PERVERSE SHEAVES

Contents

The goal of this note is to provide a crash-course on constructible and perverse sheaves. These sheaves play a crucial role in the geometric representation theory and, in particular, in the computation of characters of irreducible objects in interesting representation theoretic categories. We will briefly sketch this application.

1. Sheaves

We consider "reasonable" (=Hausdorff, paracompact, locally compact, locally contractible, etc.) spaces. In our applications we will be dealing with quasi-projective algebraic varieties over $\mathbb{C}$ (with *complex* topology), those are reasonable. By $\mathbb{K}$ we denote a commutative Noetherian ring (a coefficient ring). In this section X, Y always denote reasonable topological spaces and $f : X \to Y$ is a continouous map.

1.1. **Sheaves.** We consider the category $\mathrm{Sh}(X, \mathbb{K})$ of sheaves of $\mathbb{K}$-modules on X. This is an abelian category. For example, when X is a point, we have $\mathrm{Sh}(X, \mathbb{K}) = \mathbb{K}$-Mod.

Let us give some examples of sheaves.

Example 1.1. Let $x \in X$ and M be a $\mathbb{K}$-module. We have the sky-scrapper sheaf $\underline{M}_x$ whose sections are given by

$$\underline{M}_x(U) = \begin{cases} M, & \text{if } x \in U, \\ \{0\}, & \text{else} \end{cases} .$$

Example 1.2. Let M be a $\mathbb{K}$-module. Then we can consider the constant sheaf $\underline{M}_X$ whose sections on every connected open subset U are M and the restriction maps for the inclusion of connected open subsets are the identity.

Example 1.3. More generally, we can consider *locally constant* sheaves a.k.a. *local systems*, i.e., sheaves $\mathcal{F}$ such that each $x \in X$ has an open neighborhood U such that $\mathcal{F}|_U$ is constant. If X is connected and reasonable, then for any $x \in X$, we have an equivalence between

- the full subcategory $\mathrm{Loc}(X, \mathbb{K}) \subset \mathrm{Sh}(X, \mathbb{K})$ of local systems
- and the category $\mathbb{K}(\pi_1(X, x))$-Mod

given by taking the monodromy representation at x.

Exercise 1.4. Prove that $\mathrm{Loc}(X)$ is closed under taking kernels and cokernels.

Let $\mathcal{F} \in \mathrm{Sh}(X, \mathbb{K})$ and $x \in X$. We can take the *stalk*

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U).$$

This is a $\mathbb{K}$-module. If X is connected and $\mathcal{F}$ is a local system, then the stalks at different points are (non-canonically) identified.

1.2. **Functor f^*.** We have the *pull-back functor* $f^* : \mathrm{Sh}(Y, \mathbb{K}) \to \mathrm{Sh}(X, \mathbb{K})$ that sends $\mathcal{G} \in \mathrm{Sh}(Y, \mathbb{K})$ to $f^*\mathcal{G}$ to the sheafification of the following presheaf

$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V),$$

here and below we write U for an open subset of X and V for an open subset of Y.

Example 1.5. $f^*\underline{\mathbb{K}}_Y = \underline{\mathbb{K}}_X$.

For $y \in Y$ and the inclusion $i_y : \{y\} \to Y$ we have $i_y^*\mathcal{G} = \mathcal{G}_y$. Also note that, for two continuous maps f and g, we have

$$(fg)^* = g^* f^*. \tag{1.1}$$

Exercise 1.6. $(f^*\mathcal{F})_y = \mathcal{F}_{f(y)}$ and f^* is exact.

1.3. **Functor $^\circ f_*$.** As usual, f^* has the right adjoint functor, the push-forward functor $^\circ f_* : \mathrm{Sh}(X, \mathbb{K}) \to \mathrm{Sh}(Y, \mathbb{K})$ (we reserve the notation f_* for the corresponding derived functor to be discussed below):

$$^\circ f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}V).$$

(1.1) implies

$$^\circ(fg)_* = {^\circ f_*}\, {^\circ g_*}. \tag{1.2}$$

When Y is a point, $^\circ f_*$ becomes the global section functor Γ given by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.

1.4. **Functor $^\circ f_!$.** We also have a different version of push-forward, the *shriek* push-forward $^\circ f_!$. While $^\circ f_*$ is a relative version of Γ, the functor $^\circ f_!$ is a relative version of the functor Γ_c that takes global sections with compact support. More precisely, $^\circ f_!\mathcal{F}$ is defined by

$$^\circ f_!\mathcal{F}(V) = \{s \in \mathcal{F}(f^{-1}(V)) | f|_{\operatorname{supp}(s)} \text{ is proper}\}.$$

Here, as usual, $\operatorname{supp}(s)$ stands for the support of s, the set of all $x \in X$ such that the image of s in $\mathcal{F}_x$ is nonzero. We also can consider $\operatorname{supp}\mathcal{F} = \{x \in X | \mathcal{F}_x \neq \{0\}\}$, it does not need to be closed.

By definition, we have a natural inclusion $^\circ f_!\mathcal{F} \hookrightarrow {}^\circ f_*\mathcal{F}$. If f is proper, then this inclusion is an isomorphism.

Exercise 1.7. Prove that $^\circ f_!$ is left exact and that $^\circ(fg)_! = {}^\circ f_! {}^\circ g_!$.

Exercise 1.8. Suppose that $h : X \to Y$ is a locally closed inclusion. Then $^\circ h_!\mathcal{F}$ is the sheafification of the presheaf given by

$$V \mapsto \begin{cases} \mathcal{F}(V), & \text{if } V \cap \overline{X} \subset X, \\ \{0\}, & \text{else} \end{cases}.$$

For the stalks, we have $({}^\circ h_!\mathcal{F})_y = \mathcal{F}_y$ for $y \in X$ and $= 0$ else (because of this, $^\circ h_!$ is called the *extension by zero functor*). In particular, $^\circ h_!$ is exact in this case.

Exercise 1.9. Let U be an open subset of X and $Z := X \setminus U$. Let $j : U \hookrightarrow X, i : Z \hookrightarrow X$ be the inclusions. Then we have the following exact sequence in $\operatorname{Sh}(X, \mathbb{K})$:

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0.$$

1.5. **Internal $\mathcal{H}om$.** Let $\mathcal{F}_1, \mathcal{F}_2 \in \operatorname{Sh}(X, \mathbb{K})$. We can define $\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \in \operatorname{Sh}(X, \mathbb{K})$ via

$$\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)(U) := \operatorname{Hom}_{\operatorname{Sh}(U,\mathbb{K})}(\mathcal{F}_1|_U, \mathcal{F}_2|_U).$$

In particular, $\Gamma(\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)) = \operatorname{Hom}_{\operatorname{Sh}(X,\mathbb{K})}(\mathcal{F}_1, \mathcal{F}_2)$.

Exercise 1.10. We have $\mathcal{H}om(\underline{\mathbb{K}}_X, \mathcal{F}) = \mathcal{F}$ and $\mathcal{H}om(\mathcal{F}, \underline{M}_x) = \underline{\operatorname{Hom}}_{\mathbb{K}}(\mathcal{F}_x, M)_x$ (another sky-scrapper sheaf at x).

We can also define the tensor product of two sheaves, $\mathcal{F}_1 \otimes_{\mathbb{K}} \mathcal{F}_2$.

1.6. **Derived versions.** Note that $\operatorname{Sh}(X, \mathbb{K})$ has enough injectives. We consider the full derived category $D(X, \mathbb{K})$ of the abelian category $\operatorname{Sh}(X, \mathbb{K})$ and its subcategories D^+, D^-, D^b. For $\mathcal{F} \in D(X, \mathbb{K})$, we write $\mathcal{H}^i(\mathcal{F})$ for the ith cohomology sheaf in $\operatorname{Sh}(X, \mathbb{K})$.

The functors introduced above all have derived functors as follows:

1) f^* is t-exact.

2) $^\circ f_*$, $^\circ f_!$ have the right derived functors denoted by $f_*, f_!$. Note that $f_* : D^+(X, \mathbb{K}) \to D^+(Y, \mathbb{K})$ is still right adjoint of f^*.

3) $\mathcal{H}om$ has right derived functor $R\mathcal{H}om$.

We note that, for the derived functors, we still have

$$f_*g_* = (fg)_*, f_!g_! = (fg)_!. \tag{1.3}$$

and we still have a functor morphism

$$f_! \to f_*. \tag{1.4}$$

Now we provide some examples of (partial) computations of push-forwards.

Example 1.11. Let $\mathcal{L}$ be a local system on X. Then

$$H^k(R\Gamma(\mathcal{L})) = H^k_{sing}(X, \mathcal{L}),$$
$$H^k(R\Gamma_c(\mathcal{L})) = H^k_{sing,c}(X, \mathcal{L}),$$

where the subscript "sing" stands for the singular (=usual) cohomology. So, roughly speaking, $f_*, f_!$ should be thought as the relative versions of taking the cohomology and compactly supported cohomology.

Example 1.12. Let $j : X \to Y$ be an open inclusion and consider $\mathcal{L} \in \mathrm{Loc}(X, \mathbb{K})$. Let us compute the cohomology of $(j_*\mathcal{L})_y \in D(\mathbb{K}\text{-Mod})$. First of all, for $x \in X$, we have $(j_*\mathcal{L})_x = \mathcal{L}_x$ and for $y \notin \overline{X}$, we have $(j_*\mathcal{L})_y = \{0\}$. Now let us take $y \in \overline{X} \setminus X$. It follows from Example 1.11 combined with (1.3) that

$$H^k((j_*\mathcal{L})_y) = \varinjlim_{V \ni y} H^k_{sing}(V \cap X, \mathcal{L}).$$

Example 1.13. Let $Y = \mathbb{C}, X = \mathbb{C} \setminus \{0\}, \mathbb{K}$ be a field, and $\mathcal{L}$ be a rank one local system on X. Via the monodromy representation, it corresponds to a nonzero scalar $\alpha \in \mathbb{K} \leftrightarrow \mathcal{L}^\alpha \in \mathrm{Loc}(X, \mathbb{K})$, so that, in particular, $\mathcal{L}^1 = \underline{\mathbb{K}}_X$. Then, for $\alpha \neq 0$, we have $j_*\mathcal{L}^\alpha = j_!\mathcal{L}^\alpha$. Further, we have $H^0(j_*\mathcal{L}^1_0) = H^1(j_*\mathcal{L}^1_0) = \mathbb{K}$ and the other cohomology spaces vanish.

Now we discuss the proper base change. Consider a Cartesian diagram.

$$\begin{tikzcd}
X' \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Proposition 1.14. *We have a natural isomorphism $g^* f_! \cong \tilde{f}_! \tilde{g}^*$ of functors $D^+(X, \mathbb{K}) \to D^+(Y', \mathbb{K})$.*

Applying this to $Y' = \{y\}$ we see that

$$(f_!\mathcal{F})_y = R\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}). \tag{1.5}$$

Exercise 1.15. Find a counterexample to Proposition 1.14 when we use f_* instead of $f_!$.

1.7. **Functor $f^!$.** Assume further that there is an integer $n \geqslant 0$ such that one of the following (equivalent) conditions are satisfied:

(1) $H^{n+1}_c(X, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathrm{Sh}(X, \mathbb{K})$,
(2) every $x \in X$ has an open neighborhood U such that $H^{n+1}_c(U, \mathbb{K}) = 0$.

For example, this holds if X is locally a closed subspace of an n-dimensional manifold.

Proposition 1.16. *Let X, Y satisfy the equivalent conditions (1),(2) above. Then the functor $f_! : D^+(X, \mathbb{K}) \to D^+(Y, \mathbb{K})$ has right adjoint, $f^! : D^+(Y, \mathbb{K}) \to D^+(X, \mathbb{K})$.*

This is an existence result based on properties of $D^+(X, \mathbb{K}), D^+(Y, \mathbb{K})$ and $f_!$. In some cases, one can describe $f^!$ more explicitly.

For example, let $h : X \to Y$ be a locally closed inclusion. As we have seen in Exercise 1.8, $h_!$ is t-exact. It turns out that $h_! : \mathrm{Sh}(X, \mathbb{K}) \to \mathrm{Sh}(Y, \mathbb{K})$ has right adjoint, ${}^{\circ}h^!$, the *functor* *of restriction with supports.* It is the sheafification of the following presheaf

$$ {}^{\circ}h^!\mathcal{G}(U) := \varinjlim_{V \mid V \cap \overline{X} = U} \{s \in \mathcal{G}(V) \mid \operatorname{supp} s \subset U\}. $$

Exercise 1.17. Check that, indeed, ${}^{\circ}h^!$ is right adjoint of $h_!$. Moreover, check that if h is an open embedding, then ${}^{\circ}h^! = h^*$.

Of course, in this case $h^!$ is the derived functor of ${}^{\circ}h^!$.

2. Constructible sheaves

Now we change a setting and consider complex quasi-projective varieties equipped with their complex topology. The maps we consider will be morphisms of varieties. So X, Y are complex algebraic varieties and f is a morphism $X \to Y$. For simplicity, we assume that $\mathbb{K}$ has finite global dimension.

2.1. Pull-backs under smooth morphisms.

Let $f : X \to Y$ be smooth. In this section we will investigate the properties of the pull-back functors $f^*, f^!$.

First of all, we have the smooth base change. Consider a Cartesian diagram.

$$\begin{tikzcd}
X' \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Proposition 2.1. *Suppose g is smooth (hence $\tilde{g}$ is smooth of dimension d). Then there is a natural isomorphism $g^*f_* \xrightarrow{\sim} \tilde{f}_*\tilde{g}_*$ of functors $D^b(X, \mathbb{K}) \to D^b(Y', \mathbb{K})$.*

Now let us discuss a connection between f^* and $f^!$.

Proposition 2.2. *Suppose $f : X \to Y$ is smooth of relative dimension d. Then we have an isomorphism of functors $f^! \cong f^*[2d]$.*

Example 2.3. Let us see what happens when Y is a point (hence X is smooth), $\mathbb{K}$ is a field, and we apply our isomorphism to $\mathbb{K} \in \operatorname{Sh}(\mathrm{pt}, \mathbb{K})$. Here the proposition claims that $f^!\mathbb{K} = \underline{\mathbb{K}}_X[2d]$. Let $\mathcal{L}$ be a finite rank local system on X. Then we have

$$ \operatorname{Hom}_{D(X,\mathbb{K})}(\mathcal{L}, f^!\underline{\mathbb{K}}) = \operatorname{Hom}_{D(\mathbb{K}\text{-Mod})}(f_!\mathcal{L}, \mathbb{K}) = R\Gamma_c(\mathcal{L})^*. $$

On the other hand, let $\mathcal{L}^\vee$ denote the dual local system of $\mathcal{L}$. We have

$$ \operatorname{Hom}_{D(X,\mathbb{K})}(\mathcal{L}, f^*\underline{\mathbb{K}}[2d]) = \operatorname{Hom}_{D(X,\mathbb{K})}(\mathcal{L}, \underline{\mathbb{K}}_X[2d]) = R\Gamma(\mathcal{L}^\vee[2d]). $$

So we get a natural isomorphism $R\Gamma_c(\mathcal{L})^* \cong R\Gamma(\mathcal{L}^\vee[2d])$, equivalently,

$$ H^j_c(X, \mathcal{L}) = H^{2d-j}(X, \mathcal{L}^\vee)^*, \forall j \in \mathbb{Z}, $$

which is the statement of Poincare duality (for local systems). In other words, Proposition 2.2 is a relative and generalized version of the Poincare duality.

Remark 2.4. There is a way to make an isomorphism of Proposition 2.2 more canonical. Note that $H^2_c(\mathbb{C}, \mathbb{K}) \cong \mathbb{K}$ as a $\mathbb{K}$-module. Define the *Tate module* $\mathbb{K}(1) := \operatorname{Hom}_{\mathbb{K}}(H^2_c(\mathbb{C}, \mathbb{K}), \mathbb{K})$. For a $\mathbb{K}$-module M we set $M(d) := M \otimes_{\mathbb{K}} \mathbb{K}(d)$. Then one can strengthen Proposition 2.2 as follows: we have a canonical isomorphism

$$ f^! \cong f^*[2d](d). \tag{2.1} $$

2.2. **Stratifications.** We will not be interested in all sheaves, we will only consider those that are, in a sense, assembled from local systems. To define that class of sheaves we need the notion of a stratification.

Definition 2.5. A partition $X = \bigsqcup_{i=0}^{k} X_i$ is called a *stratification* if:

- X_i are smooth connected locally closed subvarieties,
- and, for each $i, j = 1, \dots, k$, we have $X_i \cap \overline{X}_j = X_i$ or $\varnothing$.

Example 2.6. $X = \mathbb{P}^n, k = n$ and $X_i = \{[x_1 : \dots : x_{i-1} : 1 : 0 : \dots : 0]\} \cong \mathbb{A}^i$.

Example 2.7. Let H be a connected algebraic group acting on X with finitely many orbits. Then we have the orbit stratification of X. We will mostly care about $X = G/P$, a partial flag variety, and $H = U$, a maximal unipotent subgroup of a reductive group G. In this way we get the parabolic Bruhat stratification, of which the previous example is a special case.

2.3. **Constructible sheaves.** Let $\mathcal{S}$ be a stratification of X. For a stratum X_i, let $h_i : X_i \hookrightarrow X$ denote the inclusion. We say that

- $\mathcal{F} \in \mathrm{Sh}(X, \mathbb{K})$ is *constructible w.r.t.* $\mathcal{S}$ if $h_i^* \mathcal{F}$ is a local system of *finite type* (=the stalks are finitely generated over $\mathbb{K}$).
- $\mathcal{F} \in D^b(X, \mathbb{K})$ is called *constructible w.r.t.* $\mathcal{S}$ if $\mathcal{H}^i(\mathcal{F})$ is constructible w.r.t. $\mathcal{S}$ for all i.
- $\mathcal{F} \in D^b(X, \mathbb{K})$ is called *constructible* if it is constructible w.r.t. some stratification.

We use the notations $\mathrm{Sh}_{\mathcal{S}}(X, \mathbb{K}), D^b_{\mathcal{S}}(X, \mathbb{K}), D^b_c(X, \mathbb{K})$ for the corresponding categories. Note that the former is an abelian category, while the latter two are triangulated ones.

Example 2.8. For the trivial stratification $\mathcal{S}$ given by $X = X_0$ (here X is smooth and connected), we have $\mathrm{Sh}_{\mathcal{S}}(X, \mathbb{K}) = \mathrm{Loc}_{ft}(X, \mathbb{K})$ (“ft” for “finite type”).

Example 2.9. Let $\mathcal{S}$ be the stratification of $\mathbb{P}^1$ from Example 2.6. Let us describe the category $\mathrm{Sh}_{\mathcal{S}}(X, \mathbb{C})$. The strata are simply connected, so all local systems there are trivial. Pick $\mathcal{F} \in \mathrm{Sh}_{\mathcal{S}}(X, \mathbb{C})$. Set $W_0 := \mathcal{F}_{[1:0]}, W_1 := \mathcal{F}_{[0:1]}$. To define a sheaf with these stalks, we need to specify one restriction map: from a neighborhood U of zero to the punctual neighborhood $U^\times$. This gives $\varphi : W_0 = \Gamma(U, \mathcal{F}) \to \Gamma(U^\times, \mathcal{F}) = W_1$. So we have constructed an equivalence between $\mathrm{Sh}_{\mathcal{S}}(X, \mathbb{C})$ and the category of finite dimensional representations of the A_2 quiver.

Exercise 2.10. What changes if we replace $\mathbb{P}^1$ with $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ and hence X_1 with $\mathbb{C}^\times$ in the previous example?

2.4. **Preservation of constructibility.** It turns out that the functors we consider restrict to the constructible derived categories.

Theorem 2.11. *The following are true:*

(1) *for* $\mathcal{F} \in D^b_c(X, \mathbb{K})$*, we have* $f_* \mathcal{F}, f_! \mathcal{F} \in D^b_c(Y, \mathbb{K})$*,*
(2) *for* $\mathcal{G} \in D^b_c(Y, \mathbb{K})$*, we have* $f^* \mathcal{G}, f^! \mathcal{G} \in D^b_c(X, \mathbb{K})$*,*
(3) *for* $\mathcal{F}_1, \mathcal{F}_2 \in D^b_c(X, \mathbb{K})$*, we have* $R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) \in D^b_c(X, \mathbb{K})$*.*

Remark 2.12. A more interesting question is when the functors above preserve the constructibility with respect to given stratifications. We are going to consider this question for the orbit stratifications, where the answer is “yes”. To start with, we need a different category.

Namely, for a topological group G acting on a topological space X it makes sense to speak about G-equivariant sheaves – with the usual definition. Denote the corresponding category by $\mathrm{Sh}_G(X, \mathbb{K})$. With some work, one can define the corresponding derived category $D^b_G(X, \mathbb{K})$, which is not the same as the derived category of $\mathrm{Sh}_G(X, \mathbb{K})$[1].

The sheaf functors we have considered upgrade to the equivariant derived categories.

Now assume that G is a connected algebraic group acting on a quasi-projective variety X with finitely many orbits. Let $\mathcal{S}$ denote the orbit stratification, Example 2.7. We have the forgetful functor $\mathsf{for} : D^b_G(X) \to D^b_{\mathcal{S}}(X)$ that intertwines the push-forward and pull-back functors. Using this functor one can get the affirmative answer to the question in the beginning of this remark. In fact, when G is unipotent, for is an equivalence.

2.5. **Dualizing sheaf and Verdier duality.** Finally, let us discuss a contravariant duality functor on $D^b_c(X, \mathbb{K})$. First, we need to define the dualizing sheaf.

Definition 2.13. The dualizing sheaf $\omega_X \in D^b_c(X, \mathbb{K})$ is $a^!_X\underline{\mathbb{K}}$, where $a_X : X \to \mathrm{pt}$.

So $\mathrm{Hom}_{D^b(X,\mathbb{K})}(\mathcal{F}, \omega_X) = \mathrm{Hom}_{D^b(\mathbb{K}\text{-Mod})}(R\Gamma_c(\mathcal{F}), \mathbb{K})$.

Example 2.14. Let X be smooth of dimension d. Then, by Proposition 2.2, $\omega_X = \underline{\mathbb{K}}_X[2d](d)$.

Also note that $f^!\omega_Y = \omega_X$.

Definition 2.15. The Verdier duality functor is

$$\mathbb{D}(\bullet) := R\mathcal{H}om(\bullet, \omega_X) : D^b_c(X, \mathbb{K}) \to D^b_c(X, \mathbb{K})^{opp}.$$

If we want to indicate the underlying variety, we put it as a subscript, e.g., write $\mathbb{D}_X$.

Example 2.16. Let X be smooth and irreducible of dimension d and $\mathcal{L}$ be a finite type local system on X whose stalks are projective $\mathbb{K}$-modules. Then $\mathbb{D}(\mathcal{L})$ is naturally isomorphic to $\mathcal{L}^\vee[2d](d)$.

The following theorem summarizes important properties of $\mathbb{D}$.

Theorem 2.17. *The following claims are true:*

(1) $\mathbb{D}$ *is an equivalence, moreover,* $\mathbb{D}^2$ *is naturally isomorphic to* id.
(2) *We have* $f_! \circ \mathbb{D}_X = \mathbb{D}_Y \circ f_*$ *and* $f_* \circ \mathbb{D}_X = \mathbb{D}_Y \circ f_!$.
(3) *We have* $f^! \circ \mathbb{D}_X = \mathbb{D}_Y \circ f^*$ *and* $f^* \circ \mathbb{D}_X = \mathbb{D}_Y \circ f^!$.
(4) *We have* $R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) = R\mathcal{H}om(\mathbb{D}\mathcal{F}_2, \mathbb{D}\mathcal{F}_1)$.

3. Perverse sheaves

Here we maintain the basic assumptions of the previous section and assume, in addition, that $\mathbb{K}$ is a field.

[1]That one shouldn't take $D^b\,\mathrm{Sh}_G(X, \mathbb{K})$ is clear already for a compact Lie group action on a point. Namely, one wishes that the self-extensions of the constant sheaf in the equivariant derived category is the equivariant cohomology of the point. Of course, there are no higher self-extensions in the category of finite dimensional representations of a compact Lie group

3.1. **t-structures.** A t-structure on a triangulated category is an additional structure that allows to see an abelian subcategory (the *heart*) inside, in particular, recovering an abelian category inside its derived category.

Namely, let $\mathcal{T}$ be a triangulated category and let $\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0}$ be two full subcategories. For $n \in \mathbb{Z}$, we set $\mathcal{T}^{\leqslant n} = \mathcal{T}^{\leqslant 0}[-n]$ and $\mathcal{T}^{\geqslant n} = \mathcal{T}^{\geqslant 0}[-n]$.

Definition 3.1. We say that the pair $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ is a *t-structure* if the following conditions hold:

(i) $\mathcal{T}^{\leqslant -1} \subset \mathcal{T}^{\leqslant 0}$ and $\mathcal{T}^{\geqslant -1} \supset \mathcal{T}^{\geqslant 0}$,
(ii) If $A \in \mathcal{T}^{\leqslant -1}$ and $B \in \mathcal{T}^{\geqslant 0}$, then $\mathrm{Hom}(A, B) = 0$.
(iii) For any $C \in \mathcal{T}$, there is a distinguished triangle $A \to C \to B \xrightarrow{+1}$ with $A \in \mathcal{T}^{\leqslant -1}$ and $B \in \mathcal{T}^{\geqslant 0}$.

We define the *heart* of the t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ as $\mathcal{T}^{\heartsuit} := \mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant 0}$.

Proposition 3.2. $\mathcal{T}^{\heartsuit}$ *is an abelian category.*

Example 3.3. Let $\mathcal{A}$ be an abelian category, and $\mathcal{T} := D^b(\mathcal{A})$. Then we set $\mathcal{T}^{\leqslant 0} = \{M \in \mathcal{T} | H^i(M) = 0, \forall i > 0\}$ and define $\mathcal{T}^{\geqslant 0}$ in a similar way. This is, obviously, a t-structure whose heart is identified with $\mathcal{A}$.

Let us return to the general situation. The inclusion $\mathcal{T}^{\leqslant i} \hookrightarrow \mathcal{T}$ has a right adjoint functor, to be denoted by $\tau^{\leqslant i}$. Dually, inclusion $\mathcal{T}^{\geqslant i} \hookrightarrow \mathcal{T}$ has a left adjoint functor, to be denoted by $\tau^{\geqslant i}$.

These are the so called *truncation functors*. For example, in condition (iii) above, we get $A = \tau^{\leqslant -1}(C), B = \tau^{\geqslant 0}(C)$.

For $C \in \mathcal{T}, i \in \mathbb{Z}$, we set

$$H^i(C) := \tau^{\geqslant 0}\tau^{\leqslant 0}(C[i]) = \tau^{\leqslant 0}\tau^{\geqslant 0}(C[i]) \in \mathcal{T}^{\heartsuit}.$$

These are the cohomology functors for the t-structure. Of course, for the tautological t-structure on $D^b(\mathcal{A})$ we recover the usual cohomology.

Exercise 3.4. If we have an exact triangle $C_1 \to C_2 \to C_3 \xrightarrow{+1}$ in $\mathcal{T}$ and $\mathcal{C}_1, \mathcal{C}_3 \in \mathcal{T}^{\heartsuit}$, then $C_2 \in \mathcal{T}^{\heartsuit}$. It follows that Ext^1's in $\mathcal{T}^{\heartsuit}$ are the same as in $\mathcal{T}$.

3.2. **Perverse t-structure: motivation and definition.** So we have the tautological t-structure $({}^{\tau}D^b_c(X, \mathbb{K})^{\leqslant 0}, {}^{\tau}D^b_c(X, \mathbb{K})^{\geqslant 0})$ ("τ" from "tautological") on $D^b_c(X, \mathbb{K})$ with heart $\mathrm{Sh}_c(X, \mathbb{K})$ but often we want to consider a different one. Here is one reason why the tautological t-structure is not "the best": it is not compatible with $\mathbb{D}$ in any reasonable sense. Namely, if $\iota : Z \hookrightarrow X$ is a closed inclusion of a smooth connected subvariety of and $\mathcal{L}_Z$ is a finite type local system on Z then, as we have seen, Example 2.16, $\mathbb{D}(\mathcal{L}_Z) \cong \mathcal{L}_Z^\vee[2\dim Z]$. We would like to have a t-structure compatible with $\mathbb{D}$ (meaning that $\mathbb{D}$ sends the $\leqslant 0$-part to $\geqslant 0$-part and vice versa).

For this we define a pair of full subcategories $({}^{p}D^b_c(X, \mathbb{K})^{\leqslant 0}, {}^{p}D^b_c(X, \mathbb{K})^{\geqslant 0})$ ("p" from "perverse") as follows:

$$\begin{aligned} {}^{p}D^b_c(X, \mathbb{K})^{\leqslant 0} &:= \{\mathcal{F} \in D^b_c(X, \mathbb{K}) | \dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leqslant -i, \forall i \in \mathbb{Z}\} \\ {}^{p}D^b_c(X, \mathbb{K})^{\geqslant 0} &:= \mathbb{D}({}^{p}D^b_c(X, \mathbb{K})^{\leqslant 0}) = \{\mathcal{F} \in D^b_c(X, \mathbb{K}) | \dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}\mathcal{F}) \leqslant -i, \forall i\}. \end{aligned} \tag{3.1}$$

Then we set

$$\mathrm{Perv}(X, \mathbb{K}) := {}^{p}D^b_c(X, \mathbb{K})^{\leqslant 0} \cap {}^{p}D^b_c(X, \mathbb{K})^{\geqslant 0}. \tag{3.2}$$

The objects of $\mathrm{Perv}(X,\mathbb{K})$ are called *perverse sheaves*. Note that ${}^pD^b_c(X,\mathbb{K})^{\leqslant 0} \subset {}^\tau D^b_c(X,\mathbb{K})^{\leqslant 0}$

Example 3.5. Let Z be as in the beginning of the section, then $\mathcal{L}_Z[\dim Z] \in \mathrm{Perv}(X,\mathbb{K})$.

Example 3.6. Let $X = \mathbb{C}, U = \mathbb{C}^\times$ and $j : U \to X$ be the open embedding. We claim that $j_*\underline{\mathbb{K}}_U[1], j_!\underline{\mathbb{K}}_U[1] \in \mathrm{Perv}(X,\mathbb{K})$. Note that

$${}^pD^b_c(X,\mathbb{K})^{\leqslant 0} = \{\mathcal{F} \in {}^\tau D^b_c(X,\mathbb{K})\}^{\leqslant 0} | \dim \mathrm{Supp}\, \mathcal{H}^0(\mathcal{F}) \leqslant 0\}.$$

So $j_!\underline{\mathbb{K}}_X[1] \in {}^pD^b_c(Y,\mathbb{K})^{\leqslant 0}$ because the stalks of this sheaf are in the homological degree -1. And recall, Example 1.13, that $\mathcal{H}^0(j_*\underline{\mathbb{K}}_U[1]) = \underline{\mathbb{C}}_0$, while there are no cohomology sheaves in positive degrees. So $j_*\underline{\mathbb{K}}_U[1] \in {}^pD^b_c(X,\mathbb{K})^{\leqslant 0}$. But $j_*\underline{\mathbb{K}}_U[1], j_!\underline{\mathbb{K}}_U[1]$ are dual to each other, hence both lie also in ${}^pD^b_c(X,\mathbb{K})^{\geqslant 0}$.

Remark 3.7. More generally, let $j : U \hookrightarrow X$ be an open embedding. The functor $j_!$ maps ${}^pD^b_c(U,\mathbb{K})^{\leqslant 0}$ to ${}^pD^b_c(X,\mathbb{K})^{\leqslant 0}$ because it doesn't introduce new nonzero stalks. Dually, j_* maps ${}^pD^b_c(U,\mathbb{K})^{\geqslant 0}$ to ${}^pD^b_c(X,\mathbb{K})^{\geqslant 0}$. However, in general, neither maps perverse sheaves to perverse sheaves: look at $U = \mathbb{C}^2 \setminus \{0\}, X = \mathbb{C}^2$ and $\underline{\mathbb{K}}_U[2]$.

Exercise 3.8. If $i : Z \hookrightarrow X$ is a closed embedding, then i_* is t-exact. If $j : U \hookrightarrow X$ is an open embedding, then $j^*(= j^!)$ is t-exact.

Moreover, i_* is a full embedding $\mathrm{Perv}(Z,\mathbb{K}) \hookrightarrow \mathrm{Perv}(X,\mathbb{K})$.

Theorem 3.9. *The pair* $({}^pD^b_c(X,\mathbb{K})^{\leqslant 0}, {}^pD^b_c(X,\mathbb{K})^{\geqslant 0})$ *is a t-structure (to be called the perverse t-structure). Hence* $\mathrm{Perv}(X,\mathbb{K})$ *is an abelian category.*

Again, we can work with a fixed stratification, $\mathcal{S}$, and define $\mathrm{Perv}_{\mathcal{S}}(X,\mathbb{K}) := \mathrm{Perv}(X,\mathbb{K}) \cap D^b_{\mathcal{S}}(X,\mathbb{K})$, etc. Theorem 3.9 still holds assuming the stratification is "good". For example, the orbit stratifications, Example 2.7, are good.

3.3. **Simple objects.** The goal of this section is to classify the simple objects in the abelian category $\mathrm{Perv}(X,\mathbb{K})$.

Example 3.10. If X is smooth and connected, and $\mathcal{L}$ is an irreducible local system on X, then $\mathcal{L}[\dim X]$ is a simple object of $\mathrm{Perv}(X,\mathbb{K})$. More generally, let Z be a smooth connected closed subvariety in X and let $i : Z \hookrightarrow X$ denote the inclusion. Let $\mathcal{L}$ be an irreducible local system on Z. Then $i_*\mathcal{L}[\dim Z]$ is an irreducible object in $\mathrm{Perv}(X,\mathbb{K})$.

We now proceed to the general construction of simples based on the so called *intermediate extension* functor.

Recall the functor morphism $h_! \to h_*$, (1.4). In Remark 3.7, we have seen that $h_!$ is left t-exact (preserves $\leqslant 0$ part), while h_* is right t-exact. We get a morphism

$$ {}^p\mathcal{H}^0(h_!\bullet) \to {}^p\mathcal{H}^0(h_*\bullet) \tag{3.3}$$

of functors $\mathrm{Perv}(Z,\mathbb{K}) \to \mathrm{Perv}(X,\mathbb{K})$ (where ${}^p\mathcal{H}^0$ denotes the zeroth cohomology functor for the perverse t-structure).

Definition 3.11. The intermediate extension functor $h_{!*} : \mathrm{Perv}(Z,\mathbb{K}) \to \mathrm{Perv}(X,\mathbb{K})$ is the image of (3.3).

Exercise 3.12. For $\mathcal{F} \in \mathrm{Perv}(Z)$, the object $h_{!*}(\mathcal{F})$ is the unique $\mathcal{G} \in \mathrm{Perv}(X)$ such that

- $\mathrm{Supp}(\mathcal{G}) \subset \overline{Z}$,
- $\mathcal{G}|_Z \cong \mathcal{F}$,
- $\mathcal{G}$ has no subs and quotients supported on $\overline{Z} \setminus Z$.

Because of this result, $h_{!*}$ is sometimes called the *minimal extension*.

Definition 3.13. Let $Z \subset X$ be a smooth, irreducible and locally closed subvariety, $h : Z \hookrightarrow X$ be an inclusion, and $\mathcal{L}$ be a local system on Z. Define the intersection cohomology sheaf $\mathrm{IC}(Z, \mathcal{L})$ as $h_{!*}(\mathcal{L}[\dim Z])$.

Remark 3.14. In fact, using Exercise 3.12 we can give a description of $\mathrm{IC}(Z, \mathcal{L})$ similar to the definition of a perverse sheaf. Namely, $\mathrm{IC}(Z, \mathcal{L})$ is a unique perverse sheaf $\mathcal{F}$ supported on $\overline{Z}$ whose restriction to Z is $\mathcal{L}[\dim Z]$ that, in addition, satisfies

- $\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) < -i, \forall i > -\dim Z$, (this conditions turns out to be equivalent to the claim that $\mathcal{F}$ has no quotients supported on $\overline{Z} \setminus Z$)
- $\dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}\mathcal{F}) < -i, \forall i > \dim Z$ (the same for subobjects).

Example 3.15. Let $X = \mathbb{C}, U = \mathbb{C}^\times$ and j be the inclusion. Let us compute $\mathrm{IC}(U, \mathcal{L})$ for rank one local systems on U. Recall that these local systems are classified by $\mathbb{K}^\times$ via monodromy: $a \mapsto \mathcal{L}^a$. We have computed the stalks of $j_*\mathcal{L}^a$ in Example 1.13. For $a \neq 1$, we have $j_*\mathcal{L}^a \cong j_!\mathcal{L}^a$ hence $\mathrm{IC}(U, \mathcal{L}^a) = j_!\mathcal{L}^a[1]$. For $a = 1$, $\mathcal{F} := \underline{\mathbb{K}}_X[1]$ satisfies the conditions of Remark 3.14. So $\mathrm{IC}(U, \mathcal{L}) = \underline{\mathbb{K}}_X[1]$.

The following theorem is a consequence of Example 3.10 and Exercise 3.12.

Theorem 3.16. *The following claims are true:*

- *if $\mathcal{L}$ is irreducible, then* $\mathrm{IC}(Z, \mathcal{L})$ *is simple in* $\mathrm{Perv}(X, \mathbb{K})$,
- *every simple in* $\mathrm{Perv}(X, \mathbb{K})$ *is isomorphic to* $\mathrm{IC}(Z, \mathcal{L})$ *for some* $Z, \mathcal{L}$,
- *we have* $\mathrm{IC}(Z_1, \mathcal{L}_1) \cong \mathrm{IC}(Z_2, \mathcal{L}_2)$ *if and only if* $Z_1 \cap Z_2$ *is open in both* Z_1, Z_2 *are* $\mathcal{L}_1|_{Z_1 \cap Z_2} \cong \mathcal{L}_2|_{Z_1 \cap Z_2}$.

Exercise 3.17. We have $\mathbb{D}\,\mathrm{IC}(Z, \mathcal{L}) = \mathrm{IC}(Z, \mathcal{L}^\vee)$.

Remark 3.18. The sheaves $\mathrm{IC}(Z, \mathcal{L})$ were essentially introduced by Goresky and MacPherson who were looking for a generalization of the Poincare duality to singular varieties (and more general singular spaces). Suppose $\overline{Z} = X$. Recall that $R\Gamma_c \circ \mathbb{D}_X(\bullet) = \mathbb{D}_{\mathrm{pt}} \circ R\Gamma(\bullet) = R\Gamma(\bullet)^*$. So $R\Gamma_c(\mathrm{IC}(Z, \mathcal{L}^\vee)) = R\Gamma(\mathrm{IC}(Z, \mathcal{L}))^*$ – which is a version of the Poincare duality.

3.4. **Properties of functors.** Here we consider properties of pull-back and push-forward functors under various assumptions on the morphism.

Lemma 3.19. *Let $f : X \to Y$ be an affine morphism. Then f_* is right t-exact (and $f_!$ is left t-exact) for the perverse t-structures.*

Combining this with Remark 3.7, we see that if f is an affine open embedding, then $f_*, f_!$ are t-exact, we have seen a special case of this in Example 3.6.

Exercise 3.20. Let $f : X \to Y$ be a smooth morphism of relative dimension d. Then $f^*[d] \cong f^![-d](-d)$ is t-exact for the perverse t-structures.

Remark 3.21. Note that $f^*[d]\,\mathrm{IC}(Z, \mathcal{L}) = \mathrm{IC}(f^{-1}(Z), f^*\mathcal{L})$. In particular, if f is a locally trivial fibration with connected fibers, then $f^*[d]$ maps simples to simples.

Finally, we need to consider the special case of proper morphisms from smooth varieties.

Definition 3.22. Let f be a proper surjective morphism $X \to Y$. We say that f is *semismall*, if, for each $d \geqslant 0$, we have

$$\dim\{y \in Y \,|\, \dim f^{-1}(y) \geqslant d\} \leqslant \dim Y - 2d.$$

If for all $d > 0$ we have $<$ in the inequality above, we say that f is *small*.

Lemma 3.23. *Let X be smooth and connected and let f be a surjective proper morphism. Assume f is semismall. Then for any $\mathcal{L} \in \mathrm{Loc}_{ft}(X, \mathbb{K})$, we have $f_*\mathcal{L}[\dim X] \in \mathrm{Perv}(Y, \mathbb{K})$.*

3.5. **Decomposition theorem.** The Beilinson-Bernstein-Deligne decomposition theorem is one of the most powerful tools to study the perverse sheaves. It comes with an additional important restriction: the coefficient field $\mathbb{K}$ must have characteristic 0. This is a geometric reason why the representation theory in characteristic p is harder than in characteristic 0.

Definition 3.24. By a *semisimple complex* in $D^b_c(X, \mathbb{K})$ we mean an object that is the direct sum of simple perverse sheaves with (homological) shifts.

Theorem 3.25. *Let X be a smooth connected variety and $f : X \to Y$ be a proper morphism. Suppose that* $\mathrm{char}\, \mathbb{K} = 0$. *Then $f_*\underline{\mathbb{K}}_X$ is a semisimple complex in $D^b_c(Y, \mathbb{K})$.*

Combining this with Lemma 3.23, we get the following corollary.

Corollary 3.26. *Assume, in addition, that f is semismall. Then $f_*\underline{\mathbb{K}}_X[\dim X]$ is a semisimple perverse sheaf on Y.*

Example 3.27. Let Y be the quadratic cone $\{(a, b, c)|ab = c^2\}$ (a.k.a. the nilpotent cone for $\mathfrak{sl}_2$) and $\mathbb{K} = \mathbb{C}$. Let X be the minimal resolution of singularities for Y. It is obtained by blowing up the zero point, i.e., $X = T^*\mathbb{P}^1$. The resolution morphism $f : X \to Y$ is semismall: it is an isomorphism over $Y^\times := Y \setminus \{0\}$, while $f^{-1}(0) = \mathbb{P}^1$. By equation (1.5), for $y \neq Y^\times$, we have $(f_*\underline{\mathbb{C}}_X[2])_y = \mathbb{C}[2]$, while $(f_*\underline{\mathbb{C}}_X[2])_0 = \mathbb{C}[2] \oplus \mathbb{C}[0]$. We have $(f_*\underline{\mathbb{C}}_X[2])|_{Y^\times} = \underline{\mathbb{C}}_{Y^\times}[2]$. It follows that $\mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times})$ has to be a direct summand in $f_*\underline{\mathbb{C}}_X[2]$. Let $\mathcal{F}$ denote a complimentary summand. By Remark 3.14,

$$\dim \mathrm{supp}\, \mathcal{H}^0(\mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times})) < 0 \Rightarrow \mathcal{H}^0(\mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times})) = 0.$$

So we must have $\mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times})_0 = \mathbb{C}[2]$. It follows that $\mathcal{F}_y = \{0\}, \mathcal{F}_0 = \mathbb{C}$, i.e., $\mathcal{F} = \underline{\mathbb{C}}_0$. We conclude that $f_*\underline{\mathbb{C}}_X[2] = \mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times}) \oplus \underline{\mathbb{C}}_0$. In fact, $\mathrm{IC}(Y^\times, \underline{\mathbb{C}}_{Y^\times}) \cong \underline{\mathbb{C}}_Y[2]$ despite the fact that Y is not smooth.

Exercise 3.28. In the situation of Theorem 3.25, if f is, in addition, small, then $f_*\underline{\mathbb{C}}_X[\dim X] = \mathbb{C}_Y[\dim Y]$.

Remark 3.29. One can extend Theorem 3.25 (with the same conclusion) to an arbitrary perverse sheaf instead of $\underline{\mathbb{K}}_X[\dim X]$.

Remark 3.30. The decomposition theorem fails when $\mathrm{char}\, \mathbb{K} > 0$. The simplest example is the situation of Example 3.27 in characteristic 2. More precisely, in that example we have

$$\dim \mathrm{Hom}(\underline{\mathbb{K}}_0, f_*\underline{\mathbb{K}}_X[2]) = \dim \mathrm{Hom}(f_*\underline{\mathbb{K}}_X[2], \underline{\mathbb{K}}_0) = 1$$

but the composed homomorphism

$$\mathrm{Hom}(\underline{\mathbb{K}}_0, f_*\underline{\mathbb{K}}_X[2]) \otimes \mathrm{Hom}(f_*\underline{\mathbb{K}}_X[2], \underline{\mathbb{K}}_0) \to \mathbb{K}$$

is given by the Euler class of $\mathbb{P}^1$ (in $H^2(\mathbb{P}^1, \mathbb{K})$). This Euler class is equal to 2. So $\underline{\mathbb{K}}_0$ is not a direct summand in $f_*\mathbb{K}_X$.

4. The case of parabolic flag variety

4.1. **Case of $\mathbb{P}^1$.** Here we describe the category $\mathrm{Perv}_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$, where $\mathcal{S}$ is the standard (Bruhat) stratification: $X_0 = \{[1 : 0]\}, X_1 = \mathbb{A}^1$. Our goal is to see that this category is equivalent to the principal block for $\mathfrak{sl}_2$.

Since both strata are connected, we have two simple objects: $L_0 := \underline{\mathbb{C}}_0, L_1 := \mathrm{IC}(X_1, \underline{\mathbb{C}}_{X_1}) = \underline{\mathbb{C}}_{\mathbb{P}^1}[1]$.

Let $j : X_1 \hookrightarrow \mathbb{P}^1$ denote the open inclusion. As we have seen above, Exercise 3.6, the objects $\nabla_1 := j_*\underline{\mathbb{C}}_{X_1}[1], \Delta_1 := j_!\underline{\mathbb{C}}_{X_1}[1]$ are also perverse. We note that for $\mathcal{F} \in \mathrm{Perv}_{\mathcal{S}}(\mathbb{P}^1, \mathbb{C})$, we have

$$\mathrm{Hom}_{D^b_c(\mathbb{P}^1,\mathbb{C})}(\Delta_1, \mathcal{F}[i]) = [j^! = j^*] = \mathrm{Hom}_{D^b_c(\mathbb{A}^1,\mathbb{C})}(\underline{\mathbb{C}}_{X_1}[1], j^*\mathcal{F}[i]) = H^{i-1}(\mathbb{A}^1, \mathcal{F}|_{\mathbb{A}^1}).$$

It follows that Δ_1 is a projective cover of L_1. Dually, ∇_1 is the injective hull of L_1.

Now note that, for $\mathcal{F}' \in D^b_c(\mathbb{P}^1, \mathbb{C})$,

$$\mathrm{Hom}_{D^b_c(\mathbb{P}^1,\mathbb{C})}(\mathcal{F}', L_0) = H^0(\mathcal{F}'_x)^*.$$

Recall that $H^0(j_*\underline{\mathbb{C}}_0) = H^1(j_*\underline{\mathbb{C}}_0)$ is one dimensional, while all the other homology groups of the stalk at 0 vanish. It follows that

$$\dim \mathrm{Hom}_{\mathrm{Perv}}(\nabla_1, L_0) = \dim \mathrm{Ext}^1_{\mathrm{Perv}}(\nabla_1, L_0) = 1.$$

Let P_0 denote the universal (in $\mathrm{Perv}(\mathbb{P}^1, \mathbb{C})$) extension $0 \to L_0 \to P_0 \to \nabla_1 \to 0$.

Exercise 4.1. We have $\dim \mathrm{Ext}^1_{\mathrm{Perv}}(L_0, L_1) = \dim \mathrm{Ext}^1_{\mathrm{Perv}}(L_1, L_0) = 1$ and hence non-split exact sequences

$$0 \to L_1 \to \nabla_1 \to L_0 \to 0,$$
$$0 \to L_0 \to \Delta_1 \to L_1 \to 0.$$

Moreover, $\mathbb{D}P_0 \cong P_0$.

The exact sequences of the previous exercise imply that

$$\dim \mathrm{Hom}_{\mathrm{Perv}}(P_0, L_0) = 1, \mathrm{Ext}^1_{\mathrm{Perv}}(P_0, L_0) = \mathrm{Hom}_{\mathrm{Perv}}(P_0, L_1) = 0.$$

We claim that $\mathrm{Ext}^1_{\mathrm{Perv}}(P_0, L_1) = 0$, equivalently, $\mathrm{Ext}^1_{\mathrm{Perv}}(L_1, P_0) = 0$. We use the exact sequence $0 \to L_0 \to P_0 \to \nabla_0 \to 0$. The relevant terms are:

$$\mathrm{Hom}(L_1, \nabla_1) \to \mathrm{Ext}^1(L_1, L_0) \to \mathrm{Ext}^1(L_1, P_0) \to \mathrm{Ext}^1(L_1, \nabla_1).$$

The first two spaces are both $\mathbb{C}$ and the homomorphism between them is an isomorphism because ∇_1 realizes a nontrivial extension between L_0, L_1. So the last homomorphism is injective. The last space is zero and we are done.

Exercise 4.2. Show that $\mathrm{Perv}_{\mathcal{S}}(\mathbb{P}^1)$ is equivalent to the principal block of the category $\mathcal{O}$ for $\mathfrak{sl}_2$.

4.2. **General case.** Now let G be a connected semisimple group over $\mathbb{C}$ and let $P (= P_J)$ be the parabolic subgroup of G corresponding to a subset J in the set of simple roots of G. We will be using the notation from the Hecke algebra lecture.

Set $\mathbb{K} = \mathbb{C}$. Consider the Bruhat stratification $\mathcal{S}$ on $X := G/P$. Recall that the strata are affine spaces labelled by W^J, the set of shortest representatives of cosets in $W_J \setminus W$. This is done as follows: $x \mapsto X_x := Bx^{-1}P/P$. Let $j_x : X_x \hookrightarrow G/P$ denote the inclusion. Note that $\Delta_x := j_{x,!}\underline{\mathbb{C}}_{X_x}[\dim X_x]$ and $\nabla_x := j_{x,*}\underline{\mathbb{C}}_{X_x}[\dim X_x]$ are objects in $\mathrm{Perv}_{\mathcal{S}}(X)$.

The following theorem follows from the work of Soergel.

Theorem 4.3. *The category* $\mathrm{Perv}_{\mathcal{S}}(X)$ *is equivalent to the principal block of the category* $\mathcal{O}$ *for* P*, where* $\Delta_x \leftrightarrow \Delta_J(w_{0,J}x \cdot (-2\rho)), \nabla_x \leftrightarrow \nabla_J(w_{0,J}x \cdot (-2\rho))$.

Let us now state a geometric incarnation of the parabolic Kazhdan-Lusztig theorem. Let us write $\mathrm{IC}(x)$ for the IC sheaf $\mathrm{IC}(X_x, \underline{\mathbb{C}}_{X_x})$ and $\mathrm{IC}(x)_y$ for the stalk of this sheaf at a point of X_y (doesn't matter which because the IC sheaf is constructible with respect to $\mathcal{S}$). It turns out that the dimensions of the cohomology of $\mathrm{IC}(x)_y$ are precisely the coefficients of the spherical parabolic Kazhdan-Lusztig polynomials. More precisely, we have the following.

Theorem 4.4. *We have* $m_{y,x} = \sum_{i\in\mathbb{Z}} \dim H^{-i}(\mathrm{IC}(x)_y) v^{i-\dim X_y}$.

Example 4.5. Consider $x = w_{0,J}w_0$ so that X_x is the open stratum. Then $\mathrm{IC}(x) = \underline{\mathbb{C}}_{G/P}[\dim G/P]$ so the theorem says $m_{y,x} = v^{\dim G/P - \dim X_y}$. We have seen that this is the case for $X = G/B$ (Example 2.7 in the Hecke algebra lecture).

Exercise 4.6. Use Exercise 1.9 and Theorem 4.4 to prove that the coefficient of $[\Delta_y]$ in $[L_x]$ equals $m_{y,x}(-1)$ (mind the homological shifts!).

Sketch of proof of Theorem 4.4 for $P = B$. Assume, for simplicity, that $P = B$. The proof in the general case is analogous.

For $\mathcal{F} \in D^b_{\mathcal{S}}(X)$, define its "character" in $\mathcal{H}$ by

$$\mathrm{ch}\,\mathcal{F} = \sum_{y\in W}\sum_{i\in\mathbb{Z}} \dim H^{-i}(\mathcal{F}_y) v^{i-\ell(y)} H_y.$$

Also set $\underline{H}'_x := \mathrm{ch}\,\mathrm{IC}(x)$. What we need to prove is that $\underline{H}'_x = \underline{H}_x$. For this, we use the uniqueness part of Theorem in the Hecke algebra lecture. We need to show that

(1) $\underline{H}'_x \in H_x + v\,\mathrm{Span}_{\mathbb{Z}[v]}(H_y)$,
(2) $\underline{H}'_x$ is self-dual.

Part (1) is a direct consequence of the inequalities in Remark 3.14.

Part (2) is more subtle and is based on the decomposition theorem. As usual, we use the induction with respect to the Bruhat order and assume that (2) and also

(3) the stalks of $\mathrm{IC}(X_{x'})$ have nonzero cohomology in the same parity, i.e., only in the even degrees or only in the odd degrees

are known for all $x' \prec x$. Pick a simple reflection s such that $\ell(xs) < \ell(x)$. Let P_s be the corresponding minimal parabolic

To prove (3), consider the following variety

$$\overline{X}_{xs,s} = \overline{BxsB} \times^B P_s/B$$

We have the following two morphisms

$$p : \overline{X}_{xs,s} \to \overline{X}_{xs}, q : \overline{X}_{xs,s} \to \overline{X}_x,$$

given by $p([f_1, f_2]) = f_1B, q([f_1, f_2]) = f_1f_2$. The morphism p is a $\mathbb{P}^1$-bundle, while the morphism q is projective. One can show that q is semismall. So we have a t-exact functor $\varphi := q_*p^*[1] : \mathrm{Perv}_{\mathcal{S}}(\overline{X}_{xs}) \to \mathrm{Perv}_{\mathcal{S}}(\overline{X})$. By Remark 3.21, $p^*[1]$ sends $\mathrm{IC}(xs)$ to $\mathrm{IC}(p^{-1}(X_{xs}), \underline{\mathbb{C}}_{p^{-1}(X_{xs})})$. Then q_* sends this sheaf to the direct sum of simple perverse sheaves, thanks to the general form of the decomposition theorem. The sheaf $\mathrm{IC}(x)$ occur in this sum with multiplicity 1, while the other summands are $\mathrm{IC}(y)$'s with $y \prec x$. One can show that (3) for xs implies the analog of (3) for $p_*q^*[1](\mathrm{IC}(xs))$ hence for its direct summand $\mathrm{IC}(x)$.

To prove (2), pick a reduced decomposition $x = s_1 \dots s_k$, let $\underline{x}$ denote the word $(s_1, \dots, s_k)$. The variety $\overline{X}_x$ has a resolution of singularities, the Bott-Samelson variety $\overline{X}_{\underline{x}} = P_{s_1} \times^B P_{s_2} \dots \times^B P_{s_k}/B$. Apply the Decomposition theorem to $\underline{\mathbb{C}}_{\overline{X}_{\underline{x}}}$ and the resolution of singularities morphism $\rho : \overline{X}_{\underline{x}} \to \overline{X}_x$. The image $\rho_* \underline{\mathbb{C}}_{\overline{X}_{\underline{x}}}[\ell(x)]$ is the direct sum of shifted perverse sheaves and is Verdier self-dual. Similarly to the previous paragraph, the character of $\rho_* \underline{\mathbb{C}}_{\overline{X}_{\underline{x}}}[\ell(x)]$ is $C_{s_1} \dots C_{s_k}$, it is self-dual. Moreover, we have

$$\rho_* \underline{\mathbb{C}}_{\overline{X}_{\underline{x}}}[\ell(x)] = \mathrm{IC}(x) \oplus \bigoplus_{y \prec x} \mathrm{IC}(y)[?]^{\oplus ?}.$$

The second summand is self-dual. Applying (2) for this summand, we see that its character is self-dual. Hence the character of $\mathrm{IC}(x)$ is self-dual. $\square$