Injective rational representations.

The main goal of this note is to prove the following thm.

Thm: Let X_0 be a finite type affine scheme over a field $\mathbb{F}$, let G be an algebraic group over $\mathbb{F}$, and let $X \longrightarrow X_0$ be a principal G-bundle. Then $\mathbb{F}[X]$ is an injective object in the category of rational representations.

This is mentioned in Lec 19 (Sec 2.3). Below we prove this theorem

1) Injectivity criterium.

Proposition: A rational G-representation N is injective iff the functor $\mathrm{Hom}_G(\cdot, N)$ is exact on the category of <u>finite dimensional</u> rational representations.

This is morally similar to the Baer's injectivity criterium for modules over rings and is proved in the same fashion using transfinite induction: for G-modules $M \subset \tilde{M}$ we need to extend a homomorphism $M \to N$ to $\tilde{M}$. We look at the set of extensions $M' \to N$ (for $M \subset M' \subset \tilde{M}$), equip it w. a poset structure and use the Zorn lemma to show there's a maximal element, $\hat{M}$. If $\hat{M} \neq \tilde{M}$, pick $m \notin \hat{M}$, include it into a finite dimensional subrep $V \subset \tilde{M}$ and use the exactness of $Hom_G(\cdot, N)$ on the finite dimensional reps to extend $V \cap \hat{M} \to N$ to $V \to N$. This allows to extend from $\hat{M}$ to $\hat{M}+V$, a contradiction w. the maximality of $\hat{M}$.

Everybody loves transfinite induction, so the details are left as exercise.

2) Consequences.

Corollary 1: An arbitrary direct sum of injective reps is injective.

Proof is an exercise.

Corollary 2: $\mathbb{F}[G]$ is an injective rational rep (w. $G \curvearrowright \mathbb{F}[G]$ from the action $G \curvearrowright G$ on the left)

Proof: This follows from the following claim & Proposition

(*) For a finite dimensional rational G-rep V, we have $\mathrm{Hom}_G(V, \mathbb{F}[G]) \simeq V^*$, a functorial iso.

To prove (*), note that $\mathrm{Hom}_{vs}(V, \mathbb{F}[G]) \simeq \mathrm{Hom}_{Alg}(S(V), \mathbb{F}[G]) \simeq \mathrm{Mor}_{Sch}(G, V^*)$, note that the last expression is naturally a vector space b/c V^* is. ; $\mathrm{Hom}_G(V, \mathbb{F}[G]) \subset \mathrm{Hom}_{vs}(V, \mathbb{F}[G])$ corresponds to G-equivariant morphisms $G \longrightarrow V^*$ in $\mathrm{Mor}_{Sch}(G, V^*)$. Such a morphism is uniquely recovered from its value at 1, hence the claim. $\square$

The next claim follows from Corollaries 1&2.

Corollary 3: For any set I, the rational representation $\mathbb{F}[G]^{\oplus I}$ is injective.

3) Proof of the main result.

In the case when $X \simeq G\times X_0 \Leftrightarrow \mathbb{F}[X]\simeq \mathbb{F}[G]\otimes \mathbb{F}[X_0]$ follows from Cor 3. In the general case we can find a surjective etale morphism $\tilde{X}_0 \twoheadrightarrow X_0$ w. $\tilde{X}_0 \times_{X_0} X \xrightarrow{\sim} G\times \tilde{X}_0$. We'll use the injectivity of $\mathbb{F}[X]$ from that of $\mathbb{F}[\tilde{X}_0 \times_{X_0} X] = \mathbb{F}[\tilde{X}_0]\otimes_{\mathbb{F}[X_0]} \mathbb{F}[X]$.

We are going to check that the functor $V\mapsto \operatorname{Hom}_G(V,\mathbb{F}[X]) = (V^* \otimes \mathbb{F}[X])^G$ is exact on finite dimensional reps. For this, we need an interpretation of taking G-invariants.

To equip a vector space M w. a rational representation structure is the same as to equip M w. an $\mathbb{F}[G]$-comodule structure (see e.g. Lec 8.5 from MATH 603 in S22).

Consider the coaction map $\alpha_M: M \longrightarrow M\otimes \mathbb{F}[G]$. It is a $\mathbb{F}[G]$-comodule map, where on $M\otimes\mathbb{F}[G]$ the coalgebra $\mathbb{F}[G]$ coacts on the 2nd factor. The claim that this is a map of $\mathbb{F}[G]$-comodules is the coassociativity of the coaction.

On the other hand, we can consider the map $id_M\otimes \varepsilon$: $M \longrightarrow M\otimes \mathbb{F}[G]$, where $\varepsilon: \mathbb{F}\to \mathbb{F}[G]$ is the unit. Then $M^G = \ker[\alpha_M - id_M\otimes \varepsilon]$, exercise.

Now let $M = V^* \otimes \mathbb{F}[X]$. It comes w. an action of $\mathbb{F}[X_0] = \mathbb{F}[X]^G$ by G-linear endomorphisms. The map $\alpha_M - id_M \otimes \varepsilon$ is $\mathbb{F}[X_0]$-linear. Note that the analogous map for $\tilde{X}_0 \times_{X_0} X$ is obtained from $\alpha_M - id_M \otimes \varepsilon$ by applying the functor $\mathbb{F}[\tilde{X}_0] \otimes_{\mathbb{F}[X_0]} \bullet$. This functor is exact and faithful b/c $\mathbb{F}[\tilde{X}_0]$ is a fully faithful $\mathbb{F}[X_0]$-module. From the exactness it then follows that $[\mathbb{F}[\tilde{X}_0] \otimes_{\mathbb{F}[X_0]} (V^* \otimes \mathbb{F}[X])]^G \xrightarrow{\sim} (V^* \otimes \mathbb{F}[\tilde{X}_0 \times_{X_0} X])^G$. The latter is isomorphic to $(V^* \otimes \mathbb{F}[G \times \tilde{X}_0])^G \simeq V^* \otimes \mathbb{F}[\tilde{X}_0]$.

So $\mathbb{F}[\tilde{X}_0] \otimes_{\mathbb{F}[X_0]} \mathrm{Hom}_G(\cdot, \mathbb{F}[X])$ is an exact functor. And $\mathbb{F}[\tilde{X}_0] \otimes_{\mathbb{F}[X_0]} \bullet$ is exact & faithful. It follows that $\mathrm{Hom}_G(\cdot, \mathbb{F}[X_0])$ is an exact functor (this implication is an exercise).