Problem 1, 5pts. As in HW3, consider a faithful action of a torus $T$ on a vector space $V$. Recall that $V$ has an eigen-basis $v_1,\ldots,v_n$, let $\chi_1,\ldots,\chi_n$ be the corresponding eigen-characters of $T$. Let $\tilde{T} \subset \text{GL}(V)$ be the maximal torus of all operators diagonal in the basis $v_1,\ldots,v_n$. So $T \subset \tilde{T}$. Pick a character $\theta$ of $T$.

a, 2pts) Show that $V^{\theta-ss} \subset V$ is $\tilde{T}$-stable, $\tilde{T}/T$ acts on $V/\theta T$ in such a way that $\pi^\theta : V^{\theta-ss} \rightarrow V/\theta T$ is $\tilde{T}$-equivariant.

b, 3pts) Show that the fixed points of $\tilde{T}/T$ on $V/\theta T$ are in bijection with the subsets $I \subset \{1,\ldots,n\}$ satisfying the following two conditions

\begin{itemize}
    \item $\chi_i, i \in I$, are linearly independent,
    \item and there are rational numbers $n_i, i \in I$, such that $\theta = \sum_i n_i \chi_i$ and $n_i < 0$ for all $i \in I$.
\end{itemize}

Problem 2, 5pts. Let $G$ be a connected factorial reductive algebraic group (i.e., $\mathbb{C}[G]$ is a UFD), let $H$ be an algebraic subgroup of $G$. Note that we have the restriction map $\rho : \mathfrak{x}(G) \rightarrow \mathfrak{x}(H)$ between the character groups. Prove that $\text{Pic}(G/H) \cong \text{coker} \rho$.

Problem 3, 5pts. Let $X$ be an affine algebraic variety equipped with an action of a reductive algebraic group $G$. Show that there are finitely many reductive subgroups $H_1,\ldots,H_k \subset G$ such that every closed $G$-orbit in $X$ is $G$-equivariantly isomorphic to one of $G/H_i$. 

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