Formal loops

Daishi introduced a linear map $L\mathfrak{g} \to \mathrm{Vect}(\mathcal{L}U)$ and checked that it's a Lie algebra homomorphism when $\mathfrak{g} = \mathfrak{sl}_2$. The goal of this note is to explain why it's a Lie algebra homomorphism for general $\mathfrak{g}$.

1.0) Discussion

In the notes by Ivan we have seen that jets behave nicely w.r.t. gluing: if $X = U_1 \cup U_2$ is the union of opens, then $\mathcal{J}X$ is glued from $\mathcal{J}U_1, \mathcal{J}U_2$ along their common open subscheme $\mathcal{J}(U_1 \cap U_2)$.

For loops, this is not the case: $\mathcal{L}(U_1 \cap U_2)$ is not open in $\mathcal{L}(U_i)$ in any reasonable sense. Because of this, in general one cannot even define loops into a non-affine scheme (as an ind-scheme), in particular, one cannot pass from (non-existing) $\mathcal{L}(G/B^-)$ to $\mathcal{L}U$.

"Formal loops" remedy this problem.

1.1) Definition of formal loops.

Let's examine the relationship between $L\mathbb{A}^1$ (whose R-points are the Laurent series $\sum_i a_i z^i$ $(a_i \in R)$ $L\mathbb{G}_m$ (whose R-points are the invertible Laurent series $\sum_i a_i z^i$. The latter set of R-points looks very differently from $\{\sum_i a_i z^i \mid a_0$ is invertible$\}$.

Now fix $n, k \in \mathbb{Z}_{>0}$ and consider the functor $L_{n,k}\mathbb{A}^1$ sending R to $\{\sum_i a_i z^i \mid a_i \in R;\ a_i = 0\ \forall i < -k;\ a_{i_1} \ldots a_{i_n} = 0\ \forall i_1, \ldots i_n < 0\}$
If R has no nilpotents, then $L_{n,k}\mathbb{A}^1(R) = J\mathbb{A}^1(R)$ but the two functors are different. As $J\mathbb{A}^1$, $L_{n,k}\mathbb{A}^1$ is an affine scheme (highly non-reduced).

Crucial exercise: Let $\sum_i a_i z^i \in L_{n,k}\mathbb{A}^1(R)$. Then $\sum_i a_i z^i$ is invertible in $R((z))$ if $a_i \neq 0$. The inverse lies in $L_{n,nk}\mathbb{A}^1(R)$.

One can define $L_{n,k}X$ for any affine scheme. Consider the limit $L_n X = \varinjlim_k L_{n,k}X$. The corresponding topological algebra of functions in the case $X = \mathbb{A}^1$ is the completion of $\mathbb{C}[a_i]_{i\in\mathbb{Z}} / (a_j \mid j<0)^n$

with respect to the inverse system of ideals $(a_j \mid j < -k)$

Exercise: Use Crucial exercise to deduce an isomorphism between the induced completion of $\left(\mathbb{C}[a_i]/(a_j \mid j<0)^n\right)[a_0^{-1}]$ & the topological algebra corresponding to $\mathcal{L}_n \mathbb{G}_m$.

Definition: By the ind-scheme of formal loops into X (an affine scheme of finite type) we mean $\hat{\mathcal{L}}X := \varinjlim_n \mathcal{L}_n X$ $= \varinjlim_{n,k} \mathcal{L}_{n,k}(X)$.

As the previous discussion suggests, the ind-schemes $\hat{\mathcal{L}}U_i$ glue nicely over an open affine cover $X = \bigcup U_i$ for a general finite type scheme X. The geometric meaning of $\hat{\mathcal{L}}X$ for X affine is that $\hat{\mathcal{L}}X$ is the formal neighborhood of $\mathcal{J}X$ in $\mathcal{L}X$.

For more on formal loops see

M. Kapranov, E. Vasserot "Vertex algebras & formal loop space."

1.2) Application

Let G be an algebraic group acting on a smooth variety X & $U \subset X$ be an open affine. Note that $\hat{\mathcal{L}}G$ is a group ind-scheme acting on $\hat{\mathcal{L}}X$. Since $\hat{\mathcal{L}}G$ is the formal neighborhood of $\mathcal{J}G$ in $\mathcal{L}G$, the Lie algebras of $\hat{\mathcal{L}}G$ & $\mathcal{L}G$ coincide (with $\mathfrak{g}((t))$). The action of $\hat{\mathcal{L}}G$ on $\hat{\mathcal{L}}X$ gives a Lie algebra homomorphism $\mathfrak{g}((t)) \longrightarrow \mathrm{Vect}(\hat{\mathcal{L}}X)$. Also we have the restriction homomorphism $\mathrm{Vect}(\hat{\mathcal{L}}X) \longrightarrow \mathrm{Vect}(\hat{\mathcal{L}}U)$.

Now apply this construction to the situation of interest: G is a simple group, $X = G/B_-$, $U = N_+B_-/B_-$. Note that we have the restriction map $\mathrm{Vect}(\mathcal{L}U) \longrightarrow \mathrm{Vect}(\hat{\mathcal{L}}U)$ & its injective & a Lie algebra homomorphism ($\mathbb{C}[\mathcal{L}U]$ is a subalgebra in $\mathbb{C}[\hat{\mathcal{L}}U]$ & every continuous derivation of $\mathbb{C}[\mathcal{L}U]$ extends to $\mathbb{C}[\hat{\mathcal{L}}U]$ – this is an exercise on the definitions of $\mathcal{L}U$ & $\hat{\mathcal{L}}U$). It remains to observe that the Lie algebra homomorphism $\mathfrak{g}((t)) \to \mathrm{Vect}(\hat{\mathcal{L}}U)$ factors through $\mathrm{Vect}(\mathcal{L}U)$, this follows from the construction in Sec 3.4 of Daishi's talk.