COURSE INFORMATION ON INVARIANT THEORY

In this document you will find:

- A preliminary list of topics with short descriptions.
- A likely incomplete discussion of prerequisites.
- Some references.

1. List of topics with prerequisites

1.1. **Categorical quotients.** Let $\mathbb{C}$ be the field of complex numbers, V be a finite dimensional vector space and G be a group acting linearly on V. One can consider the algebra $\mathbb{C}[V]^G$ of G-invariant polynomial functions on V. An important question is when it is finitely generated. We will see that this is the case in the presence of an "averaging operator" (which is equivalent to the complete reducibility of a suitable class of representations of G). We will see that these conditions are satisfied for "rational representations" of "reductive algebraic groups". We will further see that one can replace V with an arbitrary affine variety X equipped with an algebraic action of G and still get that the algebra $\mathbb{C}[X]^G$ is finitely generated. The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ gives rise to a morphism $\pi : X \to \operatorname{Spec}(\mathbb{C}[X]^G) := X//G$. The variety $X//G$ (or, more precisely, the pair $(X//G, \pi)$) is called the *categorical quotient* of X. We will see that its points are in a natural bijection with the Zariski closed G-orbits in X.

Prerequisites: Algebraic geometry needed is mostly covered in [OV, Section 2.1], we will also need various facts about normal varieties and finite morphisms. The background on algebraic groups will be briefly recalled, it is contained in [OV, Section 3.1] as will be some structural features in the remainder of [OV, Section 3].

1.2. **Adjoint action.** We will consider an important special case of the above situation: the action of G on its Lie algebra $\mathfrak{g}$ via the adjoint representation (as well its generalizations). We will examine the structure of the action and compute the categorical quotient (via the Chevalley restriction theorem). We will also discuss a generalization known as Vinberg's θ-groups. Examples arising here are going to be of importance for much of the class.

Prerequisites: Some knowledge of the structure of semisimple Lie algebras over $\mathbb{C}$ and the corresponding algebraic groups (equivalently, Lie groups), see, e.g., [OV, Section 4.1].

1.3. **Hilbert-Mumford theorem.** Since the categorical quotient parameterizes closed orbits, it is important to known when an orbit is closed and how to determine which closed orbit is contained in the closure of a given one. The Hilbert-Mumford theorem to some extent reduces this question to an easy case of the multiplicative group. We will also consider "optimal destabilizing subgroups" and how they help to partially classify orbits in a given fiber of the quotient morphism.

Prerequisites: More or less the same as for Topic 1.

1.4. **Local structure of quotients.** We will state the Luna slice theorem, give a very brief sketch of proof and discuss applications.

Prerequisites: Same as for topic 1 plus facts about etale morphisms.

1.5. **Kempf-Ness theorem and connections to Symplectic geometry.** A Kempf-Ness theorem is a powerful tool to study the closedness of an orbit via the action of a maximal compact subgroup. We plan to discuss both some applications and a broader context of quotients in Symplectic geometry.

Prerequisites: Same as above. For a discussion of connections with Symplectic geometry, some basic familiarity with symplectic manifolds, e.g. [CdS, Sec. 1].

1.6. **GIT quotients.** This an important modification of categorical quotients, and, in a way, a far reaching generalization of the construction of the projective spaces. While the categorical quotients parameterize the closed orbits, GIT quotients require an additional datum, a character of the group acting, and parameterize so called semistable orbits. We will present the construction, discuss the Hilbert-Mumford theorem on semistability, and give examples including the Hilbert scheme of points on $\mathbb{C}^2$. We will also continue the discussion of connections with Symplectic geometry.

Prerequisites: Same as for Topic 1, plus good working knowledge of projective varieties and projective morphisms, e.g. [OV, Section 2.2]. Plus same as in Topic 4 for the discussion of connections with Symplectic geometry.

1.7. **Construction of moduli spaces.** We will discuss one of the most important applications of GIT: to constructing moduli spaces in Algebraic geometry. We will likely construct the coarse moduli space of vector bundles on a smooth projective curve.

Prerequisites: This is the most demanding part from the point of view of Algebraic geometry. Be prepared to having many black boxes but the knowledge of coherent sheaves ([Ha, Sec. 2.5]) and their cohomology (from [Ha, Ch. 3]) will likely be needed.

1.8. **Further topics.** Those may include the linearizability of line bundles for reductive group actions, the classical invariant theory, and/or the U-invariants. Details and prereqisites TBA.

2. References

We will mostly use [PV]. A reference for Topic 6 TBA. Detailed notes will be posted.

References

[B] N. Bourbaki, *Lie groups and Lie algebras.* Chapters 1-3, Springer-Verlag, Berlin, 1989, Chapters 4-6, Springer-Verlag, Berlin, 2002, Chapter 7-9, Springer-Verlag, Berlin, 2005.

[CdS] A. Cannas da Silva, *Lectures on Symplectic Geometry.* LNM 1764, Springer, 2008.

[CM] D. Collingwood, W. McGovern, *Nilpotent orbits in semisimple Lie algebras.* Van Nostrand Reinhold Math. Ser., 1993.

[Ha] R. Hartshorne, *Algebraic Geometry.* GTM 52, Springer, 1977.

[Hu] J. Humphreys, *Linear algebraic grouos.* GTM 21, Springer, 1975.

[K] V. Kac, *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, 1990, xxii+400 pp.

[Ko] B. Kostant, *Lie group representations on polynomial rings.* Amer. J. Math. 85 (1963), 327-404.

[L] I. Losev, *The Kempf-Ness theorem and Invariant theory.* Preprint (2006), arXiv:math. AG/0605756.

[LS] E. Lerman, R. Sjamaar, *Stratified symplectic spaces and reduction.* Ann. of Math. (2) 134 (1991), no. 2, 375–422.

[MF] D. Mumford, J. Fogarty, *Geometric invariant theory*, Ergeb. Math. Grenzgeb., 34, Springer, 1982.

[N] M. Nagata, *Complete reducibility of rational representations of a matric group.* J. Math. Kyoto Univ. 1 (1961/62), 87-99.

[OV] A. Onishchik, E. Vinberg, *Lie groups and algebraic groups.* Springer Ser. Soviet Math. Springer-Verlag, Berlin, 1990. xx+328 pp.

[PV] V. Popov, E. Vinberg, *Invariant theory.* In Algebraic Geometry, IV, EMS 55, Springer, 1994.

[Ver] J.-L. Verdier, *Stratifications de Whitney et theoreme de Bertini-Sard*, Invent. Math. 36 (1976), 295-312.

[V] E. Vinberg, *The Weyl group of a graded Lie algebra.* Math. USSR Izv. (1976) v. 10 N3.