BASICS ON REDUCTIVE GROUPS

Contents

1. Definitions and examples

We consider affine (a.k.a. linear) algebraic groups over an algebraically closed field $\mathbb{F}$. Our main examples are the classical groups $GL_n, PGL_n, SL_n, Sp_{2n}, SO_n$, the additive and multiplicative groups of $\mathbb{F}$ denoted by $\mathbb{G}_a, \mathbb{G}_m$, as well as the groups U_n, B_n of all uni-triangular and all upper-triangular matrices in GL_n.

Definition 1.1. *An algebraic group is called unipotent if it acts by unipotent operators in any rational representation.*

For example, U_n (or any its algebraic subgroup) is unipotent. More generally any extension of several copies of $\mathbb{G}_a$ is unipotent.

Lemma 1.2. *Let G be an algebraic group. Then G contains a unique maximal normal unipotent subgroup called the unipotent radical of G and denoted by $R_u(G)$.*

Definition 1.3. *An algebraic group G is called reductive if $R_u(G) = \{1\}$.*

Remark 1.4. *In characteristic* 0*, this is equivalent to the condition that any rational representation of* G *is completely reducible. This is not the case in positive characteristic.*

The basic examples of reductive groups are *tori* (=groups isomorphic to $\mathbb{G}_m^r$ for some $r \geqslant 1$), GL_n, SL_n, SO_n, Sp_{2n}, their products.

A connected reductive group is called *semisimple* if its center is finite. For a general connected reductive group G, the derived subgroup (G, G) is semisimple, and $G = (G, G)Z(G)^\circ$, where $Z(G)^\circ$ is the connected component of 1 in the center $Z(G)$.

2. Borel subgroups, maximal tori, Weyl group

Let G be an algebraic group.

2.1. Borel subgroups.

Definition 2.1. *A subgroup* $B \subset G$ *is called a Borel subgroup if it is maximal (w.r.t. inclusion) of all connected solvable subgroups of* G*.*

Example 2.2. *The subgroup* $B_n \subset GL_n$ *is Borel.*

Theorem 2.3. *The following claims hold:*

1. *All Borel subgroups of* G *are conjugate to each other.*
2. *If* $B \subset G$ *is a Borel subgroup, then* G/B *is a projective variety.*

2.2. **Maximal tori and Weyl group.** By a maximal torus in G we mean a subgroup of G that is a torus and is maximal among tori in G (w.r.t. inclusion). For example, the subgroup of all diagonal matrices in GL_n is a maximal torus.

Theorem 2.4. *All maximal tori are conjugate.*

From now and until the end of the section on algebraic groups we assume that G is a connected reductive group. Let $T \subset G$ be a maximal torus.

Definition 2.5. *The Weyl group* $W = W(G)$ *is, by definition,* $N_G(T)/T$*, where* $N_G(T) \subset G$ *is the normalizer of* T*.*

Note that since all maximal tori are conjugate, W is (non-canonically) independent of the choice of T.

3. Root systems and root data

We are going to define root systems and related objects for connected reductive groups. In characteristic 0 this is usually first done on the level of Lie algebras. But we also want to work in positive characteristic so cannot really rely on Lie algebras. It turns out that the story can be retold entirely on the level of groups and holds in all characteristics.

3.1. **Root subgroups and root system.**

Definition 3.1. *Let G be a connected reductive group, T its maximal torus. A subgroup $U' \subset G$ is called a root subgroup (w.r.t. T) if*

- $U' \cong \mathbb{G}_a$,
- *There is α in the character lattice $\mathfrak{X}(T)$ (called a root) such that $tut^{-1} = \alpha(t)u$ for all $t \in T, u \in U'$ (here in the r.h.s. we view u as an element of $\mathbb{G}_a$ so that the r.h.s. makes sense).*

Example 3.2. *Consider $G = GL_n$. Let $U_{ij} = \{E + tE_{ij} | t \in \mathbb{F}\}$, where we write E for the identity matrix and E_{ij} for the matrix unit in position (i, j). Then U_{ij} is a root subgroup and the corresponding root α is $\epsilon_i - \epsilon_j$, where ϵ_k maps $\operatorname{diag}(t_1, \ldots, t_n) \in T$ to $t_k \in \mathbb{G}_m$.*

Familiar results from the Lie algebra setting in characteristic 0 still hold.

Theorem 3.3. *The following claims are true.*

1. *A root $\alpha \in \Delta$ determines a root subgroup uniquely, let us write U_α for this subgroup*[1].
2. *U_α and $U_{-\alpha}$ generate a subgroup of G isomorphic to SL_2 or PGL_2.*
3. *The set Δ of all roots is a reduced root system, independent of $\mathbb{F}$, $\Delta^\vee$ is a dual root system.*
4. *$W = N_G(T)/T$ is the Weyl group of Δ.*
5. *Pick a system of positive roots $\Delta_+ \subset \Delta$ and order it in some way. The image of the product map*
$$\prod_{\alpha \in \Delta_+} U_\alpha \to G, (u_\alpha)_{\alpha \in \Delta_+} \mapsto \prod u_\alpha,$$
is a unipotent subgroup, say U of G, normalized by T. The subgroup $T \ltimes U$ is Borel. This gives a bijection between Borel subgroups containing T and systems of positive roots.

3.2. **Root data and classification of connected reductive groups.**

Our goal now is to explain the classification of connected reductive algebraic groups over $\mathbb{F}$. Somewhat surprisingly, this classification is independent of $\mathbb{F}$. To state this result we need to introduce the notion of a root datum.

Definition 3.4. *A (reduced) root datum* RD *is a quadruple $(\Lambda, \Lambda^\vee, \Delta, \Delta^\vee)$, where*

- *$\Lambda, \Lambda^\vee$ are mutually dual lattices,*
- *$\Delta \subset \Lambda, \Delta^\vee \subset \Lambda^\vee$ are dual reduced root systems.*

Example 3.5. *Let G be a connected reductive group and $T \subset G$ be a maximal torus. Then $(\mathfrak{X}(T), \mathfrak{X}(T)^*, \Delta, \Delta^\vee)$ is a root datum, where $\Delta, \Delta^\vee$ were defined in the previous section.*

[1] This is an analog of the fact that root spaces in the Lie algebra are 1-dimensional

Definition 3.6. *Let* $\mathsf{RD}_1 = (\Lambda_1, \Lambda_1^\vee, \Delta_1, \Delta_1^\vee), \mathsf{RD}_2 = (\Lambda_2, \Lambda_2^\vee, \Delta_2, \Delta_2^\vee)$ *be two root data. By a homomorphism* $\mathsf{RD}_1 \to \mathsf{RD}_2$ *we mean a lattice homomorphism* $\varphi : \Lambda_2 \to \Lambda_1$ *(yes, in the opposite direction) such that* $\varphi : \Delta_2 \xrightarrow{\sim} \Delta_1$ *and* $\varphi^*(\varphi(\alpha)^\vee) = \alpha^\vee$ *for all* $\alpha \in \Delta_2$.

Example 3.7. *Note that, for every homomorphism* $\varphi : G_1 \to G_2$*, the image of a maximal torus* $T_1 \subset G_1$ *is contained in a maximal torus* $T_2 \subset G_2$ *and hence we have the pullback map* $\mathfrak{X}(T_2) \to \mathfrak{X}(T_1)$*. If this map gives a root datum homomorphism, we say that* φ *is a* central isogeny*. Central isogenies include quotients by central subgroups (such as* $SL_n \twoheadrightarrow PGL_n$*) or inclusions like* $(G, G) \hookrightarrow G$*. We note however that the kernels of central isogenies may be nonreduced group subschemes, this is the case, for example, for* $SL_p \twoheadrightarrow PGL_p$ *in characteristic* p.

Theorem 3.8. *The following claims hold:*

1. *Isomorphism classes of reductive algebraic groups are in bijection with isomorphism classes of root data via the construction of Example 3.5.*
2. *Let* $\mathsf{RD}_1, \mathsf{RD}_2$ *be two root data and* G_1, G_2 *be the corresponding connected reductive groups. Then any homomorphism* $\mathsf{RD}_1 \to \mathsf{RD}_2$ *gives rise to a homomorphism* $G_1 \to G_2$ *and vice versa.*

Let us mention certain special classes of root data.

Example 3.9. *Suppose that* $\Lambda^\vee = \mathbb{Z}\Delta^\vee$ *(so that* Λ *is the weight lattice of the corresponding root system). The corresponding root datum and algebric groups are called* simply connected *(because this is literally so over* $\mathbb{C}$*). Examples of simply connected groups include* SL_n *or* Sp_{2n}*. If* $\mathsf{RD} = (\Lambda', \Lambda'^\vee, \Delta, \Delta^\vee)$ *is another root datum, then we have a unique homomorphism* $((\mathbb{Z}\Delta^\vee)^*, \mathbb{Z}\Delta^\vee, \Delta, \Delta^\vee) \to \mathsf{RD}$.

Example 3.10. *Dually, suppose that* $\Lambda = \mathbb{Z}\Delta$ *(so that* $\Lambda^\vee$ *is the coweight lattice). The corresponding root datum is called adjoint (or "of adjoint type"). It has a universal property dual to that of a simply connected root datum. The corresponding group is also called adjoint. Examples include* SO_{2n+1} *and* PGL_n.

Remark 3.11. *We note that if* $(\Lambda, \Lambda^\vee, \Delta, \Delta^\vee)$ *is a root datum, then so is* $(\Lambda^\vee, \Lambda, \Delta^\vee, \Delta)$ *– the dual root datum. If* G *corresponds to* $(\Lambda, \Lambda^\vee, \Delta, \Delta^\vee)$*, then the group corresponding to* $(\Lambda^\vee, \Lambda, \Delta^\vee, \Delta)$ *is called the Langlands dual of* G *and is denoted by* $G^\vee$*. Landglands dual pairs include* (Sp_{2n}, SO_{2n+1})*,* (SL_n, PGL_n)*,* (GL_n, GL_n)*. Note that Langlands dual groups have the same Weyl groups.*

Remark 3.12. *Note that by (2) of Theorem 3.8 every automorphism of a root datum corresponds to an automorphism of the corresponding group* G*. In particular, the multiplication by* -1 *in* Λ *is an automorphism of a root datum. The corresponding automorphism of* G *is called the Cartan involution, we denote it by* τ*. By definition, it restricts to* $t \mapsto t^{-1}$ *on* T *and maps every Borel containing* T *to its opposite. For example, for* $G = GL_n$*, we get* $\tau(A) = (A^T)^{-1}$.

4. Flag variety and Bruhat decomposition

We fix a Borel subgroup $B \subset G$ and consider the *flag variety* $\mathfrak{B} := G/B$. This is a smooth projective variety of dimension $|\Delta_+|$. It can be identified with the variety of Borel subgroups of G via the map $gB \mapsto gBg^{-1}$.

Example 4.1. *In the case $G = GL_n$ we get the variety Fl of full flags in $\mathbb{F}^n$.*

For each $w \in W$, fix a lift $\dot{w} \in N_G(T)$. Consider the locally closed subvariety $BwB := B\dot{w}B \subset G$. Note that $BwB = UwB$, where we write U for $R_u(B)$. Set $X_w := BwB/B \subset \mathfrak{B}$.

Proposition 4.2. *(Bruhat decomposition) The following claims hold:*
(1) $G = \bigsqcup_{w \in W} BwB$, *hence* $\mathfrak{B} = \bigsqcup_{w \in W} X_w$.
(2) X_w *is a single* U*-orbit, it is an affine space of dimension* $\ell(w)$.

4.1. Bruhat order.

Definition 4.3. *Define the Bruhat order on W by $w' \preceq w$ if $X_{w'} \subset \overline{X}_w$.*

Proposition 4.4. *The order $\preceq$ can be described combinatorially in one of the following ways:*

(1) this is a transitive closure of the relation $w' \leftarrow w$, where we write $w' \leftarrow w$ if $\ell(w) > \ell(w')$ and $w = s_\alpha w'$ for some $\alpha \in \Delta$.

(2) $w' \preceq w$ if and only if there is a reduced expression $w = s_1 s_2 \dots s_k$ in the product of simple reflections and a sequence $i_1 < i_2 < \dots < i_p$ with $w' = s_{i_1} s_{i_2} \dots s_{i_p}$.

Example 4.5. *For $W = S_3$, the Bruhat order is the order on the vertices of the following graph:*

$$\begin{tikzcd}
 & s_1 \arrow[dl] & s_1 s_2 \arrow[l] \arrow[ddl] & \\
e & & & s_1 s_2 s_1 = s_2 s_1 s_2 \arrow[ul] \arrow[dl] \\
 & s_2 \arrow[ul] & s_2 s_1 \arrow[l] \arrow[uul] &
\end{tikzcd}$$

5. Parabolic versions

As before, below G is a connected reductive group.

5.1. Parabolic and Levi subgroups.

Definition 5.1. *A subgroup $P \subset G$ is called parabolic if G/P is projective.*

Example 5.2. *For $G = GL_n$, the parabolic subgroups are precisely the stabilizers of partial flags.*

The following proposition gives some properties of parabolic subgroups.

Proposition 5.3. *The following claims hold:*

(1) *A subgroup is parabolic if and only if it contains a Borel subgroup.*
(2) *Every parabolic subgroup is connected.*
(3) *Every parabolic (in particular, Borel) subgroup coincides with its normalizer in G.*

Let us discuss a related notion: that of a Levi subgroup.

Definition 5.4. *By a Levi subgroup of G we mean a centralizer of a subtorus.*

Note that such a subgroup necessarily contains a maximal torus.

Example 5.5. *For $G = GL_n$, the Levi subgroups are precisely the stabilizers of ordered direct sum decompositions $\mathbb{F}^n = V_1 \oplus \ldots \oplus V_k$.*

Proposition 5.6. *The following claims hold:*

(1) *Every Levi subgroup is connected and reductive.*
(2) *For every Levi subgroup L, there is a parabolic subgroup P such that $P = L \ltimes R_u(P)$.*
(3) *Conversely, any parabolic subgroup P decomposes as $P = L \ltimes R_u(P)$, where L is a Levi.*

Let us now describe *standard* parabolic and Levi subgroups of G. Let I denote the set of simple roots. Pick $J \subset I$ and let $W_J \subset W$ be the subgroup generated by $s_\alpha, \alpha \in J$ (such subgroups are called *parabolic*). Let L_J be the subgroup generated by T as well as $U_\alpha, U_{-\alpha}, \alpha \in J$. This is a Levi subgroup, it is the centralizer of the connected component of $\cap_{\alpha \in J} \ker \alpha$. Next, $P_J := L_J B$ is a parabolic subgroup. These Levi and parabolic subgroups are called *standard*.

Proposition 5.7. *Every parabolic subgroup $P \subset G$ is conjugate to P_J for a unique $J \subset I$.*

5.2. **Partial flag varieties.** Fix $J \subset I$ and consider a parabolic subgroup $P = P_J$. The variety G/P is smooth and projective, it will be called *partial flag variety* of G.

Proposition 5.8 (Bruhat decomposition of G/P)**.** *We have $G/P = \bigsqcup_{w \in W/W_J} X_{[w]}$, where $X_{[w]}$ is $U\dot{w}P \subset G/P$. The varieties $X_{[w]}$ are affine spaces of dimensions $\ell(u)$, where u is the shortest element in the coset .*

Example 5.9. *Let $G = GL_n$ and P be the stabilizer of a line in $\mathbb{F}^n$. Then we have $G/P = \mathbb{P}^{n-1}$. The Bruhat decomposition in this case is the well-known decomposition of $\mathbb{P}^{n-1}$ into affine cells. And for grassmannians, we recover the classical Schubert decomposition.*

So we can also talk about the *parabolic* Bruhat order on W/W_J, denote it by $\preceq_J$.

Proposition 5.10. *$w'W_J \preceq_J wW_J$ if and only if $u' \preceq u$, where $u' \in w'W_J, u \in uW_J$ are the shortest representatives.*

6. Rational representations in characteristic 0

6.1. **Classification.** Now let $\operatorname{char} \mathbb{F} = 0$ so that every rational representation of every reductive group is completely reducible. Let us recall the classification of irreducibles. Fix $T \subset B \subset G$. For $V \in \operatorname{Rep}(G)$ we have the weight decomposition $V = \bigoplus_{\chi \in \mathfrak{X}(T)} V_\chi$, if $V_\chi \neq \{0\}$, we say that χ is a weight of T. Note that $\mathfrak{X}(T)$ comes with a partial order: $\chi_1 \leqslant \chi_2$ if $\chi_2 - \chi_1$ is a sum of positive roots. Note that for a highest (=maximal w.r.t. this order) weight λ of V, the group B acts on V_χ via (the pullback of) χ. Let us write Λ for $\mathfrak{X}(T)$ and Λ^+ for the submonoid of dominant weights i.e. $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \geqslant 0 \ \forall \alpha_i \in I\}$.

Theorem 6.1. *The irreducible rational representations of G are classified by Λ^+ via taking the highest weight.*

6.2. **Borel-Weil theorem.** For $\lambda \in \Lambda$ we denote by $\mathcal{O}(\lambda)$ the homogeneous vector bundle on G/B with fiber $\mathbb{F}_{w_0\lambda}$. For example, for $G = SL_2$ and $\lambda = n \in \mathbb{Z}$, $\mathcal{O}(\lambda)$ is the usual line bundle $\mathcal{O}(n)$.

Theorem 6.2. *For $\lambda \in \Lambda^+$, we have $L(\lambda) \simeq H^0(\mathfrak{B}, \mathcal{O}(\lambda))$. Furthermore, $\mathcal{O}(\lambda)$ has no higher cohomology.*

6.3. **Weyl character formula.** Define the character $\operatorname{ch} V := \sum_{\chi \in \Lambda} \dim(V_\chi) e^\chi$, where e^χ is a formal symbol. Since, for each $w \in W$, we have $\dot{w} V_\chi = V_{w(\chi)}$, the character is W-invariant.

Theorem 6.3. *For $\lambda \in \Lambda^+$ we have*

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}. \tag{6.4}$$

This theorem has several proofs. In fact, one can use localization in equivariant K-theory to prove that the Euler characteristic of $\mathcal{O}(\lambda)$ (viewed as a representation of T) is given by (6.4). Then one deduces Theorem 6.3 from Theorem 6.2.