HINTS FOR HOMEWORK 1

Here are hints for Homework 1. You are welcome to use them not allowed to refer to them as to facts in your solutions, in particular, you need to elaborate.

Problem 1 (3pts). For every positive integer n, give an example of an order n group G and its finite dimensional representation V such that $\mathbb{C}[V]^G$ is not generated by invariants of degree less than n.

Hint: The example is VERY easy.

Problem 2 (6pts). Let V be a vector space over $\mathbb{C}$ and G a finite subgroup of $\mathrm{GL}(V)$. Prove the following to show that the generic rank of the $\mathbb{C}[V]^G$-module $\mathbb{C}[V]$ equals $|G|$, i.e.

$$\dim_{\mathrm{Frac}(\mathbb{C}[V]^G)} \mathrm{Frac}(\mathbb{C}[V]^G) \otimes_{\mathbb{C}[V]^G} \mathbb{C}[V] = |G|.$$

a, 2pts) $\dim_{\mathbb{C}(V)^G} \mathbb{C}(V) = |G|$.

b, 2pts) $\mathbb{C}(V) = \mathbb{C}(V)^G \otimes_{\mathbb{C}[V]^G} \mathbb{C}[V]$.

c, 2pts) $\mathbb{C}(V)^G = \mathrm{Frac}(\mathbb{C}[V]^G)$.

Hint: a) is a Galois-theoretic statement, b) just means that every rational function can be represented as a ratio with a G-invariant denominator, and c) is a corollary of b).

Problem 3 (4pts). Let X be a factorial affine algebraic variety (factorial means that $\mathbb{C}[X]$ is a unique factorization domain) and G a connected algebraic group that has no nontrivial homomorphisms to $\mathbb{C}^\times$.

a, 2pts) Show that $\mathbb{C}[X]^G$ is a unique factorization domain as well.

b, 2pts) Show that there are finitely many elements $f_1, \ldots, f_k \in \mathbb{C}[X]^G$ and a Zariski open G-stable subset $X' \subset X$ such that for two points $x_1, x_2 \in X'$, the following are equivalent:

- $f_i(x_1) = f_i(x_2)$ for all i,
- and $Gx_1 = Gx_2$.

Hint: a) what you need to show is that the prime factors of a G-invariant are G-invariant. In b) you need to use Rosenlicht's theorem.

Extra-credit problem. Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. Suppose that the G-orbits in V are separated by G-invariant polynomials. Prove that G is finite.