HIGHEST WEIGHT CATEGORIES

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Let $\mathbb{F}$ be a field, Λ be a finite poset and $\mathcal{C}$ be an $\mathbb{F}$-linear artitian category with simple objects parameterized by Λ. By a highest weight category structure on $\mathcal{C}$ with respect to Λ we mean a collection of *standard* objects $\Delta(\lambda)$ in $\mathcal{C}$, one for each $\lambda \in \Lambda$ such that

- $\operatorname{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leqslant \mu$.
- $\operatorname{End}(\Delta(\lambda)) = \mathbb{F}$ for any λ.
- $\mathcal{C}$ has enough projectives. The indecomposable projectives are parameterized by $\Lambda, \lambda \mapsto P(\lambda)$, where $P(\lambda)$ is the projective cover of $L(\lambda)$. The object $P(\lambda)$ surjects onto $\Delta(\lambda)$ and the kernel admits a filtration whose quotients are of the form $\Delta(\mu)$ with $\mu > \lambda$.

We write $\mathcal{C}^{\Delta}$ for the full subcategory of all standardly filtered objects in $\mathcal{C}$, i.e., all objects that admit a filtration with standard quotients.

0.1. **Simple constituents of standards.** Let $L(\lambda)$ denote the simple corresponding to $\lambda \in \Lambda$. Show that if $L(\lambda)$ occurs in $\Delta(\mu)$, then $\lambda \leqslant \mu$. Moreover, show that the multiplicity of $L(\lambda)$ in $\Delta(\lambda)$ is 1, and $\Delta(\lambda) \twoheadrightarrow L(\lambda)$.

0.2. **Ext's between standards.** Show that if $\operatorname{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$, then $\lambda < \mu$.

0.3. **Subcategories.** Let $\Lambda' \subset \Lambda$ be an ideal in the sense that if $\lambda \in \Lambda'$ and $\mu \leqslant \lambda$, then $\mu \in \Lambda'$. Consider the Serre subcategory $\mathcal{C}' \subset \mathcal{C}$ spanned by the simples $L(\lambda)$ with $\lambda \in \Lambda'$.

1. Show that $\mathcal{C}'$ is a highest weight category with standard objects $\Delta(\lambda), \lambda \in \Lambda'$.
2. Show that the left adjoint functor $\iota^!$ to the embedding $\mathcal{C}' \hookrightarrow \mathcal{C}$ is exact on $\mathcal{C}^{\Delta}$.
3. Deduce that there is a projective resolution $P^{\bullet}$ in $\mathcal{C}$ of $M \in \mathcal{C}'^{\Delta}$ such that $\iota^!(P^{\bullet})$ is a projective resolution of M in $\mathcal{C}'$ (in fact, this is true for any projective resolution).

0.4. **Quotients.** Now let $\mathcal{C}'' := \mathcal{C}/\mathcal{C}'$ and π be the quotient functor $\mathcal{C} \twoheadrightarrow \mathcal{C}''$. Let $\pi^!$ denote its left adjoint functor.

1. Show that $\mathcal{C}''$ is a highest weight category with standard objects $\pi(\Delta(\lambda))$.
2. Moreover, show that the natural morphism $\pi^!(\pi(M)) \to M$ is an isomorphism provided M admits a filtration whose quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.
3. Deduce that $\pi^!$ gives rise to an equivalence between $\mathcal{C}''^{\Delta}$ and a full subcategory in $\mathcal{C}^{\Delta}$ consisting of all objects whose filtration quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.

0.5. **Characterization of projectives.** Show that a standardly filtered object $P \in \mathcal{C}$ is projective if and only if $\operatorname{Ext}^1(P, \Delta(\lambda)) = 0$ for all $\lambda \in \Lambda$.

0.6. **Axiomatic characterization of standards.** Show that the objects $\Delta(\lambda)$ are uniquely recovered from the poset structure on Λ as follows. Pick $\lambda \in \Lambda$. Let $\Lambda_{\leqslant\lambda} = \{\mu \in \Lambda | \mu \leqslant \lambda\}$. Consider the Serre subcategory $\mathcal{C}_{\leqslant\lambda}$ spanned by the simples $L(\mu)$ with $\mu \in \Lambda_{\leqslant\lambda}$. Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{C}_{\leqslant\lambda}$.

0.7. **Costandard objects.** Let $\nabla(\lambda)$ stand for the injective hull of $L(\lambda)$ in $\mathcal{C}_{\leqslant\lambda}$. This is a so called *costandard object*. Show that $\dim \operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$, while $\operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ for $i > 0$.

0.8. **Highest weight structure on $\mathcal{C}^{opp}$.** Prove that the collection of costandard objects $\nabla(\lambda)$ makes the opposite category $\mathcal{C}^{opp}$ into a highest weight category with respect to the poset Λ.

0.9. **Characterization of (co)standardly filtered objects.** Prove the object $M \in \mathcal{C}$ is Δ-filtered if and only if $\operatorname{Ext}^i(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda, i > 0$, if and only if $\operatorname{Ext}^1(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. Deduce the kernel of an epimorphism of standardly filtered objects is standardly filtered. State and prove the dual statements.

0.10. **BGG reciprocity.** Show that the multiplicity of $\Delta(\lambda)$ in $P(\mu)$ coincides with the multiplicity of $L(\mu)$ in $\nabla(\lambda)$.

0.11. **Tilting objects.** An object in $\mathcal{C}$ is called *tilting* if it is both standardly and costandardly filtered.

(1) For $\lambda \in \Lambda$ consider an object $T(\lambda)$ constructed as follows. Order linearly elements of $\{\mu \in \Lambda | \mu \leqslant \lambda\}$ refining the original poset structure, say $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_k$. Construct the object $T^i(\lambda), i = 1, \ldots, k$ inductively as follows. Set $T^1(\lambda) = \Delta(\lambda)$. Further, if $T^{i-1}(\lambda)$ is already defined let $T^i(\lambda)$ be the extension of $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$ by $T^{i-1}(\lambda)$ corresponding to the unit endomorphism of $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$. Show that $T(\lambda) := T^k(\lambda)$ is an indecomposable tilting.
(2) Prove that any other tilting in $\mathcal{C}$ is isomorphic to the direct sum of the objects $T(\lambda)$.

0.12. **Ringel duality.** Set $T := \bigoplus_{\lambda\in\Lambda} T(\lambda)$. Let $\mathcal{C}^\vee$ be the category of finitely generated $\operatorname{End}(T)$-modules. Show that this category is highest weight with respect to the opposite poset Λ^{opp} with standard objects $\operatorname{Hom}(\Delta(\lambda), T)$. Show that $(\mathcal{C}^\vee)^\Delta \cong (\mathcal{C}^\Delta)^{opp}$ and that, under this identification, the projective objects in $\mathcal{C}$ correspond to tilting objects in $\mathcal{C}^\vee$, while tilting objects in $\mathcal{C}$ correspond to projective objects in $\mathcal{C}^\vee$. Finally, identify $(\mathcal{C}^\vee)^\vee$ with $\mathcal{C}^{opp}$.

1. Bonus!

1.1. **Quasi-hereditary algebras.** The goal of this problem is to characterize algebras whose categories of modules are highest weight. Still, $\mathbb{F}$ is a field, Λ is a finite poset. Recall that by a *coideal* in Λ we mean a subset of $\Lambda' \subset \Lambda$ such that $\lambda' \in \Lambda', \lambda \geqslant \Lambda'$ implies $\lambda \in \Lambda'$.

Let A be a finite dimensional $\mathbb{F}$-algebra. A structure of a *(split) quasi-hereditary algebra* on A is a collection of ideals $I(\Lambda')$ indexed by coideals $\Lambda' \subset \Lambda$ and satisfying the following conditions:

- If $\Lambda' \subset \Lambda''$, then $I(\Lambda') \subset I(\Lambda'')$.
- We have $I(\Lambda) = A, I(\varnothing) = \{0\}$.
- Suppose $\Lambda' \subset \Lambda''$, and $\Lambda'' \setminus \Lambda'$ consists of one element. Then $I(\Lambda'') = I(\Lambda') + I(\Lambda'')^2$, $I(\Lambda'')/I(\Lambda')$ is a projective $A/I(\Lambda')$-module whose endomorphism algebra is a matrix algebra over $\mathbb{F}$.

Obviously, the last bullet is the most important condition.

Prove that giving A a quasi-hereditary algebra structure is the same as giving A-mod a structure of a highest weight category.

1.2. **Alternative characterization of a highest weight category.** Let $\mathcal{C}$ be the category of modules over a finite dimensional $\mathbb{F}$-algebra, and let Λ be a finite poset identified with the set of simples in $\mathcal{C}$. So we can consider the Serre subcategory $\mathcal{C}_{\leqslant\lambda} \subset \mathcal{C}$. Inside we can consider the projective cover $\Delta(\lambda)$ and the injective hull $\nabla(\lambda)$ of the simple $L(\lambda)$ labelled by λ.

a) Prove that $\mathcal{C}$ is a highest weight category (with poset Λ and standard objects $\Delta(\lambda)$) if and only if $\dim \operatorname{Ext}^i_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) = \delta_{i0}\delta_{\lambda\mu}$.

b) Suppose that the equivalent conditions of a) hold. Show that the natural functor $D^b(\mathcal{C}_{\leqslant\lambda}) \hookrightarrow D^b(\mathcal{C})$ is full (i.e., it does not matter whether we take Ext's of objects of $\mathcal{C}_{\leqslant\lambda}$ in $\mathcal{C}_{\leqslant\lambda}$ or in $\mathcal{C}$).