HW3 Solutions

1. a) Each eigenspace for $\tilde{T}$ in $\mathbb{C}[V]$ is 0- or 1-dim'l and $\mathbb{C}[V]^T \underset{T}{\hookrightarrow} \mathbb{C}[V]$ so the same holds for $\tilde{T}/T \cap \mathbb{C}[V]^T$. By Lec 10, $\mathbb{C}[V]^T = \mathrm{Span}(x_1^{m_1} \dots x_n^{m_n})$ w. $(m_1 \dots m_n) \in \mathcal{M}$, which implies the claim of (a). Set $f_\psi = x_1^{m_1} \dots x_n^{m_n}$ for $\psi = (m_1 \dots m_n) \in \mathcal{M}$ so that $f_{\psi_1} f_{\psi_2} = f_{\psi_1 + \psi_2}$ for $\psi_1, \psi_2 \in \mathcal{M}$

b) The previous equality and $\mathbb{C}[V]^T = \mathrm{Span}_{\psi \in \mathcal{M}}(f_\psi)$ shows that $\psi_1 \dots \psi_k$ gen'te $\mathcal{M} \Leftrightarrow f_{\psi_1} \dots f_{\psi_k}$ gen'te $\mathbb{C}[V]^T$. Now we know $\mathbb{C}[V]^T$ is fin. gen'd, then we can find fin. many gen'rs that are homog's in the $\mathfrak{X}(\tilde{T}/T)$-grading. We can assume the gen'rs are $f_{\psi_1} \dots f_{\psi_k}$, then $\psi_1 \dots \psi_k$ generate $\mathcal{M}$ (of course it's not difficult to show directly that a submonoid of a fin. gen'd monoid is fin. gen'd)

c) We consider the algebra $\mathbb{C}[y_1 \dots y_k]$ and define a $\mathfrak{X}(\tilde{T}/T)$-grading by putting y_i in degree ψ_i. Let's consider the ideal $I \subset \mathbb{C}[y_1 \dots y_k]$ spanned by $\prod y_i^{r_i} - \prod y_i^{s_i}$ for $\sum_i (r_i - s_i)\psi_i = 0$. This is a graded ideal. Moreover it lies in the kernel of $\mathbb{C}[y_1 \dots y_k] \twoheadrightarrow \mathbb{C}[V]^T$ given by $y_i \mapsto f_{\psi_i}$. So $\mathbb{C}[y_1 \dots y_k]/I \twoheadrightarrow \mathbb{C}[V]^T$. Note that deg ψ component of $\mathbb{C}[y_1 \dots y_k]/I$ is 1-dim'l for $\psi \in \mathcal{M}$ by the constr'n of I and is 0 for $\psi \notin \mathcal{M}$. By (a), $\mathbb{C}[y_1 \dots y_k]/I \twoheadrightarrow \mathbb{C}[V]^T$ is an iso

2) Given a one-parameter subgroup γ, let U_+ (resp. U_-, U_0) denote the sum of e-spaces for γ corresponding char'r $t \mapsto t^i$ w $i > 0$ (resp $t \mapsto 1$ & $t \mapsto t^i$ w $i < 0$) so that $U = U_+ \oplus U_0 \oplus U_-$. Recall that $\lim_{t \to 0} \gamma(t) u = 0 \Leftrightarrow u \in U_+$

a) $(u_1 \dots u_k, u^1 \dots u^\ell)$ is nilp. $\Leftrightarrow \exists\, \gamma: \mathbb{C}^\times \to GL_n$ w $\lim_{t \to 0} \gamma(t)(u_1, \dots u_k, u^1, \dots, u^\ell) = 0$ $\Leftrightarrow u_i \in U_+,\ u^j \in (U^*)_+ \cong (U_-)^*\ \forall i,j \Rightarrow \langle u_i, u^j \rangle = 0$. Conversely, suppose $\langle u_i, u^j \rangle = 0\ \forall i, j$. Then set $U_1 = \mathrm{Span}_{\mathbb{C}}(u_i)_{i=1,k}$ and let U_2 be a compl't to U_1 in U so that $u^j \in U_2^*\ \forall j$. For any $\gamma: \mathbb{C}^\times \to GL_n$ w $U_+ = U_1$, $U_- = U_2$ we have $\lim_{t \to 0} \gamma(t)(u_1 \dots u_k, u^1 \dots u^\ell) = 0$ & such γ clearly exists.

b) Here both U_+ & U_- are nontrivial (if $\gamma \neq 1$) b/c det = 1. Moreover, for any proper $U_1 \subset U$ we can find γ w. $U_1 = U_+$. This finishes the proof

c) We have $U_+ \oplus U_0 = U_+^\perp$, $U_- \oplus U_0 = U_-^\perp$, $U_0^\perp = U_0$. So U_+ is isotropic. So if $(u_1, \dots u_k)$ is nilp't, then $\mathrm{Span}(u_1, \dots u_k)$ is isotropic. Conversely, for any isotropic $U_1 < U$ $\exists$ τ s.t $U_1 = U_+$. So if $\mathrm{Span}(u_1, \dots u_k)$ is isotropic, then $(u_1, \dots u_k)$ is nilpotent.