^c}(\mathcal{A}'^c/\mathcal{J}_i'^c,\mathcal{B})_{\dagger,(m^{q+1-i})} = \operatorname{Tor}_j^{\bar{\mathcal{A}}'(m^{q+1-i})}((\bar{\mathcal{A}}'(m)/\mathcal{J}'(m))^{\otimes q+1-i},\bar{\mathcal{B}}^{\otimes q+1-i}).
\end{aligned}\tag{10.1}$$

The first equality is a special case of (3.11). The second and third equalities follows from the first and Lemma 3.10.

Proof of Theorem 10.1. First, we assume that c is Weil generic. By (b) of Lemma 10.10 combined with (2) of Lemma 10.11, for any HC $\mathcal{A}'^c$-$\mathcal{A}^c$ bimodule $\mathcal{X}$ the following are equivalent

- $\mathcal{X}_{\dagger,(m^{q+1-i})}=0$,
- $\mathcal{X}\mathcal{J}^c_{i-1}=0$,
- $\mathcal{J}'^c_{i-1}\mathcal{X}=0$.

This, combined with (10.1) and Section 10.3, yields (4) and (6). Further, the following conditions are equivalent as well:

- $\dim\mathcal{X}_{\dagger,(m^{q+1-i})}<\infty$,
- $\mathcal{X}\mathcal{J}^c_i=0$,
- $\mathcal{J}'^c_i\mathcal{X}=0$.

This yields (2).

Let us prove (3) and (5). Suppose that (3) is false. Pick the minimal i such that there is $j<d_i$ with $\mathcal{X}:=\operatorname{Tor}_j^{\mathcal{A}^c}(\mathcal{B}^c,\mathcal{A}^c/\mathcal{J}_i^c)\neq 0$. Next, let j be minimal for the given i. Since $\mathcal{X}_{\dagger,(m^{q+1-i})}=0$, we see that

$$\mathcal{X}\mathcal{J}^c_{i-1}=0.\tag{10.2}$$

Consider the derived tensor product

$$\mathcal{B}^c \otimes^L_{\mathcal{A}^c} \mathcal{A}^c/\mathcal{J}^c_{i-1} = (\mathcal{B}^c \otimes^L_{\mathcal{A}^c} \mathcal{A}^c/\mathcal{J}^c_i) \otimes^L_{\mathcal{A}^c/\mathcal{J}^c_i} \mathcal{A}^c/\mathcal{J}^c_{i-1}. \tag{10.3}$$

By the choice of j, the jth homology of the right hand side of (10.3) equals $\mathcal{X} \otimes_{\mathcal{A}^c/\mathcal{J}^c_i} \mathcal{A}^c/\mathcal{J}^c_{i-1} = \mathcal{X}/\mathcal{X}\mathcal{J}^c_{i-1}$. The latter equals $\mathcal{X}$ by (10.2). Since $j < d_{i-1}$ and the left hand side of (10.3) has non-vanishing jth homology, we get a contradiction with our choice of i. This proves $\mathcal{X} = 0$.

The equality $\operatorname{Tor}^{\mathcal{A}'^c}_j(\mathcal{A}'^c/\mathcal{J}'^c_i, \mathcal{B}) = 0$ for $j < d_i$ is proved in the same way (using that $\mathcal{B}$ is a long wall-crossing bimodule also when viewed as a $\mathcal{A}^{c,opp}$-$\mathcal{A}'^{c,opp}$-bimodule, Remark 8.5). This completes the proof of (3).

Let us proceed to (5) and prove $\mathcal{B}^c_i\mathcal{J}^c_{i-1} = \mathcal{B}^c_i$. Assume the converse, then $\mathcal{B}^c_i \otimes \mathcal{A}^c/\mathcal{J}^c_{i-1} \neq 0$. Similarly to the previous paragraph, this implies that $\operatorname{Tor}^{\mathcal{A}^c}_{d_i}(\mathcal{B}^c, \mathcal{A}^c/\mathcal{J}^c_{i-1}) \neq \{0\}$ that contradicts (3). This completes the proof of Theorem 10.1 for a Weil generic c.

Let us prove (2)-(6) for a Zariski generic c. We will do (2), the other claims are similar. Consider the HC $\mathcal{A}'^{\mathfrak{p}_0}$-$\mathcal{A}^{\mathfrak{p}_0}$ bimodule $\mathcal{J}'^{\mathfrak{p}_0}_i \operatorname{Tor}^{\mathcal{A}^{\mathfrak{p}_0}}_j(\mathcal{B}^{\mathfrak{p}_0}, \mathcal{A}^{\mathfrak{p}_0}/\mathcal{J}^{\mathfrak{p}_0}_i)$. Its specialization to a Zariski generic parameter c coincides with $\mathcal{J}'^c_i \operatorname{Tor}^{\mathcal{A}^c}_j(\mathcal{B}^c, \mathcal{A}^c/\mathcal{J}^c_i)$. So a Weil generic specialization of this bimodule vanishes. Therefore the same is true for a Zariski generic specialization, this is a consequence of (2) of Corollary 3.6. $\square$

10.6. **Proof of Theorem 10.2.** Let us check (i) in the definition of a perverse equivalence. Recall that $\mathfrak{WC}_{\theta\to\theta'}$ is $\mathcal{B}^c \otimes^L_{\mathcal{A}^c} \bullet$ and hence $\mathfrak{WC}^{-1}_{\theta\to\theta'}$ is $R\operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}^c, \bullet)$. For example, let us prove $\mathfrak{WC}^{-1}_{\theta\to\theta'} D^b_{\mathcal{C}'_i}(\mathcal{C}') \subset D^b_{\mathcal{C}_i}(\mathcal{C})$. For M' annihilated by $\mathcal{J}'^c_i$, we have $R\operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}^c, M') = R\operatorname{Hom}_{\mathcal{A}'^c/\mathcal{J}'^c_i}(\mathcal{A}'^c/\mathcal{J}'^c_i \otimes^L_{\mathcal{A}'^c} \mathcal{B}^c, M')$. Now we use (2) of Theorem 10.1 which says, in particular, that all homology of $\mathcal{A}'^c/\mathcal{J}'^c_i \otimes^L_{\mathcal{A}'^c} \mathcal{B}^c$ are annihilated by $\mathcal{J}^c_i$ on the right. This checks (i).

(ii) follows from (3) of Theorem 10.1 and the observation that, for M annihilated by $\mathcal{J}^c_i$ we have $\mathcal{B}^c \otimes^L_{\mathcal{A}^c} M = (\mathcal{B}^c \otimes^L_{\mathcal{A}^c} \mathcal{A}^c/\mathcal{J}^c_i) \otimes^L_{\mathcal{A}^c/\mathcal{J}^c_i} M$.

Let us prove (iii). By (6) of Theorem 10.1, the functor $\mathcal{B}^c_i \otimes_{\mathcal{A}^c/\mathcal{J}^c_i} \bullet : \mathcal{A}^c/\mathcal{J}^c_i\text{-mod} \to \mathcal{A}'^c/\mathcal{J}'^c_i\text{-mod}$ induces an equivalence $\mathcal{C}_{q+1-i}/\mathcal{C}_{q+2-i} \xrightarrow{\sim} \mathcal{C}'_{q+1-i}/\mathcal{C}'_{q+2-i}$ (for example, a right inverse is given by tensoring with $\operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}^c_i, \mathcal{A}'^c/\mathcal{J}'^c)$). (6) also implies that, for $M \in \mathcal{C}_{q+1-i}$, we have $\operatorname{Tor}^{\mathcal{A}^c/\mathcal{J}^c_i}_j(\mathcal{B}^c_i, M) \in \mathcal{C}'_{q+2-i}$ for all $j > 0$. Together with (4) of Theorem 10.1 this completes the proof of (iii). This finishes the proof of (1) of Theorem 10.2 and also establishes the first claim in (2).

To complete the proof of (2) we need to check that $\mathcal{J}'^c_{q-i}(\mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} S) = \mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} S$. By (5) of Theorem 10.1, the natural homomorphism $\mathcal{J}'^c_{q-i} \otimes_{\mathcal{A}'^c} \mathcal{B}^c_{q+1-i} \to \mathcal{B}^c_{q+1-i}$ is surjective. It follows that the natural homomorphism $\mathcal{J}'^c_{q-i} \otimes_{\mathcal{A}'^c} \mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} S \to \mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} S$ is surjective as well. This finishes the proof of (2) of Theorem 10.2.

To show (3) – that the associated varieties of the annihilators are preserved – one can argue as follows. Let $\mathcal{I}$ denote the annihilator of S. So $\mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} S$ is a quotient of $\mathcal{B}^c_{q+1-i} \otimes_{\mathcal{A}^c} \mathcal{A}^c/\mathcal{I}$, a HC bimodule annihilated by $\mathcal{I}$ on the right. From Corollary 3.2 one can now deduce that the associated variety of the annihilator $\mathcal{I}'$ of S' is contained in that of $\mathcal{I}$. On the other hand, S is a submodule of $\operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}^c_{q+1-i}, S') = \operatorname{Hom}_{\mathcal{A}'^c}(\mathcal{B}^c_{q+1-i}/\mathcal{I}'\mathcal{B}^c_{q+1-i}, S')$. So the right annihilator of $\mathcal{B}^c_{q+1-i}/\mathcal{I}'\mathcal{B}^c_{q+1-i}$ is contained in $\mathcal{I}$. This shows that the associated variety of $\mathcal{I}'$ contains that of $\mathcal{I}$ and completes the proof of (3).

Theorem 10.2 is now proved.

10.7. **Connection to Heisenberg categorical actions.** Recall that wall-crossing functors through walls defined by real roots are related to a categorical $\mathfrak{sl}_2$-action, see Theorem 5.10.

Let us discuss a conjectural relation of the affine wall-crossing functor to a categorical Heisenberg action. A possible definition of a categorical Heisenberg action appeared in [CL], see Section 3 there in particular. On the other hand, some concrete examples of categorical Heisenberg actions were constructed in both classical (by Cautis and Licata, [CL], in the case when $\mathcal{M}^\theta(v) = \operatorname{Hilb}_n(\widetilde{\mathbb{C}^2/\Gamma_1})$, where $\widetilde{\mathbb{C}^2/\Gamma_1}$ is a minimal resolution of $\mathbb{C}^2/\Gamma_1$) and quantum (Shan and Vasserot, [SV, Section 5], when Q is cyclic) situations. Cautis, Licata and Sussan, [CLS], defined a complex from a categorical Heisenberg action that should be thought as a Heisenberg version of the Rickard complex, let us denote this complex by Θ_δ. This gives rise to a derived self-equivalence of a category with a categorical Heisenberg action.

Let λ, λ' be parameters with integral difference separated only by the affine wall (meaning that the corresponding stability conditions θ, θ' are separated by that wall only).

Conjecture 10.13. There is a categorical Heisenberg action on $\bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ with the following property: the t-structure on $D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ coming from the identification with $D^b(\mathcal{A}^{\theta'}_\lambda(v)\text{-mod})$ is obtained from the initial one by applying Θ_δ.

11. Proof of counting result and some conjectures

11.1. **Extremal simples.** Let us start by proving Lemma 5.16.

Proof of Lemma 5.16. Let $L \in \operatorname{Irr}(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ be extremal. By the minimality assumption on v, we see that $[L] \notin \sum_\alpha \operatorname{im}[F_\alpha]$. In particular, L is α-singular for all $\alpha \in \Pi^\theta$. $\square$

Proof of Proposition 5.17. We will need to consider the following three cases separately. Fix $\mathfrak{a}$. Then the set of λ with $\mathfrak{a}^\lambda = \mathfrak{a}$ looks as follows: we take the union of a countable discrete collection of affine subspaces in $\mathfrak{P}$ and remove another countable discrete collection of affine subspaces. We say that λ is *generic with respect to* $\mathfrak{a}$ if λ is Weil generic in a connected component in the closure of a connected component.

The three cases we consider are as follows:

(1) $\lambda \in \mathbb{Q}^{Q_0}$,
(2) λ is generic with respect to $\mathfrak{a}$.
(3) λ is arbitrary with $\mathfrak{a}^\lambda = \mathfrak{a}$.

We will also see that in (1) and (2) the endomorphisms $[E_\alpha], [F_\alpha]$ of $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ give rise to an action of $\mathfrak{a}$.

Case 1. Consider the case when λ is rational. In this case, by Proposition 6.2, we see that $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \hookrightarrow \bigoplus_v K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$. By Proposition 7.16, we see that the operators $[E_\alpha], [F_\alpha]$ give rise to an $\mathfrak{a}$-action on $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ and the embedding above is equivariant. So we see that $K_0(\mathcal{C}) = \bigoplus_\sigma U(\mathfrak{a}) K_0(\mathcal{A}^\theta_\lambda(\sigma \bullet w)\text{-mod})$, where the summation is taken over all $\sigma \in W(Q)$ such that $\sigma\omega$ is dominant for $\mathfrak{a}$.

Recall, Proposition 7.11, that the wall-crossing functor $\mathfrak{WC}_{\theta\to\theta'}$ intertwines the embeddings of $K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}), K_0(\mathcal{A}^{\theta'}_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ into $K_0(\operatorname{Coh}_{\rho^{-1}(0)}(\mathcal{M}^\theta(v)))$. It follows that $\mathfrak{WC}_{\theta\to\theta'}$ maps $D^b_{\mathcal{C}}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ to $D^b_{\mathcal{C}}(\mathcal{A}^{\theta'}_\lambda(v)\text{-mod})$, where the subscript $\mathcal{C}$ means that we consider the objects with homology in $\mathcal{C}$.

Now consider the wall-crossing functor $\mathfrak{WC}_{\theta\to\theta'}$ through $\ker\alpha$. Let L be extremal in $\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}$. Corollary 5.16 implies that ν is dominant for $\mathfrak{a}$. So all constituents of $H_*(\mathfrak{WC}_{\theta\to\theta'}L)$ but L' lie in the image of $\tilde{f}_\alpha$ and hence, by the minimality assumption on v, in $\mathcal{C}$. We deduce that $L'\not\in\mathcal{C}$. So L' is extremal provided the minimality assumption on v holds for θ' as well. But if v is not minimal for θ', then by switching θ',θ in the argument above in this paragraph, we see that v is not minimal for θ either.

Case 2. Now let Γ be a connected component of the closure of $\{\lambda|\mathfrak{a}^\lambda=\mathfrak{a}\}$. Let $\lambda_1\in\mathbb{Q}^{Q_0}$. Pick a Weil generic $\lambda\in\Gamma$. The algebra $\mathcal{A}_\Gamma(v)$ and the sheaf $\mathcal{A}^\theta_\Gamma(v)$ are defined over $\mathbb{Q}$. It follows that we have a specialization map $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})\to K_0(\mathcal{A}_{\lambda_1}(v)\text{-mod}_{fin})$. The functors E_α,F_α are also defined over the rationals, so the specialization map intertwines $[E_\alpha],[F_\alpha]$. The same is true for the wall-crossing functor $\mathfrak{WC}_{\theta\to\theta'}$ and hence the specialization map intertwines $[\mathfrak{WC}_{\theta\to\theta'}]$.

We claim that there is $\lambda_1\in\mathbb{Q}^{Q_0}$ with $\mathfrak{a}^\lambda=\mathfrak{a}$ such that the degeneration map $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})\to K_0(\mathcal{A}_{\lambda_1}(v)\text{-mod}_{fin})$ is an embedding.

Let d be the maximal dimension of an irreducible finite dimensional $\mathcal{A}_\lambda(v)$-module. Since λ is Weil generic, d is also the maximal dimension of a finite dimensional irreducible for any other Weil generic parameter. Let A_Γ be the quotient of $\mathcal{A}_\Gamma(v)$ by the ideal generated by the elements
$$\sum_{\sigma\in S_{2d}}\operatorname{sgn}(\sigma)a_{\sigma(1)}\ldots a_{\sigma(2d)}.$$
The algebra A_Γ is a finitely generated $\mathbb{C}[\Gamma]$-module. This is proved using (2.5) similarly to the proof of [L3, Theorem 7.2.1], compare to the proof of [L11, Lemma 5.1].

So the module of traces, $A_\Gamma/[A_\Gamma,A_\Gamma]$ is finitely generated over $\mathbb{C}[\Gamma]$. Because of this there is a Zariski open subset Γ^0 such that the specializations A_{λ_2} with $\lambda_2\in\Gamma^0$ have the same number of irreducible representations. So we can take any $\lambda_1\in\Gamma^0\cap\mathbb{Q}^{Q_0}$. This shows that the degeneration map $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})\to K_0(\mathcal{A}_{\lambda_1}(v)\text{-mod}_{fin})$ is an inclusion that sends classes of irreducibles to classes of irreducibles.

Since the degeneration map intertwines the operators $[E_\alpha],[F_\alpha]$, we see that $K_0(\mathcal{C}^\theta_\lambda(v))$ (the summand corresponding to the dimension v in the category $\mathcal{C}$ for (λ,θ)) gets mapped onto $K_0(\mathcal{C}^\theta_{\lambda_1}(v))$ for any v from a given fixed finite set. It follows that, under the degeneration map the class of an extremal object goes to the class of an extremal object. Since the degeneration map is compatible with wall-crossing functors, we reduce the present case to Case 1. In particular, we get an $\mathfrak{a}$-action on $\bigoplus_v K_0(\mathcal{A}_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$.

Case 3. Now consider the general case. Let $\tilde{\lambda}$ denote the Weil generic element in the connected component of $\overline{\{\lambda|\mathfrak{a}^\lambda=\mathfrak{a}\}}$ containing λ. We still have the injective degeneration maps $K_0(\mathcal{A}_{\tilde{\lambda}}(v)\text{-mod}_{fin})\to K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$ intertwining the maps $[E_\alpha],[F_\alpha]$ as well as the maps given by wall-crossing functors. So again $K_0(\mathcal{C}^\theta_{\tilde{\lambda}}(v))$ maps bijectively onto $K_0(\mathcal{C}^\theta_\lambda(v))$ for any v from a given fixed finite set. It follows that $\mathfrak{WC}_{\theta\to\theta'}$ sends $D^b_{\mathcal{C}}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ to $D^b_{\mathcal{C}}(\mathcal{A}^{\theta'}_{\lambda'}(v)\text{-mod})$ for any v from a fixed finite set. Arguing as in Case 1, we see that $L\mapsto L'$ sends extremal objects to extremal objects. We also see that the operators $[E_\alpha],[F_\alpha]$ give an action of $\mathfrak{a}$ on $K_0(\mathcal{C})$. $\square$

11.2. **Absence of extremal simples.** In this section we will use Proposition 5.13, Theorem 10.2 and Proposition 5.17 to complete the proof of (II) in the following two cases.

(a) The quiver Q is of finite type.
(b) Q is an affine quiver, $v=n\delta, w=\epsilon_0$.

Lemma 11.1. *Let θ, θ' be two stability conditions. Suppose that θ, θ' are not separated by* $\ker\alpha$, *where $\alpha \leqslant v$ is an imaginary root with $\langle\alpha,\lambda\rangle \in \mathbb{Z}$. Further, if there is an imaginary root $\beta \leqslant v$ with $\langle\beta,\lambda\rangle \in \mathbb{Z}$, then $\langle\theta,\beta\rangle > 0$. Let M be an extremal simple $\mathcal{A}^\theta_\lambda(v)$-module. Then $H_0(\mathfrak{WC}_{\theta\to\theta'}M)$ has a quotient that is an extremal simple $\mathcal{A}^{\theta'}_\lambda(v)$-module.*

Proof. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_q = \theta'$ be stability conditions such that θ_i and θ_{i+1} are separated by $\ker\alpha_i$, where α_i is a real root with $\langle\alpha_i,\lambda\rangle \in \mathbb{Z}$ and $\alpha_i \leqslant v$. We assume that q is minimal with this property. It follows from Proposition 5.17 that if M_i is an extremal simple $\mathcal{A}^{\theta_i}_\lambda(v)$-module, then the head of $H_0(\mathfrak{WC}_{\theta_i\to\theta_{i+1}}M_i)$ again contains an extremal simple, say M_{i+1}. We start with M_1 and produce the extremal simples $M_2, \ldots, M_q$. By the construction, M_q is a quotient of $H_0(\mathfrak{WC}_{\theta_1\to\theta_q}M)$. □

Now we are ready to prove (II) from the beginning of Section 5.

Proof of (II). We need to prove that there are no extremal simples in $\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}$ (in the affine case we assume that $\langle\delta,\theta\rangle > 0$). Assume the converse.

Lemma 11.1 together with Proposition 5.13 lead to a contradiction in case (a).

Now let us deal with case (b) – the SRA case. Pick an extremal simple $M \in \mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}$. We can pick stability conditions $\theta_1, \ldots, \theta_q$ with the following properties:

(a) $\theta = \theta_j$ for some j.
(b) $-\theta_q$ and θ_1 lie in chambers separated by $\ker\delta$ and $\langle\theta_1,\delta\rangle > 0$.
(c) θ_i and θ_{i+1} are separated by a single wall defined by a real root.
(d) q is minimal with this property.

Let $M_j := M$ and find extremal simples $M_i \in \mathcal{A}^{\theta_i}_{\lambda_i}(v)\text{-mod}, i = 1, \ldots, q,$ (where $\lambda_i \in \lambda + \mathbb{Z}^{Q_0}$ is such that $(\lambda_i,\theta_i) \in \mathfrak{AL}(v)$) such that M_i and M_{i+1} are in bijection produced by crossing the wall between θ_i and θ_{i+1} (with $M = M_i$ and $M' = M_{i+1}$), see Proposition 5.17.

Using LMN isomorphisms, we can identify $\mathcal{M}^{\theta_i}(v)$ with $\mathcal{M}^{\tilde{\theta}_i}(n\delta)$ and $\mathcal{A}^{\theta_i}_{\lambda_i}(v)$ with $\mathcal{A}^{\tilde{\theta}_i}_{\tilde{\lambda}_i}(n\delta)$ for appropriate $n, \tilde{\theta}_i, \tilde{\lambda}_i$. Note that $\tilde{\theta}_i, \tilde{\theta}_{i+1}$ are still separated by a single wall, for $i = 0$, this wall is $\ker\delta$, and for $i > 0$, this is the wall defined by a real root. Moreover, the weight ν defined by v is extremal if and only if $n = 0$. Let M_0 be the simple in $\mathcal{A}^{\tilde{\theta}_0}_{\tilde{\lambda}_0}(n\delta)\text{-mod} = \mathcal{A}^{\theta_0}_{\lambda_0}(v)\text{-mod}$ corresponding to M_1 under the bijection in (3) of Theorem 10.2.

Consider the complex $\mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_q}(M_0)$. By Theorem 5.3,

$$\mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_q}(M_0) = \mathfrak{WC}_{\tilde{\theta}_1\to\tilde{\theta}_q} \circ \mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_1}(M_0).$$

By Proposition 5.13, the left hand side has vanishing H_k for $k < n$ because $\Gamma^{\tilde{\theta}_0}_{\tilde{\lambda}_0}(M_0)$ is finite dimensional. On the other hand, by Theorem 10.2, we have $H_j(\mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_1}(M_0)) \twoheadrightarrow M_1$ for some $j < n$ and $H_k(\mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_1}(M_0)) = 0$ for $k < j$. From Lemma 11.1, we deduce that $H_j(\mathfrak{WC}_{\tilde{\theta}_1\to\tilde{\theta}_q} \circ \mathfrak{WC}_{\tilde{\theta}_0\to\tilde{\theta}_1}(M_0)) \twoheadrightarrow M_q$. We arrive at a contradiction that completes the proof. □

11.3. **Injectivity of CC.** In this section we prove (III): the map $\mathsf{CC} : \bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \to L_\omega$ is injective.

A key step is as follows.

Lemma 11.2. *The operators $[E_\alpha], [F_\alpha]$ on $\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$ give an $\mathfrak{a}$-action.*

Proof. It was shown in the proof of Proposition 5.17 (Section 11.1) that the restrictions of $[E_\alpha],[F_\alpha]$ to $K_0(\mathcal{C})$ define an action of $\mathfrak{a}$. On the other hand (II) proved in the previous section shows that $\mathcal{C}=\bigoplus_v \mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}$. This finishes the proof. $\square$

Proof of (III). By Proposition 5.9, the map $\mathsf{CC}:\bigoplus_v K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})\to L_\omega$ is $\mathfrak{a}$-linear. The image coincides with $L^{\mathfrak{a}}_\omega$ by (I) and (II). It follows from the construction of $\mathcal{C}$ and 9.2 that the $\mathfrak{a}$-module $K_0(\mathcal{C})$ is generated by $\bigoplus_\sigma K_0(\mathcal{A}^\theta_\lambda(\sigma\bullet w)\text{-mod}_{\rho^{-1}(0)})$, where the summation is taken over $\sigma\in W(Q)$ such that $\sigma\omega$ is dominant for $\mathfrak{a}$. It follows that CC is injective. $\square$

This finishes the proof of Theorem 1.2.

Remark 11.3. Let us deduce the original conjecture of Etingof, [Et, Conjectures 6.3,6.8], from Conjecture 1.1. According to results of [GL, Section 5], we have a derived equivalence $D^b(\mathcal{H}_{\kappa,c}(n)\text{-mod})\xrightarrow{\sim} D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ that restricts to $D^b_{fin}(\mathcal{H}_{\kappa,c}(n)\text{-mod})\xrightarrow{\sim} D^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$. So the number of finite dimensional irreducible $\mathcal{H}_{\kappa,c}(n)$-modules coincides with the number of irreducible $\mathcal{A}^\theta_\lambda(v)$-modules supported on $\rho^{-1}(0)$. It is easy to see that the number given by Conjecture 1.1 is the same as conjectured by Etingof.

11.4. **Supports of arbitrary conjectures.** In the remainder of the section we would like to discuss two counting problems that are more general than the problem studied in this paper.

One can pose a problem of counting $\mathcal{A}_\lambda(v)$-irreducibles with given (positive) dimension of support. An obvious difficulty here is that the number of such modules is infinite. There are, at least, three different approaches to the counting problem: to deal with a filtration by support on K_0, to work in characteristic $p\gg 0$ or to restrict to a suitable category of modules in characteristic 0.

11.4.1. *Category $\mathcal{O}$.* An easier special case is when there is a Hamiltonian $\mathbb{C}^\times$-action on $\mathcal{M}^\theta(v)$ with finitely many fixed points. This action deforms to a Hamiltonian action on $\mathcal{A}^\theta_\lambda(v)$ and hence on $\mathcal{A}_\lambda(v)$. Let $h\in\mathcal{A}_\lambda(v)$ denote the corresponding hamiltonian so that $[h,\cdot]$ coincides with the derivation of $\mathcal{A}$ induced by the $\mathbb{C}^\times$-action. The algebra $\mathcal{A}_\lambda(v)$ acquires an internal grading by eigenspaces of $\operatorname{ad}h$, $\mathcal{A}_\lambda(v):=\bigoplus_{i\in\mathbb{Z}}\mathcal{A}_\lambda(v)^i$. Then we can consider the category $\mathcal{O}_\lambda(v)$ of $\mathcal{A}_\lambda(v)$-modules consisting of all finitely generated modules with locally nilpotent action of $\bigoplus_{i>0}\mathcal{A}_\lambda(v)^i$, compare with [BGK, L4, GL, BLPW2]. The simples in this category are in one-to-one correspondence with the irreducible $\mathcal{A}_\lambda(v)^+:=\mathcal{A}_\lambda(v)^0/(\bigoplus_{i>0}\mathcal{A}_\lambda(v)^{-i}\mathcal{A}_\lambda(v)^i)$-modules. It is not difficult to see that the algebra $\mathcal{A}_\lambda(v)^+$ is finite dimensional, compare to [GL, Lemma 3.1.4]. Moreover, for some non-empty Zariski open subset of λ, the algebra $\mathcal{A}_\lambda(v)^+$ is naturally identified with $\mathbb{C}[\mathcal{M}^\theta(v)^{\mathbb{C}^\times}]$, see, e.g., [BLPW2, Section 5.1]. So we may assume that the irreducibles in our category $\mathcal{O}$ are parameterized by $\mathcal{M}^\theta(v)^{\mathbb{C}^\times}$. Also to every fixed point p we can assign the corresponding Verma module, $\Delta_p:=\mathcal{A}_\lambda(v)\otimes_{\mathcal{A}_\lambda(v)^{\geqslant 0}}\mathbb{C}_p$. Here $\mathbb{C}_p$ stands for the 1-dimensional $\mathcal{A}_\lambda(v)^+$-module corresponding to p, we view $\mathbb{C}_p$ as an $\mathcal{A}_\lambda(v)^{\geqslant 0}$-module via the epimorphism $\mathcal{A}_\lambda(v)^{\geqslant 0}\twoheadrightarrow\mathcal{A}_\lambda(v)^+$. For λ in some Zariski open subset the category $\mathcal{O}$ is highest weight with standard objects Δ_p, see [BLPW2, Section 5.2]. We identify $K_0(\mathcal{O}_\lambda(v))$ with $\mathbb{C}[\mathcal{M}^\theta(v)^{\mathbb{C}^\times}]$ by sending the class $[\Delta_p]$ of Δ_p to the basis vector corresponding to p. For example, suppose we consider $\mathcal{M}^\theta(n\delta)$ for a cyclic quiver Q with ℓ vertices and $w=\epsilon_0$. Then we get the category $\mathcal{O}_{\kappa,c}(n)$ for cyclotomic Rational Cherednik algebra $H_{\kappa,c}(\Gamma_n)$ with $\Gamma_n:=\mathfrak{S}_n\ltimes(\mathbb{Z}/\ell\mathbb{Z})^n$ (at least for some Zariski open subset in $\mathfrak{p}$; it was conjectured in [GL, Section 3] that the subset coincides with the set of all spherical parameters). The Verma modules Δ_τ in that category are indexed by the irreducible representations τ of Γ_n that are in a natural bijection with $\mathcal{M}^\theta(n\delta)^{\mathbb{C}^\times}$, as pointed out by Gordon in [Go, Section 5.1], let us denote the fixed point corresponding to τ by $p(\tau)$. It follows from results of [GL, Section 3] that Δ_τ coincides with $\Delta_{p(\tau)}$.

One can ask the question to compute the number of the irreducibles in $\mathcal{O}_\lambda(v)$ with given dimension of support. In the cyclotomic Cherednik algebra case this problem was solved by Shan and Vasserot in [SV]. We will state a conjecture in the case when $X = \mathcal{M}^\theta(v)$ and Q is a cyclic quiver (in this case we do have a Hamiltonian $\mathbb{C}^\times$-action with finitely many fixed points). We remark that different choices of $\mathbb{C}^\times$ lead to different choices of the categories $\mathcal{O}$, but our answer should not depend on the choice. More precisely, there are derived equivalences relating categories $\mathcal{O}$ for different choices of $\mathbb{C}^\times$, see [L13], these equivalences can be seen to preserve the supports.

Set $\mathcal{O}_\lambda := \bigoplus_v \mathcal{O}_\lambda(v)$. We also write $\mathcal{O}_\lambda^w$ if we want to indicate the dependence on w.

A description of the $\mathbb{C}^\times$-stable points in $\mathcal{M}^\theta(v)$ follows, for example, from the work of Nakajima, [Nak2, Sections 3,7]. Namely, consider a maximal torus $T \subset \prod_{k\in Q_0} \mathrm{GL}(w_k)$. Then the T-invariant points on $\mathcal{M}^\theta\{w\} := \bigsqcup_v \mathcal{M}^\theta(v,w)$ are naturally identified with

$$\prod_{k\in Q_0} \mathcal{M}^\theta\{\epsilon_k\}^{w_k}$$

The $\mathbb{C}^\times$-fixed locus in $\mathcal{M}^\theta\{w\}$ is then the union of the fixed points in $\prod_{k\in Q_0} \mathcal{M}^\theta\{\epsilon_k\}^{w_k}$, in each dimension there are finitely many of those. The fixed points in $\mathcal{M}^\theta(v,\epsilon_k)$ are indexed by ℓ-multipartitions of n_v such that $n_v\delta \in W(Q)\nu$.

We want to state a conjecture on the filtration of $K_0(\mathcal{O}_\lambda)$ by the homological shifts under the wall-crossing functor $\mathfrak{WC}$ through the affine wall.

We again start with the case when Q is a single loop. Then $K_0(\mathcal{O}_\lambda) = \mathcal{F}^{\otimes r}$, where r stands for the framing and $\mathcal{F}$ is the Fock space, i.e., the space with a basis indexed by partitions. Consider the r copies of the Heisenberg Lie algebras $\mathfrak{heis}^i$ with bases $b^i_j, j \in \mathbb{Z}\setminus\{0\}$ and one more copy of the Heisenberg, $\mathfrak{heis}_\Delta$, with basis $b_j, j \in \mathbb{Z}\setminus\{0\}$, embedded into $\prod_{i=1}^r \mathfrak{heis}^i$ diagonally. Inside $\mathcal{O}_\lambda(n)$ consider the Serre subcategory $\mathrm{F}_j\,\mathcal{O}_\lambda(n)$ spanned by all simples with support of codimension at least j so that $\mathrm{F}_j\,\mathcal{O}_\lambda(n)$ is a decreasing filtration on $\mathcal{O}_\lambda(n)$. We view each of the r copies of $\mathcal{F}$ as a standard Fock space representation of the corresponding Heisenberg algebra $\mathfrak{heis}^i$. So $K_0(\mathcal{O}_\lambda)$ becomes a $\prod_{i=1}^r \mathfrak{heis}^i$-module and hence a $\mathfrak{heis}_\Delta$-module.

Conjecture 11.4. Let m denote the denominator of λ (equal to $+\infty$ if $\langle\lambda,\delta\rangle \notin \mathbb{Q}$). The subspace $K_0(\mathrm{F}_j\,\mathcal{O}_\lambda) \subset K_0(\mathcal{O}_\lambda)$ is the sum of the images of the operators $b_{mj_1}\dots b_{mj_k}$ with $j_1,\dots,j_k \in \mathbb{Z}_{>0}$ and $(rm-1)(j_1+\dots+j_k) \geqslant j$.

Now let us proceed to the general case: when Q is a cyclic quiver with ℓ vertices and the framing w is arbitrary. We again want to describe the filtration on $K_0(\mathcal{O}_\lambda^w)$ relative to the affine wall. The description will still be given in terms of some Heisenberg action on $K_0(\mathcal{O}_\lambda^w)$.

Let us specify that action. As we have seen above, $K_0(\mathcal{O}_\lambda^w) = \bigotimes_{k\in Q_0} K_0(\mathcal{O}_\lambda^{\epsilon_k})^{\otimes w_k}$. The space $K_0(\mathcal{O}_\lambda^{\epsilon_k})$ can be thought as an integrable highest weight module $\tilde{L}_{\omega_k}$, where ω_k is the fundamental weight corresponding to k, for the Lie algebra $\hat{\mathfrak{gl}}_\ell$ (so that $\tilde{L}_{\omega_k} = L_{\omega_k} \otimes \mathcal{F}$), compare to [Et, Section 6]. Inside $\hat{\mathfrak{gl}}_\ell$ consider the Heisenberg subalgebra corresponding to the center of $\mathfrak{gl}_\ell$. It has a basis b_j with $j \in \mathbb{Z}$.

Conjecture 11.5. Let m denote the denominator of $\langle\lambda,\delta\rangle$ (equal to $+\infty$ if $\lambda\not\in\mathbb{Q}$). Consider the subcategory $\mathrm{F}_j^{aff}\,\mathcal{O}^w$ consisting of all modules M with $H_i(\mathfrak{WC}M)=0$ for $i<j$, where $\mathfrak{WC}$ stands for the short wall-crossing functor through the affine wall. The subspace $K_0(\mathrm{F}_j^{aff}\,\mathcal{O}^w_\lambda)\subset K_0(\mathcal{O}^w_\lambda)$ is the sum of the images of the operators $b_{mj_1}\dots b_{mj_k}$ with $j_1,\dots,j_k\in\mathbb{Z}_{>0}$, and $(\overline{w}m-1)(j_1+\ldots+j_k)\geqslant j$, where $\overline{w}:=\sum_{k\in Q_0}w_k$.

Modulo Conjecture 11.5, one can state a conjecture regarding the filtration by dimension of support.

Conjecture 11.6. The span in $K_0(\mathcal{O}^w_\lambda)$ of the classes of all modules with dimension of support $\leqslant i$ is the sum of $\mathfrak{a}$-submodules generated by the singular vectors in $\mathrm{F}^{aff}_{s(\overline{w}m-1)}\,\mathcal{O}^w(v)$, where v and i are subject to the following condition:

$$w\cdot v-(v,v)/2-s(\overline{w}m-1)\leqslant i.$$

We expect that Conjecture 11.6 should be an easy corollary of Conjecture 11.5 and techniques developed in Sections 11.1,11.2. We remark that it is compatible with Conjecture 1.1 and also with the main result of [SV].

11.4.2. *Filtration on K_0.* One can also work with a filtration on $K_0(\mathcal{A}_\lambda(v)\text{-mod})$ as in [Et, Conjectures 6.1,6.7]. As before, we assume that $\mathcal{A}_\lambda(v)$ has finite homological dimension so that K_0 of the category of finitely generated $\mathcal{A}_\lambda(v)$-modules is naturally identified with the split K_0 of the category of projective $\mathcal{A}_\lambda(v)$-modules. Let $\mathrm{F}_j\,K_0(\mathcal{A}_\lambda(v)\text{-mod})$ stand for the subspace in $K_0(\mathcal{A}_\lambda(v)\text{-mod})$ generated by all objects $\mathcal{M}$ such that $\text{GK-}\dim(\mathcal{A}_\lambda(v)/\operatorname{Ann}\mathcal{M})\leqslant\dim\mathcal{M}^\theta(v)-2j$. Then one can state a conjecture similar to Conjecture 11.6.

11.4.3. *Characteristic p.* Yet another setting where one can state counting conjectures is in characteristic $p\gg 0$. We use the notation of Section 6.1. In particular, $\mathbb{F}$ stands for an algebraically closed field of characteristic p. To simplify the statement we consider the case of a rational parameter λ such that the algebra $\mathcal{A}_\lambda(v)_{\mathbb{C}}\otimes\mathcal{A}_\lambda(v)^{opp}_{\mathbb{C}}$ has finite homological dimension.

Set $K^0_p:=K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod})$ and $K^0_\infty=K_0(\mathcal{A}_\lambda(v)_{\mathbb{C}}\text{-mod})$ (recall that we consider the K_0 groups over $\mathbb{C}$). We have the specialization map $\mathrm{Sp}:K^0_\infty\to K^0_p$ for every prime $p\gg 0$ and it is an isomorphism.

Consider the category $\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0$ of finitely generated modules with zero generalized p-character. Set $K^p_0:=K_0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0)$. Similarly to Section 7.2, we have an identification $K^p_0\cong K_0(\mathrm{Coh}_{\rho^{-1}(0)}\,\mathcal{M}^\theta(v))$.

We have the Ext pairing $\chi:K^0_p\times K^p_0\to\mathbb{C}$, compare to the proof of Proposition 7.6. We have basically seen in Section 7.2 that this pairing is non-degenerate.

Conjecture 11.7. a) There exist polynomials in one variable $D_i(t)\in\mathbb{Q}[t]$, $i=1,\dots,\dim K^p_0$, such that for $p\gg 0$ the dimensions of the irreducible modules equal $D_i(p)$.

b) Let $\mathcal{A}_\lambda(v)\text{-mod}_0^{\leq d}$ be the Serre subcategory generated by irreducible objects L_i such that the corresponding polynomial D_i satisfies: $\deg(D_i)\leq d$. Then the induced filtration on $K^0(\mathcal{A}_\lambda(v)_{\mathbb{F}}\text{-mod}_0)$ is dual to the filtration on $K^0(\mathcal{A}_\lambda(v)_{\mathbb{C}}\text{-mod})$ considered in 11.4.2 with respect to the pairing χ.

Let us speculate on a possible scheme of proof. First, we need an analog of Proposition 5.13 that is not available yet, we believe this is the most important thing missing. Second, we need an analog of Webster's construction in positive characteristic. The latter is not expected to be difficult. Theorem 10.1 and an analog of Conjecture 11.5 should carry over to positive characteristic without significant modifications. This should be sufficient to prove the counting conjecture.

11.5. **Infinite homological dimension.** In this subsection we will state a conjecture on the number of irreducible finite dimensional $\mathcal{A}_\lambda(v)$-modules in the case when the homological dimension of $\mathcal{A}_\lambda(v)$ is infinite. Similar in spirit conjectures can be stated for categories $\mathcal{O}$ (or their replacements) or in positive characteristic, but we are not going to elaborate on that.

Consider the functor $R\Gamma^\theta_\lambda : D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \to D^b(\mathcal{A}_\lambda(v)\text{-mod})$. It should be a quotient functor, at least, this is so in the SRA situation thanks to an equivalence $D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}) \cong D^b(\mathcal{H}_{\kappa,c}(n)\text{-mod})$ established in [GL, 5.1] (see 5.1.6, in particular). Under this equivalence the functor $R\Gamma^\theta_\lambda$ becomes the abelian quotient functor $\mathcal{M} \mapsto e\mathcal{M}$. According to Conjecture 4.8, this quotient is proper if and only if λ lies in the finite union of hyperplanes (to be called "singular"), the singular hyperplanes can be (conjecturally) described explicitly when Q is of finite or of affine type, see Section 4.3.

So $K_0(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$ becomes a quotient of $K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)})$. Our goal is to provide a conjectural description of this quotient. Our conjecture will consist of two parts. The first (easier) will deal with the case when λ is a Zariski generic point of a singular hyperplane. The second (much harder) will handle the general case.

Let us deal with the Grassmanian case first. So let Q be a quiver with a single vertex and no arrows. The singular locus is $\lambda = 1-w, 2-w, \ldots, -1$. Assume, for convenience, that $\theta > 0$ and $2v \leqslant w$. Identify $\mathcal{A}^\theta_\lambda(v)\text{-mod}$ with $\mathcal{A}_0(v)\text{-mod}$. The ideals in the latter form a chain: $\{0\} = \mathcal{J}_{v+1} \subsetneq \mathcal{J}_v \subsetneq \ldots \subsetneq \mathcal{J}_1 \subsetneq \mathcal{J}_0 = \mathcal{A}_0(v)$. The kernel of the functor $R\Gamma^\theta_\lambda$ can be shown to consist of all modules annihilated by $\mathcal{J}_i$, where

$$i = v + 1 - \min(v, -\lambda, w + \lambda) \tag{11.1}$$

(or, more precisely, the complexes with such homology). On the level of the categorical $\mathfrak{sl}_2$-action, those should be precisely the complexes lying in the image of F^{v+1-i}.

Let us return to the general setting.

Conjecture 11.8. (1) Let α be a real root and λ be a Zariski generic parameter on a singular hyperplane $\langle \lambda, \alpha \rangle = s$. Then the complexified K_0 of the kernel of $D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \twoheadrightarrow D^b(\mathcal{A}_\lambda(v)\text{-mod}_{fin})$ coincides with the image of f^i_α, where i is determined from s and $\underline{v}, \underline{w}$ as in (11.1).

(2) Let $\langle \alpha_j, \cdot \rangle = s_j, j = 1, \ldots, k$, be all singular hyperplanes with <u>real</u> α_j that contain λ. Then $\ker K_0(D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \twoheadrightarrow D^b(\mathcal{A}_\lambda(v)\text{-mod}_{fin}))$ is spanned (as a vector space) by the sum of the images of $f^{i_j}_{\alpha_j}, j = 1, \ldots, k$, where the numbers i_j are determined as in (1).

We believe that one should not include the singular hyperplanes defined by imaginary roots. The reason is that there are no finite dimensional irreducibles for a Weil generic λ on a hyperplane defined by an imaginary root. When one deals with modules with higher dimensional support on should modify the conjecture to account for imaginary roots. We are not going to elaborate on that.

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