HINTS FOR HOMEWORK 2

Problem 1 (5pts). Let $G_1, G_2$ be algebraic groups such that $G_1 = G_n^a$ and the connected component $G_2^\circ$ is a torus. Show that there are no nontrivial algebraic group homomorphisms between $G_1, G_2$ in either direction.

Hint: Show that there are no algebraic variety morphisms from $G_1$ to $G_2$. To show that there are no algebraic group homomorphisms from $G_2$ to $G_1$ look at finite order elements.

Problem 2 (4pts). Let $H$ be a connected algebraic group and $Z$ be a finite normal subgroup of $H$. Show that $Z$ is in the center of $H$.

Hint: Look at the conjugation action of $H$ on $Z$.

Problem 3 (4pts). Let $G$ be a semisimple algebraic group, $g$ its Lie algebra and $x \in g$. Assume that the centralizer $Z_G(x)$ is reductive. Show that $x$ is semisimple. You are allowed to use facts proved in Lecture 9.

Hint: Reduce to the case when $x$ is nilpotent. Then you could get an inspiration from the proof of Kostant’s theorem in Lecture 9 (that for two $\mathfrak{sl}_2$-triples $(e, h, f), (e, h', f')$ there is $g \in Z_G(e)$ mapping $h$ to $h'$). Or you could use the fact that the centralizer of a reductive subgroup in a reductive group is reductive.

Extra-credit problem. This problem explains the classification of nilpotent orbits in the classical Lie algebras $\mathfrak{so}_n, \mathfrak{sp}_n$ (in the latter case $n$ is even, of course).

a) Show that the nilpotent $O_n$-orbits in $\mathfrak{so}_n$ and the nilpotent $Sp_n$-orbits in $\mathfrak{sp}_n$ are uniquely recovered form their Jordan types (a partition of $n$).

b) The partitions appearing for $\mathfrak{so}_n$ (resp., $\mathfrak{sp}_n$) are precisely those where the multiplicity of every even (resp., odd) part is even.

c) Show that a nilpotent $O_n$-orbit splits into two $SO_n$-orbits if and only if the parts of the corresponding partition are all even.