INVARIANT THEORY, HINTS FOR HW4

Problem 1 (5pts). Recall that $\mathcal{D}_n^+ \subset \mathrm{Mat}_k^+(\mathbb{C})$ denotes the scheme theoretic intersection of the determinantal scheme $\mathcal{D}_n$ of all matrices of rank $\leqslant n$ with the subspace $\mathrm{Mat}_k^+(\mathbb{C})$ of symmetric $k \times k$-matrices. Assuming the scheme $\mathcal{D}_n^+$ is reduced and normal, prove that $(\mathbb{C}^n)^{\oplus k} /\!/ \mathrm{O}_n(\mathbb{C}) \xrightarrow{\sim} \mathcal{D}_n^+$.

Hint: Show that every fiber of $A \mapsto A^T A : \mathrm{Mat}_{n,k}(\mathbb{C}) \to \mathcal{D}_n$ contains a single orbit of matrices B such that $\operatorname{im} B \oplus \ker B^T = \mathbb{C}^n$. Then argue as in the proof of Theorem 1 from Lecture 13.

Problem 2 (5pts). Show that $\mathrm{Mat}_n^+(\mathbb{C}) /\!/ \mathrm{O}_n(\mathbb{C})$ (the quotient under the action $g.A := gAg^{-1}$) is the polynomial algebra in the polynomials $F_i, i = 1, \ldots, n$, given by $F_i(A) := \mathrm{Tr}(A^i)$.

Hint: We can identify $\mathbb{C}^n \otimes \mathbb{C}^n$ with $\mathrm{Mat}_n(\mathbb{C})$. Show that under this identification swapping the tensor factors corresponds to taking the transpose of a matrix. Deduce that $\mathbb{C}[\mathrm{Mat}_n(\mathbb{C})]^{\mathrm{O}_n(\mathbb{C})}$ is generated by traces of the monomials in A, A^T. Then argue as in Remark after Theorem 1 of Lecture 14.

Problem 3 (5pts). Consider the Grassmanian cone $\mathrm{CGr}(k,n) \subset \Lambda^k \mathbb{C}^n$ of decomposable wedges with the reduced scheme structure. Define $f \in \mathbb{C}[\mathrm{CGr}(k,n)]$ that sends $\sum_\alpha c_\alpha e_{\alpha_1} \wedge e_{\alpha_2} \wedge \ldots \wedge e_{\alpha_k}$, where the summation is taken over all increasing sequences $\alpha = (\alpha_1 < \alpha_2 < \ldots < \alpha_k)$, to $c_{(1,2,\ldots,k)}$. Show that $\mathbb{C}[\mathrm{CGr}(k,n)]^U$ (where U is the subgroup of the unitriangular matrices in GL_n) is the polynomial algebra in the single variable f.[1]

Hint: Show that the restriction map from $\mathbb{C}[\mathrm{CGr}(k,n)_f]^U$ to $\mathbb{C}[\mathbb{C}^\times e_1 \wedge e_2 \wedge \ldots e_k]$ is an isomorphism. Then use an argument similar to a part of the proof of Theorem 2' in Lecture 16.

Extra-credit problem. Here we describe a canonical G-algebra with given algebra of U-invariants. We also show that for any G-algebra, there is a G-equivariant filtration whose associated graded is this canonical algebra.

a) Let A be a G-algebra, i.e., a commutative associative unital algebra equipped with a locally finite algebraic action of G. For $\lambda \in \mathfrak{X}^+$, let

[1] This shows that $\mathrm{CGr}(k,n)$ is a normal variety. In fact, one can also show that it is reduced with its natural scheme structure given by the Plücker equations.

A_λ denote the isotypic component of $V(\lambda)$ in A. We say that A is $\mathfrak{X}^+$-graded if $A = \bigoplus_\lambda A_\lambda$ is an algebra grading. Show that the categories of $\mathfrak{X}^+$-graded G-algebras is equivalent to the category of $\mathfrak{X}^+$-graded algebras via $A \mapsto A^U$. Moreover, show that a quasi-inverse functor is given by $B \mapsto (\mathbb{C}[G/U] \otimes B)^T$, where the action of T on $V(\mu) \otimes A_\lambda$ is by $t \mapsto \mu(t)\lambda(t)^{-1}$.

b) Now let A be any G-algebra. Define an ascending $\mathbb{Z}_{\geqslant 0}$-filtration $A_{\leqslant i}$ on A by $A_{\leqslant i} := \bigoplus_{\langle \lambda, \rho^\vee \rangle \leqslant i} A_\lambda$. Show that it is an algebra filtration and that $\operatorname{gr} A = (\mathbb{C}[G/U] \otimes A^U)^T$. In particular, $\operatorname{gr} A$ is finitely generated.