HOMEWORK 1

Problem 1 (3pts). For every positive integer n, give an example of an order n group G and its finite dimensional representation V such that $\mathbb{C}[V]^G$ is not generated by invariants of degree less than n.

Problem 2 (6pts). Let V be a vector space over $\mathbb{C}$ and G a finite subgroup of $\mathrm{GL}(V)$. Prove the following to show that the generic rank of the $\mathbb{C}[V]^G$-module $\mathbb{C}[V]$ equals $|G|$, i.e.

$$\dim_{\mathrm{Frac}(\mathbb{C}[V]^G)} \mathrm{Frac}(\mathbb{C}[V]^G) \otimes_{\mathbb{C}[V]^G} \mathbb{C}[V] = |G|.$$

a, 2pts) $\dim_{\mathbb{C}(V)^G} \mathbb{C}(V) = |G|$.
b, 2pts) $\mathbb{C}(V) = \mathbb{C}(V)^G \otimes_{\mathbb{C}[V]^G} \mathbb{C}[V]$.
c, 2pts) $\mathbb{C}(V)^G = \mathrm{Frac}(\mathbb{C}[V]^G)$.

Problem 3 (4pts). Let X be a factorial affine algebraic variety (factorial means that $\mathbb{C}[X]$ is a unique factorization domain) and G a connected algebraic group that has no nontrivial homomorphisms to $\mathbb{C}^\times$.

a, 2pts) Show that $\mathbb{C}[X]^G$ is a unique factorization domain as well.
b, 2pts) Show that there are finitely many elements $f_1, \ldots, f_k \in \mathbb{C}[X]^G$ and a Zariski open G-stable subset $X' \subset X$ such that for two points $x_1, x_2 \in X'$, the following are equivalent:

- $f_i(x_1) = f_i(x_2)$ for all i,
- and $Gx_1 = Gx_2$.

Extra-credit problem. Let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup. Suppose that the G-orbits in V are separated by G-invariant polynomials. Prove that G is finite.